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Delay-range-dependent synchronization of drive and response systems under input delay and saturation

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ABSTRACT

This paper addresses the synchronization of nonlinear drive and response systems under input saturation and subject to input time-delay. In considering generalized forms of the systems, their dynamics are assumed to satisfy the one-sided Lipschitz condition along with the quadratic inner-boundedness rather than the conventional Lipschitz condition. Further, the time-delays are handled by application of the delay-range-dependent methodology, rather than the delay-dependent one, utilizable for both short and long time-delays. Synchronization controller designs are provided by application of the Lyapunov–Krasovskii functional, local sector condition, generalized Lipschitz continuity, quadratic inner-boundedness criterion and Jensen's inequality. To the best of the authors' knowledge, a delay-range-dependent synchronization control approach for the one-sided Lipscitz nonlinear systems under input delay and saturation constraints is studied for the first time. A convex-routine-based solution to the controller gain formulation by application of recursive nonlinear optimization using cone complementary linearization is also provided. The proposed methodology is validated for synchronization of modified Chua's circuits under disturbances by considering the input delay and saturation constraints.

1. Introduction

Synchronization of complex nonlinear systems, made possible by means of a feedback controller, has vast applications in robotics, secure communications, image processing, avionics, information processing, and biomedical networks [1–5]. The main purpose of synchronization control is to establish a coherent behavior between the drive and response systems by applying a feedback of the difference between the states or outputs [6–8]. Different control schemes and tools for synchronization of nonlinear systems have been realized: Nonetheless, selection of a synchronization controller and type of control methodology depend on the circumstances and actual environment, which often vary from case to case. For instance, adaptive controllers are used for adaptation of wide-ranging unknown parameters, as seen in [9]. Robust controllers, meanwhile, are applicable to fast-varying changes and perturbations, as demonstrated in [10]. Constrained controllers are employed to deal with input, state or output restraints such as saturation. Likewise, consensus controllers are designed to deal with specific communication and network protocols (see [11,12]). Output and state feedback controllers are employed according to the availability of states and outputs. Observer-based controllers are preferable to attain the advantages of the state feedback approaches when the states are unknown [13]. Disturbance-observer-based controllers are utilized for adaptive cancellation of unknown matching disturbances [14]. Controller design for effectual synchronization remedy of nonlinear systems is still a thought-provoking research area, especially in view of system dynamics, uncertainties, various constraints, and overall performance goals.

Controllers for synchronization of nonlinear time-delay systems are designed to utilize time-delay data such as lower and upper bounds, the rate of delay, and the number of delays appearing in the state, input or output. Conventional controllers for synchronization of nonlinear systems might not guarantee synchronization, because time-delays can cause oscillations and instability in the response of the synchronization error. Several attempts to synthesize synchronization controllers for time-delay systems have been made, exclusively by employing delay-independent and delay-dependent methods and by applying elusive delay-range-dependent techniques. For instance, two delay-dependent synchronization control methods for Lur'e systems based on delayed

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feedback via the partitioning-interval approach were developed in [15]. Zhang et al. [16] utilized range-of-delay information to develop a global synchronization methodology for complex networks under stochastic disturbances. Fei and coauthors [17] followed the delay-partitioning approach in studying the coherent behavior of a complex network with interval time-varying delay coupling. Li et al. [18] utilized a novel Lyapunov function in their evaluation of a delay-range-dependent synchronization control mechanism for Lur'e systems. The work in [19] achieved the synchronization of chaotic systems with time-varying state delays and delayed nonlinear coupling between the drive and response systems. Recently, Ma and Jing [20] developed, by means of a local sector condition, a delay-independent state-feedback control approach for synchronization of uncertain nonlinear systems with time-varying state delays and input saturation. More recently, Cai and coworkers [21] have reported delay-dependent synchronization conditions of singularly perturbed systems with coupling delays.

Works on the delay-range-dependent stability investigation, control and synchronization proficiencies (owing to their utilities for dealing with short as well as long time-delays in the state, output, input or coupling between nonlinear systems) are proceeding apace. The literature on synchronization of the nonlinear systems using a delay-range-dependent approach by incorporating the input saturation nonlinearity and time-delays, however, is deficient. There are two major issues with the existing synchronization techniques for the nonlinear time-delay systems. First, most of the above-mentioned studies employ a conservative continuity condition like the conventional Lipschitz condition for the derivation of the synchronization control strategies. The literature of mathematics has developed a less conservative one-sided Lipschitz condition, which can be used to represent the Lipschitz nonlinear systems as a specific case of the one-sided Lipschitz nonlinear systems. Moreover, the one-sided Lipschitz constant may have a smaller value than the Lipschitz constant, which fact can be more effectively applied for derivation of the controllers to synchronize nonlinear oscillators with large or region dependent Lipschitz constants. Second, the input saturation nonlinearity cannot be ignored in practical systems because an untreated saturation nonlinearity can lead to oscillations, lags, overshoots, undershoots, performance abasement, and divergence of the closed-loop system response. For synchronization of the nonlinear systems under input time-delays, dealing with the saturation consequences is a non-trivial research dilemma owing to simultaneous considerations of the input saturation and the input delay effects.

This paper introduces controller design for synchronization of nonlinear systems under input saturation and subject to input time-delay varying within an interval of known lower and upper bounds. By utilizing the Lyapunov–Krasovskii (LK) functional, one-sided Lipschitz condition, quadratic inner-boundedness, the range of the input delay, the limit on the derivative of the delay, the local sector condition for input saturation and Jensen's inequality, nonlinear matrix inequalities are derived to determine an appropriate controller gain matrix, specifically by providing an estimate of the region of stability in terms of synchronization error. From these principal design conditions, novel synchronization controller design conditions for Lipschitz nonlinear systems, both for the delay-dependent case and for the scenario of an unknown bound on the delay-rate, are derived. Moreover, the proposed method is extended for robust synchronization of nonlinear systems under input lag and saturation by considering the L_2 norm-bounded perturbations in evaluating the allowable bound of the disturbance and disturbance attenuation level at the state estimation error.

The main contributions of the paper are summarized as follows: (i) To the best of our knowledge, delay-range-dependent synchronization of the nonlinear systems under input saturation and input delay, to deal with the practical limitations of actuators, is addressed for the first time. (ii) An inaugural treatment of synchronization of time-delay in one-sided Lipschitz nonlinear systems is provided. Such an approach is less conservative and can be employed to synchronize a broader class of nonlinear systems than the conventional Lipschitz systems. (iii) An estimate of the region of stability in terms of the difference between initial conditions of the nonlinear master-slave systems under input delay and saturation is provided. (iv) A robust synchronization method for time-delay nonlinear systems with one-sided Lipschitz nonlinearities, input delay, input saturation, and external perturbations is explored. In this regard, an upper bound on the L_2 norm of the synchronization error in terms of the initial condition and disturbances and the region in which the synchronization error remains bounded are revealed.

Additionally, a numerically tractable approach is outlined for determining the synchronization controller gain matrix, parameters representing the ellipsoidal region of stability, and scalars to constitute bounds on the synchronization error by utilizing the cone complementary linearization algorithm. Finally, a numerical simulation example is provided to demonstrate the effectiveness of the proposed methodology for synchronization of input-constrained modified chaotic Chua's circuits in the presence of input delays and disturbances.

This paper is organized as follows: the drive and response systems are described in Section 2. In Section 3, various synchronization controller designs for dealing with nonlinearities, delays, input saturation and disturbances are introduced. In Section 4, simulation results are provided. Concluding remarks are rendered in Section 5.

Standard notation is used in this paper. A block diagonal matrix is denoted as $\mathrm{diag}(x_1, x_2, \ldots, x_m)$, where $x_1, x_2, \ldots, x_m$ are entries at the corresponding diagonal blocks. L_2 norm for a vector $x \in R^n$ is represented as $\|x\|_2$ and the ith row of a matrix A is assigned as $A_{(i)}$. $\langle w, v\rangle$ represents the inner product between two vectors w and v of matching dimensions. The saturation nonlinearity is defined by $\Psi_{(i)}(u_{(i)}) = \mathrm{sgn}(u_{(i)})\min(\bar{u}_{(i)}, |u_{(i)}|)$ for the saturation bound given as $\bar{u}_{(i)} > 0$. Positive definite and positive semi-definite matrices are represented as $Y > 0$ and $Y \geq 0$, respectively, for a symmetric matrix Y.

