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Distributed adaptive consensus control of Lipschitz nonlinear multi-agent systems using output feedback

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ABSTRACT

This paper addresses output-feedback-based distributed adaptive consensus control of multi-agent systems having Lipschitz nonlinear dynamics. Distributed dynamic protocols are designed based on the relative outputs of neighbouring agents and the adaptive coupling weights, under which consensus is reached between the nonlinear systems for all undirected connected communication topologies. Extension to the case of Lipschitz nonlinear multi-agent systems subjected to external disturbances is further studied, and a robust adaptive fully distributed consensus protocol is suggested. By application of a decoupling technique, necessary and sufficient conditions for the existence of these consensus protocols are provided in terms of linear matrix inequalities. Finally, numerical simulation results are demonstrated to validate the effectiveness of the theoretical results.

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Consensus control; multi-agent systems; distributed adaptive protocol; Lipschitz nonlinearity; decoupling technique

1. Introduction

Multi-agent system consensus is a major problem in the field of cooperative control. The main objective of consensus is to develop a distributed interaction rule that specifies the exchange of information between sets of agents, such that all agents' states converge to the common value. In recent years, consensus control has received considerable attention from numerous researchers from diverse scientific fields, and has achieved rapid development due to the large number of applications in many areas such as surveillance and monitoring, multi-vehicle rendezvous, attitude alignment of spacecraft, formation control, distributed estimation, sensor networks, flocking and formulated coordination of multi-agent dynamic systems (Du, Wen, Yu, Li, & Chen, 2015; Li, Ren, Liu, & Xie, 2013; Olfati-Saber, Fax, & Murray, 2007; Yu & Xia, 2012). Early well-known control policies were formulated by Jadbabaie, Lin, and Morse (2003), Moreau (2005), Ren and Beard (2005), and Arcak (2007) to solve the consensus problem. In Jadbabaie et al. (2003), graph theory is introduced to the consensus problem to illustrate the theoretical explanation of the linearised Vicsek model developed in Vicsek, Czirók, Ben-Jacob, Cohen, and Shochet (1995). A distributed protocol is presented in Cortés (2008) for multi-agent networks to achieve consensus in a finite time. The consensus problem with switching topologies and time-delays is addressed by Olfati-Saber and Murray (2004) and Ren and Beard (2005) for networks of integrator agents. Further, finite-time consensus protocols are proposed by Shang (2012) for fast convergence of consensus error by multi-agent systems with fixed topologies. Consensus control for networks of double integrators and higher order multi-agent systems is discussed in the work of Ding, Yu, Liu, Guan, and Feng (2013), Ren and Beard (2008), Jiang and Wang (2010) and Ren, Moore, and Chen (2007). Furthermore, edge- and node-based adaptive dynamic protocols for linear multi-agent systems, allowing construction of fully distributed protocols using output feedback, have been designed by Li et al. (2013). In Huang, Zeng, and Sun (2015), robust consensus control protocols are developed for synchronisation of linear multi-agents in dealing with polytopic uncertainties and external disturbances. The recent work of Wen, Zhao, Zhisheng, Yu, and Chen (2015) considered the containment control of a general form of linear systems under directed communication topologies by exploiting multiple leaders and multiple agents and by employing dynamic output feedback control.

The study of nonlinear multi-agent systems is the focus of growing research attention and increasingly acknowledged practical importance, owing to the existence of abundant nonlinear systems in practice and the numerous applications of multi-agent consensus under different communication protocols. The consensus problem for a network of agents with nonlinear dynamics has

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been discussed in Das and Lewis (2010) and Yu, Chen, Cao, and Kurths (2010). In Yu, Chen, and Cao (2011), both local and global consensus problems are investigated for multi-agent systems having intrinsic nonlinearities. An interesting work on second-order nonlinear multi-agent systems by incorporating the delayed nonlinearity and communication constraint for a strongly connected and balanced topology is performed by Wen, Duan, Yu, and Chen (2013). In Ding (2014) meanwhile, consensus control is proposed for a class of nonlinear multi-agent systems with Lipschitz nonlinearities. Li, Liu, Fu, and Xie (2012) designed a two-step consensus algorithm for Lipschitz nonlinear multi-agents under a strongly connected directed graph topology. In the more recent work of Wen, Duan, Chen, and Yu (2014), a distributed consensus tracking control methodology was studied for multi-agent systems having Lipschitz-type node dynamics. Li et al. (2013) designed a distributed consensus protocol with adaptive coupling weights for both linear and Lipschitz nonlinear systems. In Li et al. (2012) and Wen et al. (2014), however, the consensus control is static and the coupling weight for network topologies is non-adaptive, which facts can limit fully distributed synchronisation control of agents. Li et al. (2013) actually proposed a state-feedback-based adaptive consensus protocol; however, the control is non-dynamic, and cannot be used if the state vector is unavailable. Previous work on Lipschitz nonlinear multi-agent system consensus (for example, Ding, 2014; Li et al., 2012; Li et al., 2013; Wen et al., 2014) cannot be applied to attain a fully distributed (adaptive) protocol if the relative states of the neighbouring agents are unrevealed.

In this paper, we consider the consensus problem for nonlinear multi-agent systems with Lipschitz nonlinearities and undirected graph topologies. Based on the relative output information of the neighbouring agents, various conditions for the design of fully distributed adaptive dynamic protocols with adaptive coupling weights for each edge, as based on graph theory, Lyapunov stability, linear matrix inequality (LMI) tools and decoupling procedures, are proposed. The main contributions of the consensus protocol proposed in this paper are fourfold. First, our protocol is fully distributed, and unlike the existing protocols (Ding, 2014; Li et al., 2012; Wen et al., 2014), does not require any global connection information for Lipschitz nonlinear multi-agent systems. In other words, the requirement for a known second communication graph eigenvalue in contrast to Li et al. (2012), Wen et al. (2014) and Ding (2014) is relaxed. Second, contrary to the work of Li et al. (2012), Li et al. (2013), Wen et al. (2014), and Ding (2014), output-feedback-based information on the neighbouring agent is employed in the proposed work for consensus between agents. Third, a decoupling methodology is provided that can be used to determine the gains of the dynamic consensus protocol. Last, an extension to the present case for development of a robust adaptive distributed consensus protocol is provided for Lipschitz nonlinear multi-agent systems under external disturbances using the L_2 stability theory. To the best of our knowledge, a dynamic consensus protocol using output feedback and allowing adaptive weights for the communication links is proposed herein for the first time for Lipschitz nonlinear systems. Simulation results on the adaptive consensus of a network of one-link flexible-joint robots using output feedback in the absence and presence of disturbances also are available in these pages.

The rest of this paper is organised as follows. Some basic preliminaries on graph theory and the system description are provided in Section 2. Designed distributed adaptive consensus protocols for nonlinear multi-agent systems without and with external disturbances are presented in Sections 3 and 4, respectively. For validation of the theoretical analysis, numerical simulation examples are shown in Section 5. Finally, conclusions are drawn in Section 6.

In this paper, the following notations are used. $R^{n\times m}$ represents the set of real matrices where n and m are the sizes of rows and columns, respectively. The superscript T indicates the transpose of real matrices. I_n is the identity matrix of dimension n. $0_{n\times m}$ represents the zero matrix with n rows and m columns. $\mathbf{1}_N = [1, 1, \ldots, 1]^T \in \boldsymbol{R}^N$ denotes the unit column vector. $diag(D_1, \ldots, D_N)$ represents a block diagonal matrix with diagonal entries D_i, $i = 1, \ldots, N$ and zero off-diagonal entries. For $X \in R^{n\times n}$, $X > 0$ means that X is positive-definite. $\|x\|$ and $\|x\|_2$ denote the Euclidean norm and the L_2 norm for a vector x. The L_2 gain between vectors d and y is defined as $\sup_{\|d\|_2\neq 0}(\|y\|_2/\|d\|_2)$ by assuming a zero initial condition of a system. Finally, $X \otimes Y$ represents the Kronecker product of matrices X and Y.

2. Graph theory and system description

Mathematically, a graph is defined as a pair of sets $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \ldots, v_N\}$ represents the set of vertices and $\mathcal{E}$ denotes the set of edges of a communication network. The two vertices v_i and v_j are the end vertices of an edge $(v_i, v_j) \in \mathcal{E}$. Edges with the same ends are known as loop or parallel edges. A simple graph is defined as a graph with no parallel edges or loop. A complete graph is a type of simple graph that contains all the possible edges between nodes. A complete graph with N nodes is denoted as K_N. Nodes are represented as dots or circles, while edges are expressed as either lines or arrows, according to the type of graph. Information sharing between the nodes of a network can be either unidirectional or bidirectional. And based on the type of information sharing, there are two types of graphs: directed and undirected. A graph $\mathcal{G}$ is said to be a directed graph if the set of edges $\mathcal{E}$ is an ordered pair, that is, $(v_i, v_j) \neq (v_j, v_i)$. A graph is undirected if $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ for any $v_i, v_j \in \mathcal{V}$. A digraph is known as strongly connected if there is a directed path between any two distinct nodes. A connected undirected graph means that there is a path between any two distinct nodes. A path is a trail or sequence of vertices provided that each of the vertices is visited once except the starting and ending nodes when they are the same. A closed path is known as a circuit. For N nodes $v_1, v_2, \ldots, v_N$, a directed path is such that $(v_i, v_{i+1}) \in \mathcal{E}$, for all $i = 1, \ldots, N$. A tree is a type of connected digraph, in which each node has an in-degree equal to one except the root node. In a path, there can be a circuit, but a tree does not contain any circuit. A graph $\mathcal{G}$ is said to have a spanning tree if a subset of edges forms a directed tree that contains all of the vertices of the graph $\mathcal{G}$.

The adjacency matrix of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is $N \times N$ matrix, given by

$$\mathcal{A} = [a_{ij}],\ \mathrm{i}, j = 1, \ldots, N, \tag{1}$$

where N is the total number of nodes in $\mathcal{G}$ such that $\mathcal{V} = \{v_1, \ldots v_N\}$. In the case of an undirected graph, a_{ij} is total number of edges between nodes v_i and v_j; for a directed graph meanwhile, a_{ij} is the total number of edges that comes out of the node v_i and enters into the node v_j. An isolated node has $a_{ij} = 0$. The adjacency matrix is symmetric for undirected graphs, but this property does not hold in the case of a directed graph. Because of the symmetric nature of an undirected graph, its eigenvalues are real.

The Laplacian matrix for any undirected graph can be calculated as $\mathcal{L} = [\ell_{ij}]_{N\times N} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(\mathcal{D}_1, \ldots, \mathcal{D}_N)$ is the degree matrix, $\mathcal{A}$ is the adjacency matrix, and ℓ_{ij} is an element of the Laplacian matrix . Mathematically, it can also be defined as

$$\ell_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j, \\ -1, & if\ i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0, & \text{Otherwise.} \end{cases} \tag{2}$$

The Laplacian matrix is symmetric and positive-semi-definite for an undirected graph.

Lemma 2.1 (Olfati-Saber et al., 2007)**:** *Laplacian matrix $\mathcal{L}$ always has a zero eigenvalue. This zero eigenvalue $\lambda_1 = 0$ corresponds to the right unit eigenvector $\mathbf{1} = [1, \ldots, 1]^T$ such that $L\mathbf{1} = 0$. Furthermore, rank of the Laplacian matrix is $N - 1$, if and only if $\mathcal{G}$ is a strongly connected directed graph and has a spanning tree.*

The second least eigenvalue of Laplacian matrix λ_2 is known as the Fiedler eigenvalue or the algebraic connectivity of a graph. The Fiedler eigenvalue is very useful in measuring the speed of consensus algorithms.

Lemma 2.2 (Lewis, Zhang, Hengster-Movric, & Das, 2013)**:** *Network topologies having large values of $\lambda_2$depict faster convergence to the consensus. For connected undirected graph topologies, the bound on the Fiedler eigenvalue is $\lambda_2 \geq \frac{1}{\mathrm{Diam}(\mathcal{G})\times \mathrm{Vol}(\mathcal{G})}$, where $\mathrm{Vol}(\mathcal{G})$ is the sum of the in-degree of each node and $\mathrm{Diam}(\mathcal{G})$ is the largest distance between the two nodes in a graph $\mathcal{G}$.*

Consider N identical nonlinear agents, described by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + D_1 f(x_i) + D_2 d_i, \\ y_i &= Cx_i,\ i = 1, \ldots, N, \end{aligned} \tag{3}$$

where $x_i \in R^n$, $u_i \in R^p$ and $y_i \in R^q$ are the state, control input and output vectors of the ith nonlinear agent, respectively, in the dynamics of nonlinear multi-agent systems. A, B, C, D_1 and D_2 are constant matrices having appropriate dimensions. Let the pair (A, B) be stabilisable and the pair (A, C) be detectable. The symbol $f(x_i)$ represents a nonlinear function, and d_i denotes the external disturbances to the agents.

Assumption 2.1: *The communication topology between these interacting nonlinear agents is represented by an undirected graph G.*

Assumption 2.2: *The function $f(x_i)$ satisfies the Lipschitz condition, for the Lipschitz constant $\gamma > 0$, given by*

$$||f(x_a) - f(x_b)|| \leq \gamma ||x_a - x_b||,\ \forall x_a, x_b \in R^n. \tag{4}$$

We employ the dynamic consensus protocol in Li et al. (2013), given by

$$\begin{aligned} \dot{z}_i &= (A + BF)z_i + L\sum_{j=1}^{N} c_{ij}a_{ij}[C(z_i - z_j) - (y_i - y_j)], \\ \dot{c}_{ij} &= \eta_{ij}a_{ij}\begin{bmatrix} y_i - y_j \\ C(z_i - z_j)\end{bmatrix}^T \Gamma \begin{bmatrix} y_i - y_j \\ C(z_i - z_j)\end{bmatrix}, \\ u_i &= Fz_i i = 1, \ldots, N, \\ \Gamma &= \begin{bmatrix} I_q & -I_q \\ -I_q & I_q \end{bmatrix}, c_{ij}(0) = c_{ji}(0), \eta_{ij} = \eta_{ji}, \end{aligned} \tag{5}$$

where z_i is the state of the consensus protocol, a_{ij} is the element of the adjacency matrix, c_{ij} is the time-varying coupling weight of edges between adjacent nonlinear agents, η_{ij} is a positive constant that can be appropriately set for adaptation, and F and L are gain matrices of the protocol having appropriate dimensions.

In order to design a consensus control protocol, global information of the second eigenvalue of Laplacian matrix of a communication graph is required. This information is employed by all the agents to compute the desired coupling weight for a consensus protocol, which destroys the fully distributed nature of a consensus control methodology. In the present work, we have employed an edge-based time-varying coupling weight c_{ij} for adaptation of the fixed coupling weight, based on the second eigenvalue of Laplacian matrix. This feature allows a fully distributed consensus control protocol synthesis for the nonlinear multi-agent systems in (3) and relaxes the requirement of a known second eigenvalue of Laplacian matrix for a communication graph.

3. Consensus protocol design

In the following theorem, we provide a nonlinear matrix inequality-based condition to determine the proper consensus protocol gain matrices for designing a consensus protocol (5).

Theorem 3.1: *Consider the nonlinear agents in (3) under $d_i = 0$ satisfying Assumptions 2.1–2.2. An asymptotic consensus using protocol (5) can be achieved between the agents, if there exist scalars $\tilde{\alpha} \geq 1$, $\tau_1 > 0$ and $\tau_2 > 0$ as well as symmetric matrices $\bar{Q} > 0$ and $Q > 0$ such that for a given matrix F, the matrix inequality*

$$\begin{bmatrix} \Pi_1 & \bar{Q}BF & I_n & \sqrt{2}\gamma\bar{Q}D_1 & 0_{n\times n} \\ * & \Pi_2 & 0_{n\times n} & 0_{n\times n} & \sqrt{2}\gamma QD_1 \\ * & * & -\tau_1 I_n & 0_{n\times n} & 0_{n\times n} \\ * & * & * & -\tau_1^{-1} I_n & 0_{n\times n} \\ * & * & * & * & -\tau_2^{-1} I_n \end{bmatrix} < 0 \quad (6)$$

is satisfied, where $\Pi_1 = \bar{Q}(A + BF) + (A + BF)^T\bar{Q}$, $\Pi_2 = QA + A^TQ - 2\tilde{\alpha}C^TC$ and $L = -Q^{-1}C^T$.