2. System description

Consider a master (or drive) system

$$\frac{dx_m}{dt} = Ax_m + f(t, x_m) + d_1,$$
$$y_m(t) = Cx_m, \tag{1}$$

where $x_m \in R^n$, $y_m \in R^p$ and $d_1 \in R^m$ represent the state, output and disturbance vectors, respectively. A and C are constant matrices of appropriate dimensions, and $f(t, x_m) \in R^n$ denotes the nonlinear dynamics in the system. The slave (or response) system is given by

$$\frac{dx_s}{dt} = Ax_s + f(t, x_s) + B\Psi(u(t-\tau)) + d_2,$$
$$y_s(t) = Cx_s, \tag{2}$$

where $x_s \in R^n$, $y_s \in R^p$, $u \in R^q$ and $d_2 \in R^m$ are the state, output, control input and disturbance to the response system, respectively, B is the input matrix, and $\Psi(u)$ is the input saturation vector-function. The input time-delay satisfies

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \tag{3}$$

$$\dot{\tau}(t) \leq \mu. \tag{4}$$

Assumption 1. The function $f(t, x_m)$ satisfies the one-sided Lipschitz condition given as

$$\langle f(t, x_m) - f(t, x_s), x_m - x_s\rangle \leq \rho\|x_m - x_s\|^2 \tag{5}$$

for a scalar ρ, and the quadratic inner-boundedness condition given as

$$(f(t,x_m)-f(t,x_s))^T(f(t,x_m)-f(t,x_s)) \leq \delta\|x_m-x_s\|^2+\sigma\langle x_m-x_s, f(t,x_m)-f(t,x_s)\rangle \quad (6)$$

for scalars δ and σ.

Both the one-sided Lipschitz condition and the quadratic inner-boundedness inequality have been employed to construct observers and controllers for a broader class of nonlinear systems relative to the Lipschitz continuity [22–26]. The one-sided Lipschitz and quadratic inner-boundedness conditions are the supersets of the conventional Lipschitz condition. Moreover, the conditions in (3) and (4) are less conservative even for the Lipschitz nonlinear systems as addressed in [22–26]. Therefore, the present work explores a synchronization controller design method for the generalized class of systems.

Defining the synchronization error $e = x_m - x_s$, (1) and (2) implies

$$\frac{de}{dt} = Ae + f(t,x_m) - f(t,x_s) - B\Psi(u(t-\tau)) + d_1 - d_2,$$

which, by application of $\Phi(u) = u - \Psi(u)$ and $d = d_1 - d_2$, further reveals

$$\frac{de}{dt} = Ae + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) - Bu(t-\tau) + d. \quad (7)$$

For the dead-zone function $\Phi(u)$, the sector condition

$$\Phi^T(u(t-\tau))W[w(t-\tau) - \Phi(u(t-\tau))] \geq 0 \quad (8)$$

is satisfied for a diagonal positive-definite matrix W (see [27,28]) if, for an auxiliary defined vector $w \in R^q$, the region given by

$$S(\bar{u}) = \left\{w \in R^m, -\bar{u}_{(i)} \leq u_{(i)}(t-\tau) - w_{(i)}(t-\tau) \leq \bar{u}_{(i)}\right\} \quad (9)$$

remains valid for the saturation limit $\bar{u}$.

3. Controller synthesis

The proposed controller for synchronization of (1)and (2) is given by

$$u(t) = Ke(t), \quad (10)$$

for an appropriate controller gain matrix K. The overall closed-loop system by considering the drive system (1), response system (2), and the proposed controller (10) is shown in Fig. 1. The drive and the response systems are under external disturbances d_1 and d_2, respectively. The synchronization error state vector is computed via equation $e = x_m - x_s$ and then sent as a feedback to the proposed controller (10). The control signal is computed via $u(t) = Ke(t)$ and assigned to the response system, which undergoes the saturation nonlinearity and the input delay, inherently present in the response system. The aim of the present study is to compute the controller gain matrix K for synchronization of the drive and the response systems subject to the input saturation and time-varying unknown input time-delay $\tau(t)$ in the absence or in the presence of the disturbances.

By fixing $w(t-\tau) = Je(t-\tau)$ for an auxiliary matrix J of matching dimensions, we obtain

$$\Phi^T(u(t-\tau))W[Je(t-\tau) - \Phi(u(t-\tau))] \geq 0, \quad (11)$$

$$S(\bar{u}) = \left\{w \in R^m, -\bar{u}_{(i)} \leq \left(K_{(i)} - J_{(i)}\right)e(t-\tau) \leq \bar{u}_{(i)}\right\}. \quad (12)$$

Application of (7) and (10) obtains

$$\frac{de}{dt} = Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d. \quad (13)$$

For positive definite matrices P, Q_1, Q_2, Q_3, Z_1 and Z_2, we define an LK functional ([19,23], and [29]) as

$$\begin{aligned} V(t,e) =\ & e^T(t)Pe(t) + \sum_{i=1}^{2}\int_{t-\tau_i}^{t} e^T(\theta)Q_ie(\theta)d\theta \\ & + \int_{t-\tau(t)}^{t} e^T(\theta)Q_3e(\theta)d\theta + \tau_1\int_{-\tau_1}^{0}\int_{t+s}^{t}\dot{e}^T(\theta)Z_1\dot{e}(\theta)d\theta ds \\ & + \tau_{21}\int_{-\tau_2}^{-\tau_1}\int_{t+s}^{t}\dot{e}^T(\theta)Z_2\dot{e}(\theta)d\theta ds. \end{aligned} \quad (14)$$

A sufficient condition for synchronization of the master-slave systems given by (1) and (2) using the delayed controller (10) under the input saturation constraint is provided in the form of the following theorem.

Theorem 1. *Consider the drive and response systems (1) and (2) under delayed and saturated control signal $\Psi(u(t-\tau))$ satisfying the time-delay properties (3) and (4), $d(t) = 0$, and Assumption 1. Suppose that there exist matrices K and J, symmetric matrices P, Q_1, Q_2, Q_3, Z_1 and Z_2, diagonal matrix W and scalars υ_1 and υ_2 such that the inequalities*

$$\begin{aligned} & P>0,\ Q_1>0,\ Q_2>0,\ Q_3>0,\ Z_1>0,\ Z_2>0, \\ & W>0,\ \upsilon_1>0,\ \upsilon_2>0, \end{aligned} \quad (15)$$

$$\begin{bmatrix} P & K_{(i)}^T - J_{(i)}^T \\ * & \bar{u}_{(i)}^2 \end{bmatrix} \geq 0,\ \forall i = 1, \ldots, m, \quad (16)$$

$$\Lambda_1 = \begin{bmatrix} \Psi_1 & -PBK & Z_1 & 0 & (-\nu_1+\sigma\nu_2)I+P & PB+J^TW & \tau_1A^TZ_1 & \tau_{21}A^TZ_2 \\ * & \Psi_2 & Z_2 & Z_2 & 0 & 0 & \tau_{21}K^TB^TZ_1 & \tau_{21}K^TB^TZ_2 \\ * & * & -Q_1-Z_1-Z_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -Q_2-Z_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\nu_2I & 0 & \tau_1Z_1 & \tau_{21}Z_2 \\ * & * & * & * & * & -2W & \tau_1B^TZ_1 & \tau_{21}B^TZ_2 \\ * & * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & * & -Z_2 \end{bmatrix} < 0 \quad (17)$$

are satisfied, where

$$\begin{aligned} \Psi_1 &= PA + A^TP + Q_1 + Q_2 + Q_3 - Z_1 + (\rho\nu_1 + \delta\nu_2)I, \\ \Psi_2 &= -(1-\mu_1)Q_3 - 2Z_2, \\ \tau_{21} &= \tau_2 - \tau_1. \end{aligned}$$

Then, for all initial conditions holding for region $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \leq 1$ for all $\vartheta \in [-\tau(t) \quad 0]$, the synchronization error defined by $e(t) = x_m(t) - x_s(t)$ converges to the origin asymptotically.