Proof: Taking $\upsilon_i = [x_i{}^T, z_i{}^T]^T$, $\bar{\upsilon} = \frac{1}{N}\sum_{j=1}^N \upsilon_j$, $e_i = \upsilon_i - \bar{\upsilon}$, $\upsilon = [\upsilon_1^T, \dots, \upsilon_N^T]^T$, and $e = [e_1^T, \dots, e_N^T]^T$, we obtain $e = [(I_N - \frac{1}{N}11^T) \otimes I_{2n}]\upsilon$, which implies that **1** is the right eigenvector corresponding to the zero simple eigenvalue of the matrix $(I_N - \frac{1}{N}11^T)$, and the multiplicity of the nonzero eigenvalues is $N - 1$. It further ensures that $e = 0$ if and only if $\upsilon_1 = \dots = \upsilon_N$. Hence, the consensus problem for the nonlinear agents in (3) under protocol (5) can be solved by attaining the asymptotic stability of the error e. As the communication topology is undirected, $c_{ij}(t) = c_{ji}(t)$, $\forall t \geq 0$. Using (3) and (5) obtains

$$\begin{aligned} \dot{e}_i &= \bar{A}e_i + \sum_{j=1}^N c_{ij}a_{ij}\bar{B}(e_i - e_j) + \varphi(x_i, \bar{x}) + \psi(d_i), \\ \dot{c}_{ij} &= \eta_{ij}a_{ij}\left(e_i - e_j\right)^T M\left(e_i - e_j\right), \end{aligned} \quad (7)$$

where $\bar{A} = \begin{bmatrix} A & BF \\ 0_{n\times n} & A+BF \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ -LC & LC \end{bmatrix}$, $\varphi(x_i, \bar{x}) = \begin{bmatrix} D_1(f(x_i) - \frac{1}{N}\sum_{j=1}^N f(x_j)) \\ 0_{n\times 1} \end{bmatrix}$, $\psi(d_i) = \begin{bmatrix} D_2(d_i - \frac{1}{N}\sum_{j=1}^N d_j) \\ 0_{n\times 1} \end{bmatrix}$, and $M = (I_2 \otimes C^T)\Gamma(I_2 \otimes C)$. □

Consider the Lyapunov function given as

$$V(t, e_i, c_{ij}) = \frac{1}{2}\sum_{i=1}^N e_i^T Pe_i + \sum_{i=1}^N \sum_{j=1, j\neq i}^N \frac{(c_{ij} - \alpha)^2}{4\eta_{ij}}, \quad (8)$$

where $P = \begin{bmatrix} \bar{Q}+Q & -Q \\ -Q & Q \end{bmatrix}$. Note that $Q > 0$ and $\bar{Q} > 0$ imply $P > 0$ and that α is a positive scalar. The time-derivative of (8) along (7) becomes

$$\dot{V}(t, e_i, c_{ij}) = \sum_{i=1}^N e_i^T P\dot{e}_i + \sum_{i=1}^N \sum_{j=1, j\neq i}^N \frac{(c_{ij} - \alpha)}{2\eta_{ij}}\dot{c}_{ij}. \quad (9)$$

Substituting (7) into (9) produces

$$\begin{aligned} \dot{V}(t, e_i, c_{ij}) = &\sum_{i=1}^N e_i^T P\Big(\bar{A}e_i + \sum_{j=1}^N c_{ij}a_{ij}\bar{B}(e_i - e_j) \\ &+ \varphi(x_i, \bar{x}) + \psi(d_i, \bar{d})\Big) \\ &+ \frac{1}{2}\sum_{i=1}^N \sum_{j=1, j\neq i}^N (c_{ij} - \alpha)a_{ij}(e_i - e_j)^T \\ &\times M(e_i - e_j). \end{aligned} \quad (10)$$

As $c_{ij}(t) = c_{ji}(t)$, $\forall t \geq 0$, we have

$$\begin{aligned} &\sum_{i=1}^N \sum_{j=1, j\neq i}^N (c_{ij} - \alpha)a_{ij}(e_i - e_j)^T M(e_i - e_j) \\ &= 2\sum_{i=1}^N \sum_{j=1, j\neq i}^N (c_{ij} - \alpha)a_{ij}e_i^T M(e_i - e_j). \end{aligned} \quad (11)$$

Using (10)–(11), $QL = -C^T$ and the value of Γ from (5) yields

$$\begin{aligned} \dot{V}(t, e_i, c_{ij}) = &\sum_{i=1}^N e_i^T P(\bar{A}e_i + \varphi(x_i, \bar{x}) + \psi(d_i, \bar{d})) \\ &- \alpha\sum_{i=1}^N \sum_{j=1, j\neq i}^N a_{ij}e_i^T M(e_i - e_j). \end{aligned} \quad (12)$$

Employing the transformation $\tilde{e}_i = Te_i$, we have

$$\dot{V}(t, \tilde{e}_i, c_{ij}) = \sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{A}\tilde{e}_i - \alpha \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\tilde{e}_i^T \tilde{M}(\tilde{e}_i - \tilde{e}_j) + \sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{\varphi}(x_i, \bar{x}) + \sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{\psi}(d_i, \bar{d}). \tag{13}$$

where

$$T = \begin{bmatrix} I_n & 0_{n\times n} \\ -I_n & I_n \end{bmatrix}, \tilde{P} = T^{-T}PT^{-1} = \begin{bmatrix} \bar{Q} & 0_{n\times n} \\ 0_{n\times n} & Q \end{bmatrix},$$

$$\tilde{A} = T\bar{A}T^{-1} = \begin{bmatrix} A+BF & BF \\ 0_{n\times n} & A \end{bmatrix}, \tilde{M} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & C^TC \end{bmatrix},$$

$$\tilde{\varphi}(x_i, \bar{x}) = T\varphi(x_i, \bar{x}) = \begin{bmatrix} D_1\left(f(x_i) - \frac{1}{N}\sum_{j=1}^{N} f(x_j)\right) \\ -D_1\left(f(x_i) - \frac{1}{N}\sum_{j=1}^{N} f(x_j)\right) \end{bmatrix},$$

$$\tilde{\psi}(d_i) = T\psi(d_i) = \begin{bmatrix} D_2\left(d_i - \frac{1}{N}\sum_{j=1}^{N} d_j\right) \\ -D_2\left(d_i - \frac{1}{N}\sum_{j=1}^{N} d_j\right) \end{bmatrix}. \tag{14}$$

Partitioning $\tilde{\varphi}(x_i, \bar{x}) = \tilde{D}_1\tilde{\theta}_1(x_i, \bar{x}) + \tilde{D}_1\tilde{\theta}_2(x_i, \bar{x})$ and rearranging $\tilde{\psi}(d_i, \bar{d}) = \tilde{D}_2\tilde{\chi}(d_i, \bar{d})$, where

$$\tilde{D}_1 = \begin{bmatrix} D_1 & 0_{n\times n} \\ 0_{n\times n} & D_1 \end{bmatrix}, \tilde{D}_2 = \begin{bmatrix} D_2 & 0_{n\times n} \\ 0_{n\times n} & D_2 \end{bmatrix},$$

$$\tilde{\theta}_1(x_i, \bar{x}) = \begin{bmatrix} f(x_i) - f(\bar{x}) \\ -f(x_i) + f(\bar{x}) \end{bmatrix},$$

$$\tilde{\theta}_2(x_i, \bar{x}) = \begin{bmatrix} f(\bar{x}) - \frac{1}{N}\sum_{j=1}^{N} f(x_j) \\ -f(\bar{x}) + \frac{1}{N}\sum_{j=1}^{N} f(x_j) \end{bmatrix},$$

$$\tilde{\chi}(d_i, \bar{d}) = \begin{bmatrix} d_i - \frac{1}{N}\sum_{j=1}^{N} d_j \\ -d_i + \frac{1}{N}\sum_{j=1}^{N} d_j \end{bmatrix}, \tag{15}$$

we can rewrite (13) as

$$\dot{V}(t, \tilde{e}_i, c_{ij}) = \sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{A}\tilde{e}_i - \alpha \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\tilde{e}_i^T \tilde{M}(\tilde{e}_i - \tilde{e}_j) + \sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{D}_1\tilde{\theta}_1(x_i, \bar{x}) + \sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{D}_1\tilde{\theta}_2(x_i, \bar{x}) + \sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{D}_2\tilde{\chi}(d_i, \bar{d}). \tag{16}$$

Applying Lipschitz condition (4) and using the matrix algebra imply

$$\begin{aligned} ||\tilde{\theta}_1(x_i, \bar{x})|| &= \sqrt{2}\left\| f(x_i) - f(\bar{x}) \right\|, \\ &\leq \sqrt{2}\gamma||Ze_i||, \\ &= \sqrt{2}\gamma||Z\tilde{e}_i||, \end{aligned} \tag{17}$$

$$Z = \begin{bmatrix} I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}, \tag{18}$$

which entails

$$\begin{aligned} \tilde{e}_i^T \tilde{P}\tilde{D}_1\tilde{\theta}_1(x_i, \bar{x}) &\leq \left\| \tilde{e}_i^T \tilde{P}\tilde{D}_1 \mathrm{diag}(\tau_1^{1/2}I_n, \tau_2^{1/2}I_n) \right\| \\ &\quad \times \left\| \mathrm{diag}(\tau_1^{-1/2}I_n, \tau_2^{-1/2}I_n)\tilde{\theta}_1(x_i, \bar{x}) \right\|, \\ &\leq \sqrt{2}\gamma \left\| \tilde{e}_i^T \tilde{P}\tilde{D}_1 \mathrm{diag}(\tau_1^{1/2}I_n, \tau_2^{1/2}I_n) \right\| \\ &\quad \times \left\| \mathrm{diag}(\tau_1^{-1/2}I_n, \tau_2^{-1/2}I_n)Z\tilde{e}_i \right\|, \\ &\leq \frac{1}{2}\tilde{e}_i^T[2\gamma^2\tilde{P}\tilde{D}_1\mathrm{diag}(\tau_1 I_n, \tau_2 I_n)\tilde{D}_1\tilde{P} \\ &\quad + Z\mathrm{diag}(\tau_1^{-1}I_n, \tau_2^{-1}I_n)Z]\tilde{e}_i. \end{aligned} \tag{19}$$

Since $\sum_{i=1}^{N} \tilde{e}_i = 0$, we have $\sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{D}_1\tilde{\theta}_2(x_i, \bar{x}) = 0$. Incorporating the condition in (19) and $\sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{D}_1\tilde{\theta}_2(x_i, \bar{x}) = 0$ into (16) reveals

$$\begin{aligned} \dot{V}(t, \tilde{e}_i, c_{ij}) \leq \sum_{i=1}^{N} \tilde{e}_i^T \Bigg(\Big[&\tilde{P}\tilde{A} + \frac{1}{2}(2\gamma^2\tilde{P}\tilde{D}_1\mathrm{diag}(\tau_1 I_n, \tau_2 I_n) \\ &\times \tilde{D}_1\tilde{P} + Z\mathrm{diag}(\tau_1^{-1}I_n, \tau_2^{-1}I_n)Z) \Big]\tilde{e}_i \\ &- \alpha\sum_{j=1}^{N} a_{ij}\tilde{M}(\tilde{e}_i - \tilde{e}_j) \Bigg) \\ &+ \sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{D}_2\tilde{\chi}(d_i, \bar{d}), \end{aligned}$$

which, by assigning $\tilde{e} = [\tilde{e}_1^T, \tilde{e}_2^T, ..., \tilde{e}_N^T]^T$ and using $\sum_{i=1}^{N} \tilde{e}_i^T \tilde{P}\tilde{D}_2\tilde{\chi}(d_i, \bar{d}) = \tilde{e}^T(I_N \otimes \tilde{P}\tilde{D}_2)\bar{\psi}$, produces

$$\dot{V}(t,\tilde{e}_i,c_{ij}) \leq \frac{1}{2}\tilde{e}^T(I_N \otimes [\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+2\gamma^2\tilde{P}\tilde{D}_1 \times \mathrm{diag}(\tau_1 I_n, \tau_2 I_n)\tilde{D}_1\tilde{P} + Z\mathrm{diag}(\tau_1^{-1}I_n, \tau_2^{-1}I_n)Z] - 2\alpha(\mathcal{L}\otimes I_n)\tilde{M})\tilde{e} + \tilde{e}^T(I_N \otimes \tilde{P}\tilde{D}_2)\bar{\psi}(d_i,\bar{d}), \quad (20)$$

$$\bar{\psi}(d_i,\bar{d}) = \begin{bmatrix} \tilde{\chi}(d_1,\bar{d}) \\ \vdots \\ \tilde{\chi}(d_N,\bar{d}) \end{bmatrix}. \quad (21)$$

As $\mathcal{G}$ is connected under Assumption 2.1, $\tilde{e}^T(\mathcal{L}\otimes I)\tilde{e} \geq \lambda_2\tilde{e}^T\tilde{e}$ holds, which along with (20) produces

$$\dot{V}(t,\tilde{e}_i,c_{ij}) \leq \frac{1}{2}\tilde{e}^T(I_N \otimes [\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P} + 2\gamma^2\tilde{P}\tilde{D}_1\mathrm{diag}(\tau_1 I_n, \tau_2 I_n)\tilde{D}_1\tilde{P} + Z\mathrm{diag}(\tau_1^{-1}I_n, \tau_2^{-1}I_n)Z - 2\alpha\lambda_2\tilde{M}])\tilde{e} + \tilde{e}^T(I_N \otimes \tilde{P}\tilde{D}_2)\bar{\psi}(d_i,\bar{d}). \quad (22)$$

For asymptotic consensus, we need $\dot{V}(t,\tilde{e}_i,c_{ij}) < 0$. Under $d_i = 0$, (22) implies $\dot{V}(t,\tilde{e}_i,c_{ij}) < 0$, if

$$\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+2\gamma^2\tilde{P}\tilde{D}_1\mathrm{diag}(\tau_1 I_n, \tau_2 I_n)\tilde{D}_1\tilde{P} + Z\mathrm{diag}\left(\tau_1^{-1}I_n, \tau_2^{-1}I_n\right)Z - 2\alpha\lambda_2\tilde{M} < 0. \quad (23)$$

Application of the Schur complement and employing $\tilde{\alpha} = \alpha\lambda_2 \geq 1$, (14), (15) and (18) result in

$$\begin{bmatrix} \Pi_1 & \bar{Q}BF & I_n & 0_{n\times n} & \sqrt{2}\gamma\bar{Q}D_1 & 0_{n\times n} \\ * & \Pi_2 & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \sqrt{2}\gamma QD_1 \\ * & * & -\tau_1 I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & -\tau_2 I_n & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & -\tau_1^{-1}I_n & 0_{n\times n} \\ * & * & * & * & * & -\tau_2^{-1}I_n \end{bmatrix} < 0. \quad (24)$$

Constraint (6) is obtained by ignoring the 4th column and row of (24), which have zeros in non-diagonal elements, thus completing the proof.

Remark 3.1: Several researchers have addressed the issues concerning the consensus of Lipschitz nonlinear multi-agents by providing non-adaptive (Ding, 2014; Du, He, & Cheng, 2014; Li et al., 2012; Wen et al., 2014) and adaptive (Li et al., 2013) protocols using state feedback. Contrastingly, the proposed approach in Theorem 3.1 is an output-feedback-based consensus approach that is applicable when the states of the agents are not known. Additionally to Li et al. (2012), Li et al. (2013), Wen et al. (2014) and Ding (2014), our developed protocol for nonlinear agents is dynamic, employing two gain matrices F and L, to attain multiple performance objectives. In contrast to Li et al. (2012), Wen et al. (2014), Ding (2014), and Du et al. (2014), our methodology is adaptive and does not require information on the algebraic connectivity of a graph; it can, therefore, be implemented in a completely distributed manner.

The condition in Theorem 3.1 can be used to find the consensus protocol gain matrices F and L such that consensus is achieved between the nonlinear agents in (3). However, it is very difficult to solve the design condition in Theorem 3.1, because (6) is not an LMI. Therefore, Theorem 3.1 is not appropriate for the design of a suitable consensus protocol. In the next theorem, we decouple the nonlinear matrix inequality into two relatively simple constraints by extending the ideas of Huang, Huang, Chen, and Qian (2013) and Lin, Wang, Lee, He, and Chen (2008) for the consensus control case such that the gain matrices F and L can be calculated efficiently and independently.