Proof. The time derivative of $V(t, e)$ is given by

$$\begin{aligned} \dot{V}(t,e) =\ & e^T(t)P(Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) \\ & + B\Phi(u(t-\tau)) + d) + (Ae - BKe(t-\tau) + f(t,x_m) \\ & - f(t,x_s) + B\Phi(u(t-\tau)) + d)Pe(t) \\ & - \sum_{i=1}^{2} e^T(t-\tau_i)Q_ie(t-\tau_i) + \sum_{i=1}^{3} e^T(t)Q_ie(t) - (1-\mu)e^T \end{aligned}$$

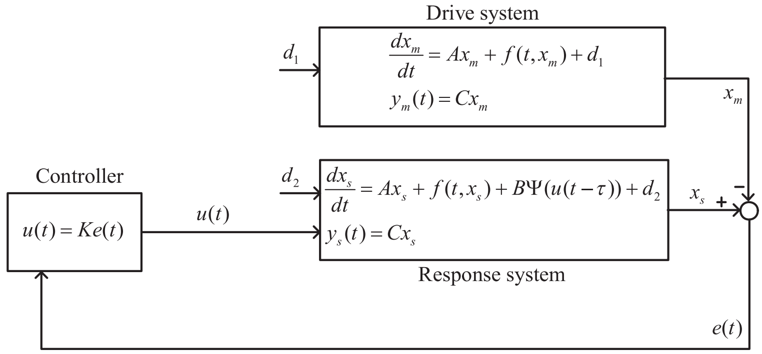

Fig. 1. Closed-loop system formed by the drive and the response systems by using the proposed control strategy.

$$\begin{aligned}&\times(t-\tau(t))Q_3e(t-\tau(t)) + (Ae - BKe(t-\tau)\\&+f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)^T\\&\times(\tau_1^2Z_1 + \tau_{21}^2Z_2)\times(Ae - BKe(t-\tau) + f(t,x_m)\\&-f(t,x_s) + B\Phi(u(t-\tau)) + d)\\&-\tau_1\int_{t-\tau_1}^{t}\dot e^T(\theta)Z_1\dot e(\theta)d\theta - \tau_{21}\int_{t-\tau_2}^{t-\tau_1}\dot e^T(\theta)Z_2\dot e(\theta)d\theta.\end{aligned} \tag{18}$$

The conditions (5) and (6) for positive scalars ν_1 and ν_2 are rewritten as

$$\nu_1e^T(t)(f(t,x_m) - f(t,x_s)) \le \rho\nu_1e^T(t)e(t), \tag{19}$$

$$\begin{aligned}&\nu_2(f(t,x_m) - f(t,x_s))^T(f(t,x_m) - f(t,x_s))\\&\le \delta\nu_2e^T(t)e(t) + \sigma\nu_2e^T(t)(f(t,x_m) - f(t,x_s)).\end{aligned} \tag{20}$$

According to (4), (18), (19) and (20), this implies that

$$\begin{aligned}&\dot V(t,e)\\&\le e^T(t)P(Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)\\&+(Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)\\&Pe(t) - \sum_{i=1}^{2}e^T(t-\tau_i)Q_ie(t-\tau_i) + \sum_{i=1}^{3}e^T(t)Q_ie(t)\\&-(1-\mu)e^T(t-\tau(t))Q_3e(t-\tau(t)) + (Ae - BKe(t-\tau)\\&+f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)^T\left(\tau_1^2Z_1 + \tau_{21}^2Z_2\right)\\&\times(Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)\\&-\tau_1\int_{t-\tau_1}^{t}\dot e^T(\theta)Z_1\dot e(\theta)d\theta - \tau_{21}\int_{t-\tau_2}^{t-\tau_1}\dot e^T(\theta)Z_2\dot e(\theta)d\theta\\&-\nu_1e^T(t)(f(t,x_m) - f(t,x_s)) + \rho\nu_1e^T(t)e(t) - \nu_2(f(t,x_m)\\&-f(t,x_s))^T(f(t,x_m) - f(t,x_s)) + \delta\nu_2e^T(t)e(t)\\&+\sigma\nu_2e^T(t)(f(t,x_m) - f(t,x_s)).\end{aligned} \tag{21}$$

Incorporation of (11) obtains

$$\begin{aligned}&\dot V(t,e)\\&\le e^T(t)P(Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau))\\&+d) + (Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau))\\&+d)Pe(t) - \sum_{i=1}^{2}e^T(t-\tau_i)Q_ie(t-\tau_i) + \sum_{i=1}^{3}e^T(t)Q_ie(t)\\&-(1-\mu)e^T(t-\tau(t))Q_3e(t-\tau(t)) + (Ae - BKe(t-\tau)\\&+f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)^T\left(\tau_1^2Z_1 + \tau_{21}^2Z_2\right)\\&\times(Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)\\&-\tau_1\int_{t-\tau_1}^{t}\dot e^T(\theta)Z_1\dot e(\theta)d\theta - \tau_{21}\int_{t-\tau_2}^{t-\tau_1}\dot e^T(\theta)Z_2\dot e(\theta)d\theta\\&-\nu_1e^T(t)(f(t,x_m) - f(t,x_s)) + \rho\nu_1e^T(t)e(t) - \nu_2(f(t,x_m)\\&-f(t,x_s))^T(f(t,x_m) - f(t,x_s)) + \delta\nu_2e^T(t)e(t)\\&+\sigma\nu_2e^T(t)(f(t,x_m) - f(t,x_s)) + \Phi^T(u(t-\tau))WJe(t-\tau)\\&-2\Phi^T(u(t-\tau))W\Phi(u(t-\tau)) + e^T(t-\tau)J^TW\Phi(u(t-\tau)).\end{aligned} \tag{22}$$

By virtue of Jensen's inequality, then, we have

$$\begin{aligned}&-\tau_1\int_{t-\tau_1}^{t}\dot e^T(\theta)Z_1\dot e(\theta)d\theta \le -(e(t)\\&-e(t-\tau_1))^TZ_1(e(t) - e(t-\tau_1)),\end{aligned} \tag{23}$$

$$\begin{aligned}&-\tau_{21}\int_{t-\tau_2}^{t-\tau_1}\dot e^T(\theta)Z_2\dot e(\theta)d\theta\\&\le -(e(t-\tau(t)) - e(t-\tau_2))^TZ_2(e(t-\tau(t)) - e(t-\tau_2))\\&-(e(t-\tau_1) - e(t-\tau(t)))^TZ_2(e(t-\tau_1) - e(t-\tau(t))).\end{aligned} \tag{24}$$

And from (22) to (24), it follows that

$$\begin{aligned}&\dot V(t,e)\\&\le e^T(t)P(Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau))\\&+d) + (Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau))\\&+d)Pe(t) - \sum_{i=1}^{2}e^T(t-\tau_i)Q_ie(t-\tau_i) + \sum_{i=1}^{3}e^T(t)Q_ie(t)\\&-(1-\mu)e^T(t-\tau(t))Q_3e(t-\tau(t)) + (Ae - BKe(t-\tau)\\&+f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)^T\left(\tau_1^2Z_1 + \tau_{21}^2Z_2\right)\\&\times(Ae - BKe(t-\tau) + f(t,x_m) - f(t,x_s) + B\Phi(u(t-\tau)) + d)\\&-(e(t) - e(t-\tau_1))^TZ_1(e(t) - e(t-\tau_1)) - (e(t-\tau(t))\\&-e(t-\tau_2))^T\times Z_2(e(t-\tau(t)) - e(t-\tau_2)) - (e(t-\tau_1)\\&-e(t-\tau(t)))^TZ_2\times(e(t-\tau_1) - e(t-\tau(t)))\\&-\nu_1e^T(t)(f(t,x_m) - f(t,x_s)) + \rho\nu_1e^T(t)e(t)\\&-\nu_2(f(t,x_m) - f(t,x_s))^T(f(t,x_m) - f(t,x_s)) + \delta\nu_2e^T(t)e(t)\\&+\sigma\nu_2e^T(t)(f(t,x_m) - f(t,x_s)) + \Phi^T(u(t-\tau))WJe(t-\tau)\\&-2\Phi^T(u(t-\tau))W\Phi(u(t-\tau)) + e^T(t-\tau)J^TW\Phi(u(t-\tau)),\end{aligned} \tag{25}$$

this further produces

$$\dot V(e,t) \le \xi_1^T\Lambda_1\xi_1, \tag{26}$$

$$\xi_1^T = \big[e(t)^T \quad e^T(t-\tau(t)) \quad e^T(t-\tau_1) \quad e^T(t-\tau_2) \quad f^T(t,x_m)-f^T(t,x_s) \quad \Phi^T(u(t-\tau))\big], \tag{27}$$

$$\Lambda_2 = \begin{bmatrix} \Psi_1 & -PBK & Z_1 & 0 & (-\nu_1+\sigma\nu_2)I+P & PB+J^TW \\ * & -(1-\mu_1)Q_3-2Z_2 & Z_2 & Z_2 & 0 & 0 \\ * & * & -Q_1-Z_1-Z_2 & 0 & 0 & 0 \\ * & * & * & -Q_2-Z_2 & 0 & 0 \\ * & * & * & * & -\nu_2 I & 0 \\ * & * & * & * & * & -2W \end{bmatrix} + \begin{bmatrix} A & BK & 0 & 0 & I & B \end{bmatrix}^T \left(\tau_1^2 Z_1 + \tau_{21}^2 Z_2\right)\begin{bmatrix} A & BK & 0 & 0 & I & B \end{bmatrix}, \tag{28}$$

for $d(t)=0$. To attain the asymptotic stability of the synchronization error system (13), the condition $\dot{V}(e,t)<0$, which can be attained through the constraint $\Lambda_2<0$, must hold. Then, the inequality (17) in Theorem 1 is achieved by application of the Schur complement to $\Lambda_2<0$.

For initial conditions bounded by $V(0, e(0)) \le 1$, we have $V(t, e) \le 1$, because $\dot{V}(t,e)<0$. $V(t, e) \le 1$ implies $e^T(t)Pe(t) \le 1$. In addition, the initial condition $V(\vartheta, x_m(\vartheta)-x_s(\vartheta)) \le 1$ implies that $e^T(\vartheta)Pe(\vartheta) \le 1$ for all $\vartheta \in [-\tau(t) \quad 0]$. Combining the constraints $e^T(t)Pe(t) \le 1$ and $e^T(\vartheta)Pe(\vartheta) \le 1$ for $t \ge 0$ and for $\vartheta \in [\ -\tau(t) \quad 0\ ]$, respectively, we obtain the region $e^T(t-\tau)Pe(t-\tau) \le 1$. By including the ellipsoidal region $e^T(t-\tau)Pe(t-\tau) \le 1$ in the sector $S(\bar{u})$ given by (12), we obtain

$$e^T(t-\tau)Pe(t-\tau) \ge \bar{u}_{(i)}^2 e^T(t-\tau)\left(K_{(i)}-J_{(i)}\right)^T \times \left(K_{(i)}-J_{(i)}\right)e(t-\tau),\ \forall i=1,\ldots,m, \tag{29}$$

this implies the inequality

$$P-\bar{u}_{(i)}^2\left(K_{(i)}^T-J_{(i)}^T\right)\left(K_{(i)}-J_{(i)}\right) \ge 0,\ \forall i=1,\ldots,m. \tag{30}$$

Applying the Schur complement, we attain the inequality (16) in Theorem 1, which completes the proof. □

The constrained synchronization control approach in Theorem 1 requires an *a priori* guess of the matrices K and J, and therefore cannot be applied to determine the controller gain matrix through convex routines. Alternatively then, in Theorem 2, we provide a sufficient condition for controlled synchronization of the drive and response systems that exhibit this feature.

Theorem 2. *Consider the drive and response systems (1) and (2) under delayed and saturated control signal $\Psi(u(t-\tau))$ satisfying the time-delay properties (3)-(4) and Assumption 1. Suppose that there exist matrices X and M, symmetric matrices $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$, $\tilde{Z}_1$, and $\tilde{Z}_2$, a diagonal matrix H and scalars υ_1 and υ_2 such that the inequalities*

$$Y>0,\ \tilde{Q}_1>0,\ \tilde{Q}_2>0,\ \tilde{Q}_3>0,\ \tilde{Z}_1>0,\ \tilde{Z}_2>0,$$
$$H>0,\ \upsilon_1>0,\ \upsilon_2>0, \tag{31}$$

$$\begin{bmatrix} Y & X_{(i)}^T - M_{(i)}^T \\ * & \bar{u}_{(i)}^2 \end{bmatrix} \ge 0,\ \forall i=1,\ldots,m, \tag{32}$$

$$\Lambda_3 = \begin{bmatrix} \Psi_3 & -BX & \tilde{Z}_1 & 0 & (-\nu_1+\sigma\nu_2)Y+I & BH+M^T & \tau_1 YA^T & \tau_{21}YA^T & \sqrt{|\rho\nu_1+\delta\nu_2|}Y \\ * & \Psi_4 & \tilde{Z}_2 & \tilde{Z}_2 & 0 & 0 & \tau_1 X^TB^T & \tau_{21}X^TB^T & 0 \\ * & * & \Psi_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\tilde{Q}_2-\tilde{Z}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\nu_2 I & 0 & \tau_1 I & \tau_{21} I & 0 \\ * & * & * & * & * & -2H & \tau_1 H^TB^T & \tau_{21}H^TB^T & 0 \\ * & * & * & * & * & * & -Y\tilde{Z}_1^{-1}Y & 0 & 0 \\ * & * & * & * & * & * & * & -Y\tilde{Z}_1^{-1}Y & 0 \\ * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{33}$$

are satisfied, where

$$\Psi_3 = AY+YA^T+\tilde{Q}_1+\tilde{Q}_2+\tilde{Q}_3-\tilde{Z}_1,$$
$$\Psi_4 = -(1-\mu_1)\tilde{Q}_3-2\tilde{Z}_2,$$
$$\Psi_5 = -\tilde{Q}_1-\tilde{Z}_1-\tilde{Z}_2.$$

Then, for all initial conditions holding for region $V(\vartheta, x_m(\vartheta)-x_s(\vartheta)) \le 1$ for all $\vartheta \in [\ -\tau(t) \quad 0\ ]$ with $P=Y^{-1}$, $Q_1=Y^{-1}\tilde{Q}_1Y^{-1}$, $Q_2=Y^{-1}\tilde{Q}_2Y^{-1}$, $Q_3=Y^{-1}\tilde{Q}_3Y^{-1}$, $Z_1=Y^{-1}\tilde{Z}_1Y^{-1}$ and $Z_2=Y^{-1}\tilde{Z}_2Y^{-1}$, the synchronization error defined by $e(t)=x_m(t)-x_s(t)$ converges to the origin asymptotically. The gain matrices can be determined by $K=XY^{-1}$ and $J=MY^{-1}$.

Proof. Applying the congruence transformation to (16) and (17) using $\mathrm{diag}(P^{-1},I)$ and $\mathrm{diag}(P^{-1},P^{-1},P^{-1},P^{-1},I,W^{-1},Z^{-1},Z^{-1})$, respectively, and substituting $Y=P^{-1}$, $\tilde{Q}_1=P^{-1}Q_1P^{-1}$, $\tilde{Q}_2=P^{-1}Q_2P^{-1}$, $\tilde{Q}_3=P^{-1}Q_3P^{-1}$, $\tilde{Z}_1=P^{-1}Z_1P^{-1}$, $\tilde{Z}_2=P^{-1}Z_2P^{-1}$, $H=W^{-1}$, $X=KY$ and $M=JY$, we obtain the constraints in (32) and

$$\Lambda_4 = \begin{bmatrix} \Psi_3+(\rho\nu_1+\delta\nu_2)Y^2 & -BX & \tilde{Z}_1 & 0 & (-\nu_1+\sigma\nu_2)Y+I & BH+M^T & \tau_1 YA^T & \tau_{21}YA^T \\ * & \Psi_4 & \tilde{Z}_2 & \tilde{Z}_2 & 0 & 0 & \tau_1 X^TB^T & \tau_{21}X^TB^T \\ * & * & \Psi_5 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\tilde{Q}_2-\tilde{Z}_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\nu_2 I & 0 & \tau_1 I & \tau_{21} I \\ * & * & * & * & * & -2H & \tau_1 H^TB^T & \tau_{21}H^TB^T \\ * & * & * & * & * & * & -Y\tilde{Z}_1^{-1}Y & 0 \\ * & * & * & * & * & * & * & -Y\tilde{Z}_1^{-1}Y \end{bmatrix} < 0. \tag{34}$$

Incorporating $(\rho\nu_1+\delta\nu_2)Y^2 \le |\rho\nu_1+\delta\nu_2|Y^2$ as used in Cai et al. [24] and applying the Schur complement to the resultant, the inequality (33) is obtained, which ends the proof. □

Remark 1. Synchronization of nonlinear systems with either the one-sided Lipschitz nonlinearities [25] or the input saturation constraint [28] is lacking in the literature. Theorems 1 and 2 provide synchronization investigation methodologies for a given controller gain matrix and synchronization controller synthesis approach, respectively, by considering one-sided Lipschitz, quadratic inner-boundedness and local sector conditions. Synchronization of the chaotic systems by simultaneous exploitation of the one-sided Lipschitz condition and the practical input saturation limitation has not been fully addressed in the relevant previous studies.

Remark 2. Another contribution of the present work is the consideration of the input time-delay in addition to the input saturation nonlinearity for formulation of the synchronization conditions in Theorems 1 and 2. Incorporation of the input delay complicates controller design, because the input saturation is already a complex nonlinearity, and control signal must consider, simultaneously, constraints arising from delay and saturation. Since $\Psi(u(t))$ is a specific case of $\Psi(u(t-\tau))$ for $\tau = 0$, delayed nonlinearities such as $\Psi(u(t-\tau))$ are always difficult to deal with, compared with non-delayed ones.

Remark 3. It should be noted that the input delay is exploited in the present work using the delay-range-dependent paradigm regarding variations in delay, rather than the traditionalistic delay-dependent methods. The delay-range-dependent methodologies allow interval time-delays with any finite zero or nonzero lower bound, whereas for delay-dependent methods, the lower bound is fixed to zero.

By taking $\nu_1 = 0$, $\nu_2 = 1$, $\sigma = 0$, and $\lambda = \sqrt{\delta}$, where λ is the Lipchitz constant for *f*(*t*, *x*), the following corollary is straightforwardly obtained from Theorem 2.