Theorem 3.2: *A necessary and sufficient condition for solving the constraints in Theorem 3.1 is that there exist scalars $\tau_1 > 0$ and $\tau_3 > 0$ as well as symmetric matrices $\bar{Q} > 0$ and $Q_1 > 0$ such that the following LMIs hold:*

$$\begin{bmatrix} Q_1 A + A^T Q_1 - 2\beta C^T C & \sqrt{2}\gamma Q_1 D_1 \\ * & -\tau_3 I_n \end{bmatrix} < 0, \quad (25)$$

$$\begin{bmatrix} AS + BV + SA^T + V^T B^T & S & \sqrt{2}\gamma\tau_1 D_1 \\ * & -\tau_1 I_n & 0_{n\times n} \\ * & * & -\tau_1 I_n \end{bmatrix} < 0. \quad (26)$$

The gain matrices F and L of the proposed consensus protocol (5) can be computed by evaluating $F = VS^{-1}$ and $L = -Q_1{}^{-1}C^T$, respectively.

Proof: *Necessity*: Let us assign

$$\Upsilon_1 = \begin{bmatrix} \bar{Q}^{-1} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & I_n & 0_{n\times n} \end{bmatrix}, \quad \Upsilon_2 = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & I_n \end{bmatrix}. \quad (27)$$

□

By pre- and post-multiplication of Υ_1 and Υ_1^T, respectively, to matrix inequality (6), we obtain

$$\begin{bmatrix} (A+BF)\bar{Q}^{-1} + \bar{Q}^{-1}(A+BF)^T & \bar{Q}^{-1} & \sqrt{2}\gamma D_1 \\ * & -\tau_1 I_n & 0_{n\times n} \\ * & * & -\tau_1^{-1}I_n \end{bmatrix} < 0. \quad (28)$$

Setting $S = \bar{Q}^{-1}$ and $V = F\bar{Q}^{-1}$ and, further, applying congruence transformation using $diag(I_n, I_n, \tau_1 I_n)$ leads to (26). In the same way, multiplying (6) by Υ_2 and Υ_2^T leads to

$$\begin{bmatrix} QA - A^TQ - 2\tilde{\alpha}C^TC & \sqrt{2}\gamma QD_1 \\ * & -\tau_2^{-1}I_n \end{bmatrix} < 0, \quad (29)$$

which further produces (25) by application of $Q_1 = Q$, $\tau_3 = \tau_2^{-1}$ and $\beta = \tilde{\alpha}$.

Sufficiency: Applying congruence transformation, using diagonal matrix $\mathrm{diag}(S^{-1}, I_n, \tau_1^{-1}I_n)$ and substituting $V = FS$ and $\bar{Q} = S^{-1}$ into (26), we have

$$\Pi_3 = \begin{bmatrix} \Pi_1 & I_n & \sqrt{2}\gamma\bar{Q}D_1 \\ * & -\tau_1 I_n & 0_{n\times n} \\ * & * & -\tau_1^{-1}I_n \end{bmatrix} < 0. \quad (30)$$

Substituting $\Pi_4 = Q_1A + A^TQ_1 - 2\beta C^TC$ into (25) implies

$$\prod_5 = \begin{bmatrix} \Pi_4 & \sqrt{2}\gamma Q_1D_1 \\ * & -\tau_3 I_n \end{bmatrix} < 0. \quad (31)$$

For a sufficiently large scalar $\varepsilon > 0$, inequalities (30) and (31) result in

$$\begin{bmatrix} \prod_3 & \prod_6^T \\ \prod_6 & \varepsilon\prod_5 \end{bmatrix} < 0, \quad (32)$$

$$\prod_6^T = \begin{bmatrix} \bar{Q}BF & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}. \quad (33)$$

Let $\Im_i$ represents a matrix with five partitions of $R^{n\times n}$. The ith partition is an identity matrix, and all other partitions are zero. For example, $\Im_3 = [0_{n\times n}, 0_{n\times n}, I_n, 0_{n\times n}, 0_{n\times n}]$. Substituting (30), (31) and (33) into (32) and employing pre- and post-multiplication with $[\Im_1^T, \Im_4^T, \Im_2^T, \Im_3^T, \Im_5^T]^T$ and its transpose, respectively, we have

$$\begin{bmatrix} \Pi_1 & \bar{Q}BF & I_n & \sqrt{2}\gamma\bar{Q}D_1 & 0_{n\times n} \\ * & \varepsilon\Pi_4 & 0_{n\times n} & 0_{n\times n} & \sqrt{2}\gamma\varepsilon Q_1D_1 \\ * & * & -\tau_1I_n & 0_{n\times n} & 0_{n\times n} \\ * & * & * & -\tau_1^{-1}I_n & 0_{n\times n} \\ * & * & * & * & -\varepsilon\tau_3I_n \end{bmatrix} < 0. \quad (34)$$

The above resultant inequality implies (6) for $Q = \varepsilon Q_1$, $\tau_2 = \varepsilon^{-1}\tau_3^{-1}$ and $\alpha = \varepsilon\beta$. This completes the proof of Theorem 3.2.

Remark 3.2: By employing a decoupling technique, a necessary and sufficient condition is established in Theorem 3.2 in terms of LMIs for designing an adaptive protocol (5), by which the nonlinear agents (3) can achieve consensus for all undirected graph topologies. Now, consensus protocol gain matrices F and L can be straightforwardly and roughly computed, which addressed the limitation in Theorem 3.1. The design condition in Theorem 3.2 is easy to handle compared with Theorem 3.1, because of the LMIs and the elimination of dependency between the protocol gain matrices.

Remark 3.3: The decoupling methodology has been efficiently utilised for observer-based control of linear and Lipschitz nonlinear systems (Huang et al., 2013; Lin et al., 2008). Note, however, that the decoupling condition in Theorem 3.2 is not a straightforward extension of the observer-based linear and Lipschitz nonlinear control results in Lin et al. (2008) or Huang et al. (2013). The present work addresses a more complex problem of the consensus control of multiple nonlinear agents and provides a decoupling condition for an adaptive dynamic protocol rather than a less complicated observer-based control scenario. Moreover, both necessity and sufficiency are demonstrated in Theorem 3.2, in contrast to the previous work on Lipschitz systems by Huang et al. (2013).

4. Robust consensus control

Now, we develop conditions for the design of distributed robust adaptive protocols for the attainment of consensus in the Lipschitz nonlinear multi-agent systems in (3). The objective being to attain consensus of the multi-agents in the presence of disturbances, we define $\bar{x} = \frac{1}{N}\sum_{j=1}^{N} x_j$, $e_{xi} = x_i - \bar{x}$, $e_x = [e_{x1}^T, e_{x2}^T, ..., e_{xN}^T]^T$, $\tilde{d}_i = d_i - \frac{1}{N}\sum_{j=1}^{N} d_j$, and $\tilde{d} = [\tilde{d}_1^T, \tilde{d}_2^T, ..., \tilde{d}_N^T]^T$. The following theorem presents conditions for minimisation of disturbance effects $\tilde{d}$ at the error signal e_x.

Theorem 4.1: (*a*) *Consider the nonlinear agents in* (3) *satisfying Assumptions* 2.1–2.2 *under protocol* (5). *Suppose there exist scalars $\tilde{\alpha} \geq 1$, $\tau_1 > 0$, $\tau_2 > 0$, $\kappa_1 > 0$, $\sigma_1 > 0$ and $\sigma_2 > 0$ as well as symmetric matrices $\bar{Q} > 0$ and $Q > 0$ such that for a given matrix F, the matrix inequality*

$$\begin{bmatrix} \frac{1}{2}\Pi_1 & \frac{1}{2}\bar{Q}BF & \frac{1}{2}\bar{Q}D_2 & 0_{n\times n} & I_n & \frac{1}{\sqrt{2}}I_n & \gamma\bar{Q}D_1 & 0_{n\times n} \\ * & \frac{1}{2}\Pi_2 & 0_{n\times n} & \frac{1}{2}QD_2 & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \gamma QD_1 \\ * & * & -\sigma_1 I & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & -\sigma_2I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & -\kappa_1I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & * & -\tau_1I_n & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & * & * & -\tau_1^{-1}I_n & 0_{n\times n} \\ * & * & * & * & * & * & * & -\tau_2^{-1}I_n \end{bmatrix} < 0 \quad (35)$$

holds, where $L = -Q^{-1}C^T$. Then, an asymptotic consensus of the agents using protocol (5) *can be achieved if $d_i = 0$.*

Additionally, the L_2 gain from $\tilde{d}$ to e_x remains bounded by $\sqrt{\kappa_1(\sigma_1+\sigma_2)}$.