Corollary 1. *Consider the drive and response systems (1) and (2) under delayed and saturated control signal $\Psi(u(t-\tau))$ satisfying the time-delay properties (3), (4) and (6) for $\sigma = 0$. Suppose that there exist matrices X and M, symmetric matrices $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$, $\tilde{Z}_1$, and $\tilde{Z}_2$, and a diagonal matrix H such that inequalities (32),*

$$Y > 0,\ \tilde{Q}_1 > 0,\ \tilde{Q}_2 > 0,\ \tilde{Q}_3 > 0,\ \tilde{Z}_1 > 0,\ \tilde{Z}_2 > 0,\ H > 0, \tag{35}$$

$$\begin{bmatrix} \Psi_3 & -BX & \tilde{Z}_1 & 0 & I & BH+M^T & \tau_1 YA^T & \tau_{21} YA^T & \lambda Y \\ * & \Psi_4 & \tilde{Z}_2 & \tilde{Z}_2 & 0 & 0 & \tau_1 X^T B^T & \tau_{21} X^T B^T & 0 \\ * & * & \Psi_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\tilde{Q}_2 - \tilde{Z}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & \tau_1 I & \tau_{21} I & 0 \\ * & * & * & * & * & -2H & \tau_1 H^T B^T & \tau_{21} H^T B^T & 0 \\ * & * & * & * & * & * & -Y\tilde{Z}_1^{-1} Y & 0 & 0 \\ * & * & * & * & * & * & * & -Y\tilde{Z}_1^{-1} Y & 0 \\ * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{36}$$

are satisfied. Then, for all initial conditions holding for region $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \le 1$ for all $\vartheta \in [\,-\tau(t) \quad 0\,]$ with $P = Y^{-1}$, $Q_1 = Y^{-1}\tilde{Q}_1 Y^{-1}$, $Q_2 = Y^{-1}\tilde{Q}_2 Y^{-1}$, $Q_3 = Y^{-1}\tilde{Q}_3 Y^{-1}$, $Z_1 = Y^{-1}\tilde{Z}_1 Y^{-1}$ and *$Z_2 = Y^{-1}\tilde{Z}_2 Y^{-1}$, the synchronization error defined by $e(t) = x_m(t) - x_s(t)$ converges to the origin asymptotically. The gain matrices can be determined by $K = XY^{-1}$ and $J = MY^{-1}$.*

Remark 4. Corollary 1 is deduced from Theorem 2 by setting $\nu_1 = 0$, $\nu_2 = 1$, $\sigma = 0$, and $\lambda = \sqrt{\delta}$ for synchronization of Lipschitz nonlinear systems (1) and (2). Nevertheless, the methodology in Theorem 2, providing a controlled synchronization remedy for one-sided Lipschitz nonlinear systems, considers a more generic scenario. However, the approach in Corollary 1 is novel, as it considers the input time-delay, unlike the existing works [20] and [28] on synchronization of nonlinear systems under input saturation.

By substituting $\tau_1 = 0$, we conclude the following corollary.

Corollary 2. *Consider the drive and response systems (1) and (2) under delayed and saturated control signal $\Psi(u(t-\tau))$ satisfying the time-delay properties (3) and (4) with $\tau_1 = 0$ and Assumption 1. Suppose that there exist matrices X and M, symmetric matrices $\tilde{Q}_2$, $\tilde{Q}_3$, and $\tilde{Z}_2$, a diagonal matrix H and scalars υ_1 and υ_2 such that inequalities (32),*

$$Y > 0,\ \tilde{Q}_2 > 0,\ \tilde{Q}_3 > 0,\ \tilde{Z}_2 > 0,\ V > 0,\ \upsilon_1 > 0,\ \upsilon_2 > 0, \tag{37}$$

$$\begin{bmatrix} \Psi_6 & -BX + \tilde{Z}_2 & 0 & (-\nu_1 + \sigma\nu_2)Y + I & BH + M^T & \tau_{21} YA^T & \sqrt{|\rho\nu_1 + \delta\nu_2|}Y \\ * & \Psi_7 & \tilde{Z}_2 & 0 & 0 & \tau_{21} X^T B^T & 0 \\ * & * & -\tilde{Q}_2 - \tilde{Z}_2 & 0 & 0 & 0 & 0 \\ * & * & * & -\nu_2 I & 0 & \tau_{21} I & 0 \\ * & * & * & * & -2H & \tau_{21} H^T B^T & 0 \\ * & * & * & * & * & -Y\tilde{Z}_2^{-1} Y & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{38}$$

are satisfied, where

$$\Psi_6 = AY + YA^T + \tilde{Q}_2 + \tilde{Q}_3 - \tilde{Z}_2,$$
$$\Psi_7 = -(1-\mu_1)\tilde{Q}_3 - 2\tilde{Z}_2.$$

Then, for all initial conditions holding for region $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \le 1$ for all $\vartheta \in [\,-\tau(t) \quad 0\,]$ with $P = Y^{-1}$, $Q_1 = 0$, $Q_2 = Y^{-1}\tilde{Q}_2 Y^{-1}$, $Q_3 = Y^{-1}\tilde{Q}_3 Y^{-1}$, $Z_1 = 0$ and *$Z_2 = Y^{-1}\tilde{Z}_2 Y^{-1}$, the synchronization error defined by $e(t) = x_m(t) - x_s(t)$ converges to the origin asymptotically. The gain matrices can be determined by $K = XY^{-1}$ and $J = MY^{-1}$.*

Remark 5. In Corollary 2, delay-dependent controller design considering $0 \le \tau(t) \le \tau_2$ is derived from the approach in Theorem 2. Whereas the existing preliminary results in [20] on the synchronization of nonlinear systems under input saturation are based on delay-independent stability criteria, the proposed method in Corollary 2 is established using a relatively less conservative delay-dependent treatment. Additionally, Corollary 2, in contrast to [20], considers the input delay case, which presents greater difficulty for controller design owing to the necessary consideration of delayed saturation nonlinearity.

For $Q_3 = 0$ in (14), the following corollary is obtained.

Corollary 3. *Consider the drive and response systems (1) and (2) under delayed and saturated control signal $\Psi(u(t-\tau))$ of unknown delay-rate satisfying the time-delay property (3) and Assumption 1. Suppose that there exist matrices X and M, symmetric matrices $\tilde{Q}_1$,*

$\tilde{Q}_2$, $\tilde{Z}_1$, and $\tilde{Z}_2$, a diagonal matrix H and scalars υ_1 and υ_2 such that inequalities (32),

$$Y > 0,\ \tilde{Q}_1 > 0,\ \tilde{Q}_2 > 0,\ \tilde{Z}_1 > 0,\ \tilde{Z}_2 > 0,\ H > 0,\ \upsilon_1 > 0,\ \upsilon_2 > 0, \tag{39}$$

$$\begin{bmatrix} \Psi_8 & -BX & \tilde{Z}_1 & 0 & (-\nu_1+\sigma\nu_2)Y+I & BH+M^T & \tau_1 YA^T & \tau_{21}YA^T & \sqrt{|\rho\nu_1+\delta\nu_2|}Y \\ * & -2\tilde{Z}_2 & \tilde{Z}_2 & \tilde{Z}_2 & 0 & 0 & \tau_1 X^TB^T & \tau_{21}X^TB^T & 0 \\ * & * & \Psi_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\tilde{Q}_2-\tilde{Z}_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\nu_2 I & 0 & \tau_1 I & \tau_{21} I & 0 \\ * & * & * & * & * & -2H & \tau_1 H^TB^T & \tau_{21}H^TB^T & 0 \\ * & * & * & * & * & * & -Y\tilde{Z}_1^{-1}Y & 0 & 0 \\ * & * & * & * & * & * & * & -Y\tilde{Z}_1^{-1}Y & 0 \\ * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{40}$$

are satisfied, where

$\Psi_8 = AY + YA^T + \tilde{Q}_1 + \tilde{Q}_2 - \tilde{Z}_1.$

Then, for all initial conditions holding for region $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \le 1$ for all $\vartheta \in [\ -\tau(t)\quad 0\ ]$ with $P = Y^{-1}$, $Q_1 = Y^{-1}\tilde{Q}_1Y^{-1}$, $Q_2 = Y^{-1}\tilde{Q}_2Y^{-1}$, $Q_3 = 0$, $Z_1 = Y^{-1}\tilde{Z}_1Y^{-1}$ and $Z_2 = Y^{-1}\tilde{Z}_2Y^{-1}$, the synchronization error defined by $e(t) = x_m(t) - x_s(t)$ converges to the origin asymptotically. The gain matrices can be determined by $K = XY^{-1}$ and $J = MY^{-1}$.

Remark 6. It is difficult to identify the parameter μ if *a priori* knowledge of the rate of delay is not available. In such cases, the control methodologies provided in Theorems 1 and 2 and Corollaries 1 and 2 are inapplicable. Corollary 3 therefore is derived as a special case of the approach in Theorem 2, and can be applied to deal with the unknown delay derivative information.