(b) A necessary and sufficient condition for solving the constraints in Theorem 4.1(a) is that there exist scalars $\tau_1 > 0$, $\tau_3 > 0$ $\kappa_1 > 0$, $\sigma_1 > 0$ and $\sigma_3 > 0$ as well as symmetric matrices $\bar{Q} > 0$ and $Q_1 > 0$ such that the following LMIs hold:

$$\begin{bmatrix} \frac{1}{2}[AS+BV+SA^T+V^TB^T] & \frac{1}{2}D_2 & S & \frac{1}{\sqrt{2}}S & \gamma\tau_1 D_1 \\ * & -\sigma_1 I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & -\kappa_1 I_n & 0_{n\times n} & 0_{n\times n} \\ * & * & * & -\tau_1 I_n & 0_{n\times n} \\ * & * & * & * & -\tau_1 I_n \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} \frac{1}{2}\left[Q_1A+A^TQ_1-2\alpha_1\lambda_2 C^TC\right] & \frac{1}{2}Q_1D_2 & \gamma Q_1 D_1 \\ * & -\sigma_3 I_n & 0_{n\times n} \\ * & * & -\tau_3 I_n \end{bmatrix} < 0. \tag{37}$$

The gain matrices F and L of the proposed consensus protocol (5) can be computed by evaluating $F = VS^{-1}$ and $L = -Q_1{}^{-1}C^T$, respectively.

Proof: Defining

$$J(t,\tilde{e}_i,c_{ij},\tilde{d}) = \dot{V}(t,\tilde{e}_i,c_{ij}) + \kappa_1^{-1}e_x^Te_x - \sigma\tilde{d}^T\tilde{d}. \tag{38}$$

□

Under zero disturbance, $\tilde{d}=0$ is implied; therefore, $J(t,\tilde{e}_i,c_{ij},\tilde{d}) < 0$ ensures $\dot{V}(t,\tilde{e}_i,c_{ij}) < 0$. That is, asymptotic consensus of the agents can be achieved under zero disturbances through $J(t,\tilde{e}_i,c_{ij},\tilde{d}) < 0$. When $d_i \neq 0$, integrating (38) under zero initial condition reveals that $\|e_x\|_2^2 < \kappa_1\sigma\|\tilde{d}\|_2^2$; that is, the L_2 gain between signals $\tilde{d}$ and e_x is less than $\sqrt{\kappa_1\sigma}$. Using $\tilde{e}_i = Te_i$, $e = [e_1^T,\ldots,e_N^T]^T$, $e_i = \upsilon_i - \bar{\upsilon}$, $\upsilon_i = [x_i{}^T, z_i{}^T]^T$, $\bar{\upsilon} = \frac{1}{N}\sum_{j=1}^N \upsilon_j$, $e_{xi} = x_i - \bar{x}$, $e_x = [e_{x1}^T, e_{x2}^T, ..., e_{xN}^T]^T$, and $\bar{x} = \frac{1}{N}\sum_{j=1}^N x_j$ and solving matrix algebra, we obtain

$$\kappa_1^{-1}e_x^Te_x = \tilde{e}^T\,(I_N\otimes Y)\,\tilde{e}, \tag{39}$$

$$Y = \begin{bmatrix} \kappa_1^{-1}I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} = \begin{bmatrix} I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}\begin{bmatrix} \kappa_1^{-1}I_n & 0_{n\times n} \\ 0_{n\times n} & \kappa_2^{-1}I_n \end{bmatrix} \times \begin{bmatrix} I_n & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix}. \tag{40}$$

Partitioning the scalar σ as $\sigma = \sigma_1+\sigma_2$ and using (15), (21), $\tilde{d}_i = d_i - \frac{1}{N}\sum_{j=1}^N d_j$, and $\tilde{d} = [\tilde{d}_1^T, \tilde{d}_2^T, ..., \tilde{d}_N^T]^T$, we obtain

$$\sigma\tilde{d}^T\tilde{d} = \bar{\psi}^T(d_i,\bar{d})\,(I_N\otimes W)\,\bar{\psi}(d_i,\bar{d}),\quad W = \begin{bmatrix} \sigma_1 I_n & 0_{n\times n} \\ 0_{n\times n} & \sigma_2 I_n \end{bmatrix}. \tag{41}$$

Incorporating (39) and (41) into (38), we have

$$J(t,\tilde{e}_i,c_{ij},\tilde{d}) = \dot{V}(t,\tilde{e}_i,c_{ij}) + \tilde{e}^T\,(I_N\otimes Y)\,\tilde{e} - \bar{\psi}^T(d_i,\bar{d})\,(I_N\otimes W)\,\bar{\psi}(d_i,\bar{d}). \tag{42}$$

Substituting (22) into (42) entails

$$\begin{aligned} J(t,\tilde{e}_i,c_{ij},\tilde{d}) \le{}& \frac{1}{2}\tilde{e}^T(I_N\otimes[\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+2\gamma^2\tilde{P}\tilde{D}_1 \\ &\times \mathrm{diag}(\tau_1 I_n,\tau_2 I_n)\tilde{D}_1\tilde{P} \\ &+ Z\mathrm{diag}(\tau_1^{-1}I_n,\tau_2^{-1}I_n)Z - 2\alpha\lambda_2\tilde{M}])\tilde{e} \\ &+ \tilde{e}^T(I_N\otimes\tilde{P}\tilde{D}_2)\bar{\psi}(d_i,\bar{d}) + \tilde{e}^T(I_N\otimes Y)\tilde{e} \\ &- \bar{\psi}^T(d_i,\bar{d})(I_N\otimes W)\bar{\psi}(d_i,\bar{d}), \end{aligned} \tag{43}$$

which can be rewritten

$$J(t,\tilde{e}_i,c_{ij},\tilde{d}) \le \begin{bmatrix} \tilde{e}^T & \bar{\psi}^T(d_i,\bar{d}) \end{bmatrix}\begin{bmatrix} I_N\otimes\Pi_7 & I_N\otimes\frac{1}{2}\tilde{P}\tilde{D}_2 \\ * & -I_N\otimes W \end{bmatrix} \times \begin{bmatrix} \tilde{e} \\ \bar{\psi}(d_i,\bar{d}) \end{bmatrix}, \tag{44}$$

$$\Pi_7 = \frac{1}{2}\Big[\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+2\gamma^2\tilde{P}\tilde{D}_1\mathrm{diag}(\tau_1 I_n,\tau_2 I_n)\tilde{D}_1\tilde{P} + Z\mathrm{diag}(\tau_1^{-1}I_n,\tau_2^{-1}I_n)Z - 2\alpha\lambda_2\tilde{M} + 2Y\Big]. \tag{45}$$

For $J(t,\tilde{e}_i,c_{ij},\tilde{d}) < 0$, we require

$$\begin{bmatrix} I_N\otimes\Pi_7 & I_N\otimes\frac{1}{2}\tilde{P}\tilde{D}_2 \\ * & -I_N\otimes W \end{bmatrix} < 0. \tag{46}$$

By expanding the Kronecker product and interchanging the rows and columns with each other, constraint (46) produces

$$I_N\otimes\begin{bmatrix} \Pi_7 & \frac{1}{2}\tilde{P}\tilde{D}_2 \\ * & -W \end{bmatrix} < 0. \tag{47}$$

Note that the left sides of (46) and (47) are not equal; however, inequalities (46) and (47) are equivalent, due to the rows and columns interchange operation. Since $I_N > 0$, $J(t,\tilde{e}_i,c_{ij},\tilde{d}) < 0$ if

$$\begin{bmatrix} \Pi_7 & \frac{1}{2}\tilde{P}\tilde{D}_2 \\ * & -W \end{bmatrix} < 0. \tag{48}$$

Substituting (45), applying the Schur complement, incorporating (14), (15), (18), (40) and (41) into (48), and solving the matrix algebra, it is obvious that

$$\begin{bmatrix} \frac{1}{2}\Pi_1 & \frac{1}{2}\bar{Q}BF & \frac{1}{2}\bar{Q}D_2 & 0_{n\times n} & I_n & 0_{n\times n} & \frac{1}{\sqrt{2}}I_n & 0_{n\times n} & \gamma\bar{Q}D_1 & 0_{n\times n} \\ * & \frac{1}{2}\Pi_2 & 0_{n\times n} & \frac{1}{2}QD_2 & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \gamma QD_1 \\ * & * & -\sigma_1 I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & 0 & -\sigma_2 I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & -\kappa_1 I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & * & -\kappa_2 I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & * & * & -\tau_1 I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & * & * & * & -\tau_2 I_n & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & * & * & * & * & -\tau_1^{-1} I_n & 0_{n\times n} \\ * & * & * & * & * & * & * & * & * & -\tau_2^{-1} I_n \end{bmatrix} < 0. \quad (49)$$

By ignoring the sixth and eighth rows and columns, we obtain (35). This completes the proof of Theorem 4.1(a). The proof of Theorem 4.1(b) is analogous to Theorem 3.2.

Remark 4.1: When nonlinear agents are subject to external disturbances, Theorems 3.1 and 3.2 are not suitable for evaluating gain matrices of the consensus protocol (5). In Theorem 4.1(a) and 4.1(b), two design conditions are provided to design robust adaptive fully distributed consensus protocols based on nonlinear matrix inequalities and LMIs. These consensus protocols, in contrast to the approaches in Theorems 3.1–3.2, can deal with perturbations by ensuring stability against disturbances.

By taking $f(x_i) = 0$, the following results are obtained from Theorem 4.1(a) and 4.1(b).

Corollary 4.1: *(a) Consider the nonlinear agents in (3) satisfying Assumptions 2.1–2.2 and $f(x_i) = 0$ under protocol (5). Suppose there exist scalars $\tilde{\alpha} \geq 1$, $\kappa_1 > 0$, $\sigma_1 > 0$ and $\sigma_2 > 0$ as well as symmetric matrices $\bar{Q} > 0$ and $Q > 0$ such that for a given matrix F, the matrix inequality*

$$\begin{bmatrix} \frac{1}{2}\Pi_1 & \frac{1}{2}\bar{Q}BF & \frac{1}{2}\bar{Q}D_2 & 0_{n\times n} & I_n \\ * & \frac{1}{2}\Pi_2 & 0_{n\times n} & \frac{1}{2}QD_2 & 0_{n\times n} \\ * & * & -\sigma_1 I & 0_{n\times n} & 0_{n\times n} \\ * & * & * & -\sigma_2 I_n & 0_{n\times n} \\ * & * & * & * & -\kappa_1 I_n \end{bmatrix} < 0 \quad (50)$$

holds, where $L = -Q^{-1}C^T$. Then, an asymptotic consensus of the agents using protocol (5) can be achieved if $d_i = 0$. Additionally, the L_2 gain from $\tilde{d}$ to e_x remains bounded by $\sqrt{\kappa_1(\sigma_1 + \sigma_2)}$.

(b) A necessary and sufficient condition for solving the constraints in Corollary 4.1(a) is that there exist scalars $\kappa_1 > 0$, $\sigma_1 > 0$ and $\sigma_3 > 0$ as well as symmetric matrices $\bar{Q} > 0$ and Q_1 such that the following inequalities hold:

$$\begin{bmatrix} \frac{1}{2}[AS + BV + SA^T + V^T B^T] & \frac{1}{2}D_2 & S \\ * & -\sigma_1 I_n & 0_{n\times n} \\ * & * & -\kappa_1 I_n \end{bmatrix} < 0. \quad (51)$$

$$\begin{bmatrix} \frac{1}{2}\left[Q_1 A + A^T Q_1 - 2\tilde{\alpha} C^T C\right] & \frac{1}{2}Q_1 D_2 \\ * & -\sigma_3 I_n \end{bmatrix} < 0. \quad (52)$$

The gain matrices F and L of the proposed consensus protocol (5) can be computed by evaluating $F = VS^{-1}$ and $L = -{Q_1}^{-1}C^T$, respectively.

Remark 4.2: Specific results of Theorem 4.1(a) and 4.1(b) for robust adaptive distributed consensus of linear multi-agents are provided in Corollary 4.1. It should be noted that the robustness, requiring substantial research attention, is an important issue for consensus control of linear multi-agents when disturbances from several sources are acting on all of the agents. Compared with the approach in Li et al. (2013), the approach provided in Corollary 4.1 is more practicable for dealing with perturbations. Another distinctive feature of the proposed consensus control approach, in contrast to in Li et al. (2013), is that the inequalities (51)–(52) are shown to be both necessary and sufficient (rather than only sufficient) for obtainment of a solution from (50).

The results developed in the present study addresses distributed adaptive protocol design for the nonlinear multi-agent systems in the absence or presence of disturbances for undirected communication topologies between the multi-agents. Some exceptional works like Chu, Cai, and Zhang (2015) and Sun, Geng, and Lv (2016) can be found in the literature, which considers the directed communication topologies to formulate the adaptive consensus protocols. These control methodologies can be applied to the undirected communication topologies as a special case and are useful to the control of linear or nonlinear multi-agents. However, additional adaptation laws, parametric estimation and nonlinearities are applied in these approaches for dealing with the directed communication topologies for the adaptive consensus control, which factor complicates their application due to the requirement of additional hardware and software resources. Utilisation of these adaptive consensus control techniques to the directed communication topologies is interesting and practicable; however, application of such control protocols for the case of undirected communication topologies is not recommended due to the additional complexity. Consequently, the present approach can be applied for the case of undirected communication topologies with simple adaptation law for implementation of the adaptive consensus protocols. Adaptive output feedback consensus control of the Lipschitz nonlinear multi-agent systems can be studied in future for the case of directed communication topologies to avoid the global information of the eigenvalues of Laplacian matrix.

5. Simulation results

Consider a network of single-link robots with revolute joints (see Rajamani & Cho, 1998). The state-space dynamics of the ith robot is described by (3) with

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \\ x_{i4} \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_s}{J_m} & \frac{-l_{link}}{J_m} & \frac{k_s}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{qk_s}{J_l} & 0 & q\frac{-k_s}{J_l} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{k_\tau}{J_m} \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, f(x_i) = \begin{bmatrix} 0 & 0 & 0 & \frac{-q(mgh)}{J_l}\sin(x_{i3}) \end{bmatrix}^T,$$
$$D_1 = I_4,$$

where x_{i1} and x_{i2} denote the angular rotation and angular velocity of the motor, respectively, x_{i3} and x_{i4} represent the angular position and angular velocity of the link, respectively, for the ith robot, $q = 0.1$ is the transformation coefficient, k_s stands for the torsional spring constant having a numerical value of $0.18\,\mathrm{Nmrad}^{-1}$, $J_m = 0.0037\,\mathrm{Kgm}^2$ represents the inertia of the motor, $J_l = 0.0093\,\mathrm{Kgm}^2$ denotes the inertia of the link, $l_{\text{link}} = 0.31\mathrm{m}$ represents the length of the link, $k_\tau = 0.08\,\mathrm{NmV}^{-1}$ is the amplifier gain, $m = 0.139\,\mathrm{Kg}$ is the point mass of the arm, $g = 9.8\,\mathrm{m/s}^2$ denotes the gravity constant and h is the centre of gravity height having a numerical value of $0.015\,\mathrm{m}$. For the design of the consensus protocol, $\gamma = 0.22$ is fixed. The communication between the robots is subjected to the undirected graph topology $\mathcal{G}$ shown in Figure 1.

Case 1 ($D_2 = 0$)**:** First, we fix the matrix $D_2 = 0$ to verify the proposed methodologies in Theorems 3.1–3.2 in the absence of disturbances. Let $\eta_{ij} = 1$ for $i, j = 1, \ldots, 6$, $\Gamma = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$, and $c_{ij}(0) = c_{ji}(0)$ be randomly chosen in (5). By solving LMIs (25) and (26), we obtain the following gain matrices:

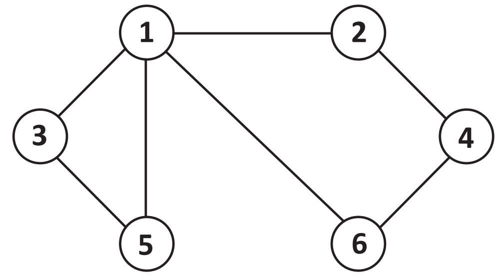

Figure 1. Undirected communication topology of robots.