A sufficient condition for a synchronization controller design that is robust against disturbances and perturbations is provided in the following theorem.

Theorem 3. *Consider the drive and response systems (1) and (2) under delayed and saturated control signal $\Psi(u(t-\tau))$ satisfying the time-delay properties (3) and (4) and Assumption 1. Suppose that there exist matrices X and M, symmetric matrices $\tilde{Q}_1$, $\tilde{Q}_2$, $\tilde{Q}_3$, $\tilde{Z}_1$, and $\tilde{Z}_2$, a diagonal matrix H and scalars υ_1 and υ_2 such that inequalities (31)*

$$\eta > 0,\ \gamma > 0,\ \begin{bmatrix} Y & X_{(i)}^T - M_{(i)}^T \\ * & \eta\bar{u}_{(i)}^2 \end{bmatrix} \ge 0,\ \forall i = 1, \ldots, m, \tag{41}$$

$$\begin{bmatrix} \Psi_3 & -BX & \tilde{Z}_1 & 0 & (-\nu_1+\sigma\nu_2)Y+I & BH+M^T & I & Y & \tau_1 YA^T & \tau_{21}YA^T & \sqrt{|\rho\nu_1+\delta\nu_2|}Y \\ * & \Psi_4 & \tilde{Z}_2 & \tilde{Z}_2 & 0 & 0 & 0 & 0 & \tau_1 X^TB^T & \tau_{21}X^TB^T & 0 \\ * & * & \Psi_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\tilde{Q}_2-\tilde{Z}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\nu_2 I & 0 & 0 & 0 & \tau_1 I & \tau_{21} I & 0 \\ * & * & * & * & * & -2H & 0 & 0 & \tau_1 H^TB^T & \tau_{21}H^TB^T & 0 \\ * & * & * & * & * & * & -\gamma I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -Y\tilde{Z}_1^{-1}Y & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -Y\tilde{Z}_1^{-1}Y & 0 \\ * & * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{42}$$

are satisfied. Then, for all initial conditions holding for region $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \le 1$ for all $\vartheta \in [\ -\tau(t)\quad 0\ ]$ with $P = Y^{-1}$, $Q_1 = Y^{-1}\tilde{Q}_1Y^{-1}$, $Q_2 = Y^{-1}\tilde{Q}_2Y^{-1}$, $Q_3 = Y^{-1}\tilde{Q}_3Y^{-1}$, $Z_1 = Y^{-1}\tilde{Z}_1Y^{-1}$ and $Z_2 = Y^{-1}\tilde{Z}_2Y^{-1}$, the synchronization error $e(t) = x_m(t) - x_s(t)$ remains bounded within the region $\eta e^T(t-\tau)Y^{-1}e(t-\tau) \le 1$, and the following holds:

(i) *the synchronization error $e(t) = x_m(t) - x_s(t)$ converges to the origin asymptotically, if $d(t) = 0$;*

(ii) *the synchronization error $e(t) = x_m(t) - x_s(t)$ satisfies $\|e(t)\|_2^2 < \gamma^2\|d\|_2^2 + \gamma V(0, e(0))$, if $\|d(t)\|_2^2 < \varsigma^{-1}$, where $\varsigma = \gamma/(\eta^{-1} - 1)$.*

(iii) *The gain matrices can be determined by $K = XY^{-1}$ and $J = MY^{-1}$.*

Proof. To achieve robustness against disturbances, we employ the following inequality

$$\dot{V}(t, e) + \gamma^{-1}e^T(t)e(t) - \gamma d^T(t)d(t) < 0. \tag{43}$$

Integrating the constraint from 0 to T, we obtain

$$V(T, e(T)) - V(0, e(0)) + \gamma^{-1}\int_0^T e^T(t)e(t)dt - \gamma\int_0^T d^T(t)d(t)dt < 0. \tag{44}$$

If $d(t) = 0$, (43) ensures asymptotic convergence of the synchronization error $e(t)$ to zero, owing to $\dot{V}(t, e) < 0$, for all initial conditions validating $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \le 1$ under $\vartheta \in [\ -\tau(t)\quad 0\ ]$. Further, $\dot{V}(t, e) < 0$ and $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \le 1$ imply $e^T(t-\tau)Pe(t-\tau) \le 1$. If $\|d(t)\|_2^2 < \varsigma^{-1}$, (44) entails $V(t, e) < 1 + \gamma\varsigma^{-1}$ under $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \le 1$. For the LK functional (14), we have $e^T(t)Pe(t) \le 1 + \gamma\varsigma^{-1}$, which for $V(\vartheta, x_m(\vartheta) - x_s(\vartheta)) \le 1$ with $\vartheta \in [\ -\tau(t)\quad 0\ ]$ implies that the ellipsoidal region $(1 + \gamma\varsigma^{-1})^{-1}e^T(t-\tau)Pe(t-\tau) \le 1$ holds. Moreover, the synchronization error satisfies $\int_0^T e^T(t)e(t)dt < \gamma^2\int_0^T d^T(t)d(t)dt + \gamma V(0, e(0))$ for all time, and minimization of γ reduces the effects of disturbances and initial conditions on the synchronization error. Note that $e^T(t-\tau)Pe(t-\tau) \le 1 \subset (1 + \gamma\varsigma^{-1})^{-1}e^T(t-\tau)Pe(t-\tau) \le 1$ holds as $1 + \gamma\varsigma^{-1} > 1$; therefore, the synchronization error $e(t)$ always remains bounded within the region $\eta e^T(t-\tau)Y^{-1}e(t-\tau) \le 1$ either $d(t) = 0$, or the disturbance is bounded as $\|d(t)\|_2^2 < \varsigma^{-1}$. Incorporating (25) into (43) obtains

$$\dot{V}(t, e) + \gamma^{-1}e^T(t)e(t) - \gamma d^T(t)d(t) \le \xi_2^T\Lambda_3\xi_2, \tag{45}$$

$$\xi_1^T = \big[e(t)^T \quad e^T(t-\tau(t)) \quad e^T(t-\tau_1) \quad e^T(t-\tau_2) \quad f^T(t, x_m) - f^T(t, x_s) \quad \Phi^T(u(t-\tau)) \quad d^T(t)\big], \tag{46}$$

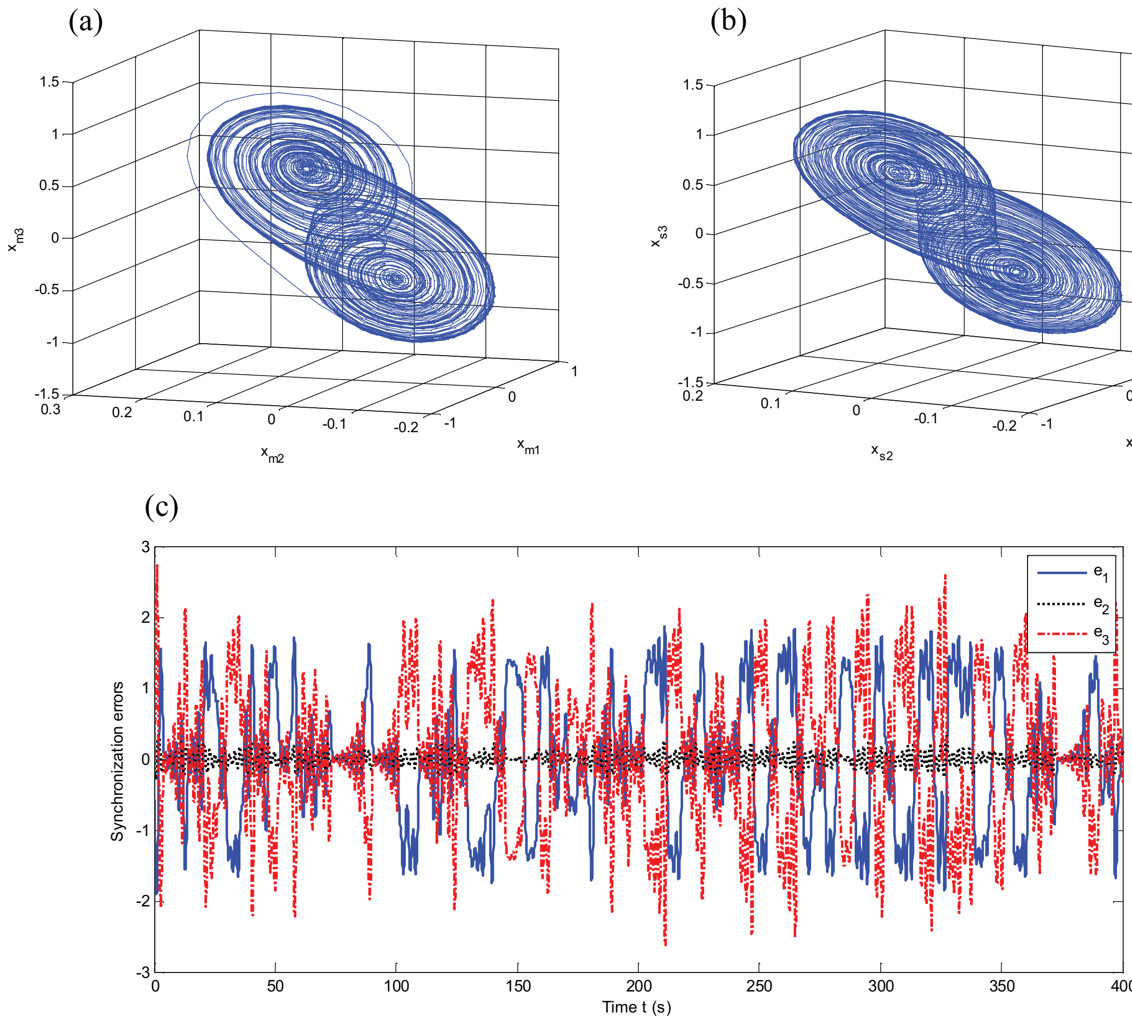

Fig. 2. Behavior of the drive and response modified Chua's circuits with zero control input: (a) response of the master system, (b) response of the slave system, (c) synchronization errors between the drive and response circuits.

$$\Lambda_3 = \begin{bmatrix} \Psi_1+\gamma^{-1}I & -PBK & Z_1 & 0 & (-\nu_1+\sigma\nu_2)I+P & PB+J^TW & P \\ * & -(1-\mu_1)Q_3-2Z_2 & Z_2 & Z_2 & 0 & 0 & 0 \\ * & * & -Q_1-Z_1-Z_2 & 0 & 0 & 0 & 0 \\ * & * & * & -Q_2-Z_2 & 0 & 0 & 0 \\ * & * & * & * & -\nu_2 I & 0 & 0 \\ * & * & * & * & * & -2W & 0 \\ * & * & * & * & * & * & -\gamma I \end{bmatrix}$$

$$+\begin{bmatrix} A & BK & 0 & 0 & I & B & I \end{bmatrix}^T\left(\tau_1^2 Z_1+\tau_{21}^2 Z_2\right)\begin{bmatrix} A & BK & 0 & 0 & I & B & I \end{bmatrix}. \quad (47)$$

Condition (43) holds for $\Lambda_3 < 0$. Applying two successive Schur complements to the matrix inequality $\Lambda_3 < 0$ produces

$$\Lambda_4 = \begin{bmatrix} \Psi_1 & -PBK & Z_1 & 0 & (-\nu_1+\sigma\nu_2)I+P & PB+J^TW & P & I & \tau_1 A^T Z_1 & \tau_{21} A^T Z_2 \\ * & \Psi_2 & Z_2 & Z_2 & 0 & 0 & 0 & 0 & \tau_{21} K^T B^T Z_1 & \tau_{21} K^T B^T Z_2 \\ * & * & -Q_1-Z_1-Z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -Q_2-Z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\nu_2 I & 0 & 0 & 0 & \tau_1 Z_1 & \tau_{21} Z_2 \\ * & * & * & * & * & -2W & 0 & 0 & \tau_1 B^T Z_1 & \tau_{21} B^T Z_2 \\ * & * & * & * & * & * & -\gamma I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & * & * & * & * & -Z_1 & 0 \\ * & * & * & * & * & * & * & * & * & -Z_2 \end{bmatrix} < 0. \quad (48)$$

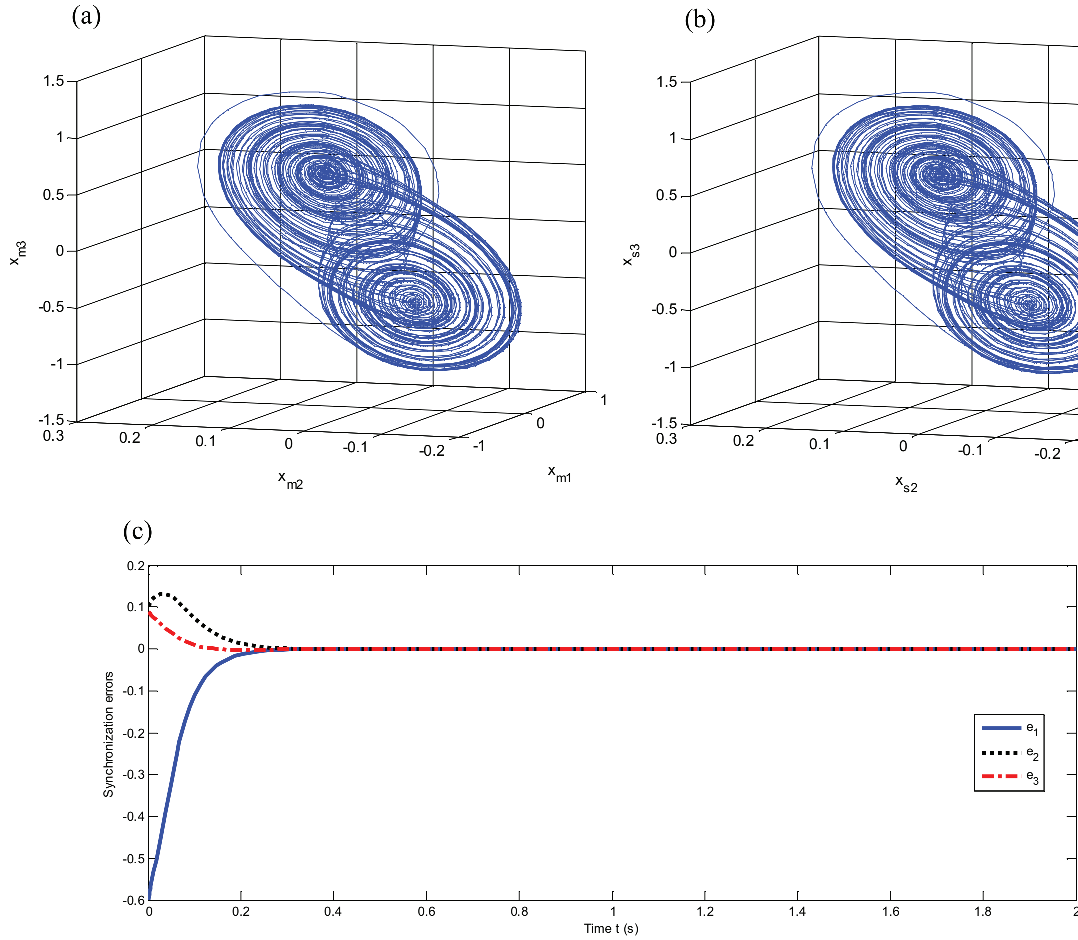

Fig. 3. Behavior of the drive and response modified Chua's circuits with the proposed controller: (a) response of the master system, (b) response of the slave system, (c) synchronization errors between the drive and response circuits.

Applying the congruence transformation by $\text{diag}(P^{-1}, P^{-1}, P^{-1}, P^{-1}, I, W^{-1}, I, I, Z^{-1}, Z^{-1})$, using $\rho\nu_1 + \delta\nu_2 \le |\rho\nu_1 + \delta\nu_2|$, applying the Schur complement, and utilizing the aforementioned substitutions, we obtain the inequality (42) in Theorem 3, whereas (41) is obtained by including region $\eta e^T(t-\tau)Y^{-1}e(t-\tau) \le 1$ in $S(\bar{u})$. □

Remark 7. An extension to Theorem 2 is provided in Theorem 3 in order to achieve robust synchronization of two nonlinear or chaotic systems under external disturbances in addition to the input time-delay and actuator saturation. The approach provided in Theorem 3 can be used to synchronize systems that are sensitive to perturbations. For instance, secure communications using synchronization of chaos cannot be attained using the controller obtained from Theorem 2, due to the highly sensitive nature of the chaotic oscillators. In point of fact, disturbances and perturbations can result in non-synchronous responses of two nonlinear systems; therefore, Theorem 3 can be applied to attain the desired robustness against perturbations when synthesizing a synchronization controller.

The constraints provided in Theorems 2 and 3 are nonlinear; however, they can be resolved by means of convex routines that convert the nonlinear constraints into linear constraints with a nonlinear objective function for optimization. For instance, the constraints in Theorem 3 are written in an equivalent form as

$$\min\ \text{trace}\begin{pmatrix} PY + Y_1\bar{Y}_1 + Y_2\bar{Y}_2 + Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + \tilde{Z}_1N_1 + \tilde{Z}_2N_2 \\ +\bar{Z}_1\bar{Y}_1\tilde{Z}_1\bar{Y}_1 + \bar{Z}_2\bar{Y}_2\tilde{Z}_2\bar{Y}_2 + Z_1Y_1N_1Y_1 + Z_2Y_2N_2Y_2 \end{pmatrix}$$
$$\text{subject to}(31),\ (41), (42)*, \tag{49}$$

$$\begin{bmatrix} P & I \\ * & Y \end{bmatrix} \ge 0, \begin{bmatrix} Y_i & I \\ * & \bar{Y}_i \end{bmatrix} \ge 0, \begin{bmatrix} Z_i & I \\ * & \bar{Z}_i \end{bmatrix} \ge 0, \begin{bmatrix} N_i & I \\ * & \tilde{Z}_i \end{bmatrix} \ge 0, \tag{50}$$

$$\begin{bmatrix} \bar{Z}_i & Y_i \\ * & \tilde{Z}_i \end{bmatrix} \ge 0, \begin{bmatrix} Z_i & \bar{Y}_i \\ * & N_i \end{bmatrix} \ge 0, i = 1, 2, \tag{51}$$

where (42)* in (49) represents (42) by substituting $\bar{Z}_1 = Y\tilde{Z}_1^{-1}Y$ and $\bar{Z}_2 = Y\tilde{Z}_2^{-1}Y$. The constraints in (50) along with the nonlinear term, given by $PY + Y_1\bar{Y}_1 + Y_2\bar{Y}_2 + Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + \tilde{Z}_1N_1 + \tilde{Z}_2N_2$, in the objective function are employed to ensure $P = Y^{-1}$, $\bar{Y}_i = Y_i^{-1}$, $\bar{Z}_i = Z_i^{-1}$, $\tilde{Z}_i = N_i^{-1}$, while the constraints in (51) along with the nonlinear term $\bar{Z}_1\bar{Y}_1\tilde{Z}_1\bar{Y}_1 + \bar{Z}_2\bar{Y}_2\tilde{Z}_2\bar{Y}_2 + Z_1Y_1N_1Y_1 + Z_2Y_2N_2Y_2$ of the objective function reveal that $\bar{Z}_1\bar{Y}_1\tilde{Z}_1\bar{Y}_1 = \bar{Z}_2\bar{Y}_2\tilde{Z}_2\bar{Y}_2 = Z_1Y_1N_1Y_1 = Z_2Y_2N_2Y_2 = I$ (see [23]). By application of the cone complementary linearization algorithm, the above-mentioned nonlinear optimization can be solved for given positive scalars ν_1 and ν_2 using the convex routines in [23] and [30].

4. Simulation results

Chua's circuit has various applications in secure communications, chaos investigation, oscillation analysis and neuronal behavior study owing to its utility in representing a wide range of dynamical behaviors (see [19] and references therein). We consider a modified Chua's circuit model containing cubic nonlinearity, which is more difficult to handle than the conventional Chua's circuit containing absolute nonlinearity.

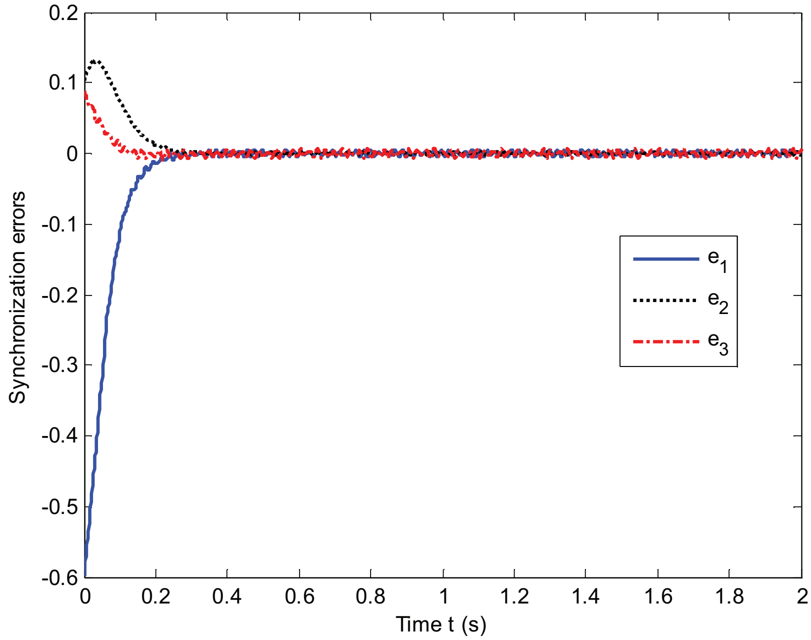

Fig. 4. Effects of disturbances on synchronization errors between the drive and response circuits.

The model of the modified Chua's circuit, as seen in [31] and references therein, is given by

$$A = \begin{bmatrix} 10/7 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -100/7 & 0 \end{bmatrix}, f(t,x) = \begin{bmatrix} -(20/7)x_1^3 \\ 0 \\ 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{52}$$

First, we show that the function $f(t, x)$ satisfies the one-sided Lipschitz condition globally. Evaluating the left side of (5) obtains

$$\langle f(t,x_m) - f(t,x_s), x_m - x_s \rangle = (20/7)\left[x_{m1}x_{s1}\left(x_m^2 + x_s^2\right) - x_m^4 - x_s^4\right]. \tag{53}$$

As we already know that

$$x_{m1}x_{s1} = 0.5\left(x_{m1}^2 + x_{s1}^2\right) - 0.5(x_{m1} - x_{s1})^2, \tag{54}$$

it reveals, by application of (53), that

$$\langle f(t,x_m) - f(t,x_s), x_m - x_s \rangle \leq 0. \tag{55}$$

Hence, the function $f(t, x)$ satisfies the one-sided Lipschitz continuity globally with $\rho = 0$. To compute the constants of the quadratic inner-boundedness, the supremum of the maximum eigenvalues of $(\partial f(t, x)/\partial x)^T(\partial f(t, x)/\partial x)$ for region $x_1 \in [\ -1 \quad 1\ ]$ is numerically calculated as 73.47. Therefore, we can select $\delta = 73.47$ and $\sigma = 0$. We can expect a 3–9 ms input delay due to the conduction of current through wires; consequently, $\tau_1 = 3$ ms and $\tau_2 = 9$ ms are fixed. The input saturation limits are taken as $\bar{u} = [\ 5 \quad 5 \quad 5\ ]^T$. The controller gain and the L_2 performance index for $\mu = 0.2$ are obtained as

$$K = \begin{bmatrix} 19.11 & 7.79 & 1.53 \\ 6.71 & 22.42 & -6.43 \\ 1.65 & -8.67 & 16.45 \end{bmatrix}, \gamma = 9.15, \tag{56}$$

by solving the constraints in Theorem 3. The open-loop responses of the modified Chua's circuits are demonstrated in Fig. 2. Phase portraits of the master and slave systems and plots of the synchronization errors, depicting the chaotic and non-synchronous behaviors of the drive and response circuits, are shown in Figs. 2(a), 1(b) and (c).

By application of the proposed controller, the above-noted responses and synchronization errors are plotted in Fig. 3(a), (b) and (c). In Fig. 3(c), it is observed that the proposed controller synchronizes all of the states of the master-slave modified Chua's circuits in the absence of disturbances.

To evaluate the robustness of the proposed approach, the disturbances are taken as

$$\begin{aligned} d_{11} &= 0.59 \sin 350t, \\ d_{12} &= 0.3 \sin 400t, \\ d_{13} &= 0.72 \sin 370t, \\ d_{21} &= 0.59 \sin 290t, \\ d_{22} &= 0.3 \sin 300t, \\ d_{23} &= 0.72 \sin 270t. \end{aligned} \tag{57}$$

Synchronization error plots by application of the proposed robust controller for a time-varying delay of $5 - 0.5 \sin 0.002t$ (in ms) are provided in Fig. 4, which shows that all of the synchronization errors e_1, e_2 and e_3 are converging in the presence of disturbance. In summary, synchronization of complex nonlinear drive and response systems under interval time-delays, input saturation and disturbances can be precisely obtained by means of the proposed control methods.

5. Conclusions

The present study formulated novel control strategies for the synchronization of nonlinear drive and response systems subjected to input delay and saturation. To deal with the delay, a delay-range-dependent methodology utilizing the LK functional and allowing for time-varying interval delays was employed. Input saturation was treated using the local sector condition, through which local synchronization schemes were developed that guarantee the regional stability of the synchronization error. Further, to consider a control scheme applicable to a wide class of systems, the concepts of one-sided Lipschitz continuity and quadratic inner-boundedness were applied, which are generalized forms of the Lipschitz continuity. Moreover, the robustness of the proposed synchronization controller against disturbances was ensured by means of L_2 stability analysis. The proposed methodology was successfully tested for synchronization of modified chaotic Chua's circuits under input time-varying delay, input saturation and disturbances.

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