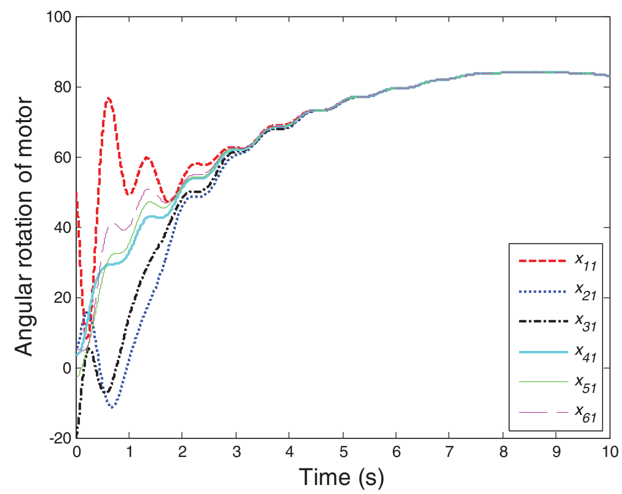

Figure 2. Consensus between angular rotations of motors of multi-agent robots.

$$F = [-0.5718 - 0.7601 - 0.4776 - 0.9670]$$

$$L = [-2.9427 - 19.8481 - 3.4632 - 0.4772]^T.$$

The proposed consensus control protocol (5) is applied for the gain matrices obtained from Theorem 3.2. The responses of the six robots are shown in Figures 2, 3, 4 and 5. The adaptive coupling weights are plotted in Figure 6. Figures 2 and 3 demonstrate that the angular rotations and velocities of all of the motors are converging. Similarly, as shown in Figures 4 and 5, the angular positions and velocities of all of the six links associated with the six robots acquire the same time profile as time increases. The coupling weights converge to constant values, as depicted by Figure 6. Hence, by using the information of the outputs, the proposed distributed adaptive control methodology in Theorem 3.2 can be

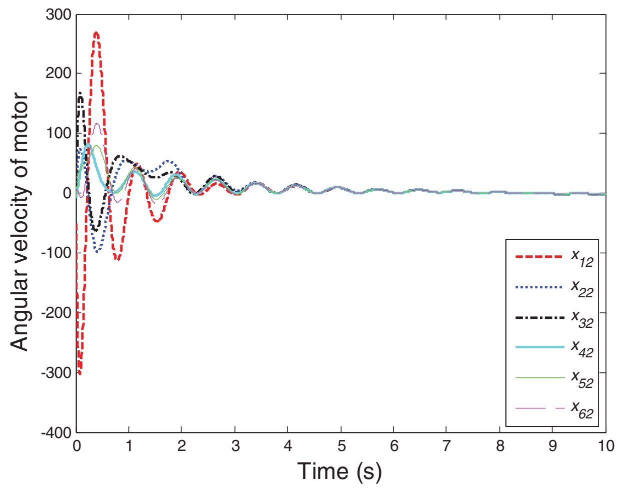

Figure 3. Consensus between angular velocities of motors of multi-agent robots.

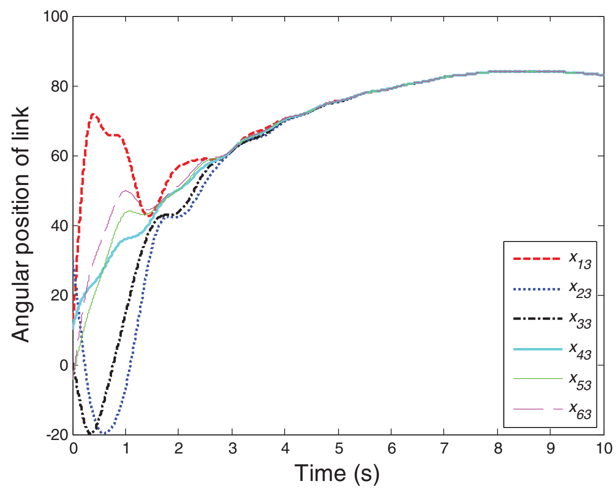

Figure 4. Consensus between angular positions of links of multi-agent robots.

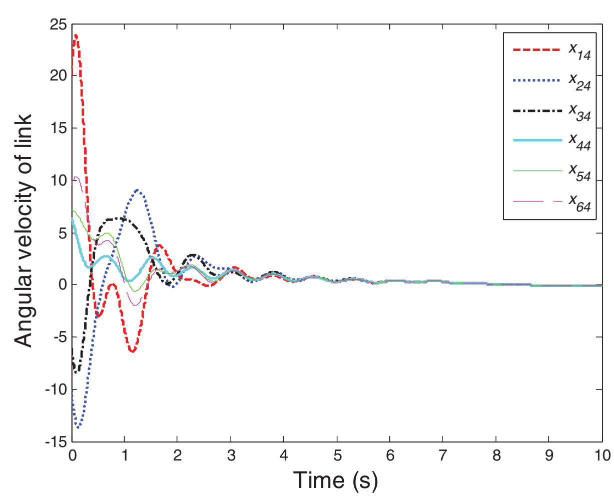

Figure 5. Consensus between angular velocities of links of multi-agent robots.

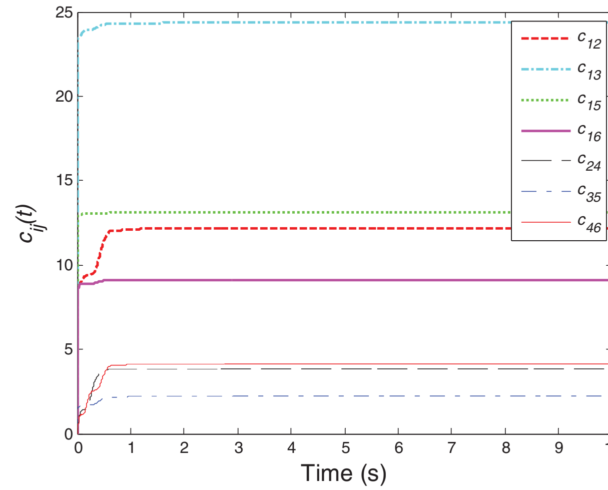

Figure 6. Adaptation of coupling weights for consensus control.

applied with the undirected graph topology to attain consensus between multiple nonlinear agents.

Case 2 $(D_2 = I_4)$**:** Now we suppose that the network of single-link flexible-joint robots is subjected to external disturbances. To evaluate the performance of the proposed consensus methodology in Theorem 4.1(a) and 4.1(b), we select $D_2 = I_4$. The disturbances are taken to be

$$
\begin{aligned}
d_1 &= \left[\, 2.5\sin 30t \;\; 4\sin 38t \;\; 3\sin 25t \;\; 3.5\sin 30t \,\right]^T, \\
d_2 &= \left[\, 5\sin 27t \;\; 3.5\sin 45t \;\; 5\sin 30t \;\; 2.5\sin 29t \,\right]^T, \\
d_3 &= \left[\, 1.5\sin 43t \;\; 4.5\sin 27t \;\; 5\sin 20t \;\; 3\sin 25t \,\right]^T, \\
d_4 &= \left[\, 2\sin 25t \;\; 2.5\sin 35t \;\; 3.5\sin 42t \;\; 1.5\sin 29t \,\right]^T, \\
d_5 &= \left[\, 3.5\sin 30t \;\; 5.5\sin 49t \;\; 4.5\sin 36t \;\; 3.8\sin 28t \,\right]^T, \\
d_6 &= \left[\, 4.5\sin 28t \;\; 3.5\sin 18t \;\; 5.5\sin 28t \;\; 2.7\sin 37.8t \,\right]^T.
\end{aligned}
$$

To illustrae Theorem 4.1(b), the same communication graph as shown in Figure 1 is used. Again taking $\eta_{ij} = 1$, $i, j = 1, \ldots, 6$, $\Gamma = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$ and $c_{ij}(0) = c_{ji}(0)$ and solving the LMIs (36) and (37), we obtain the gain matrices as

$$F = [-28.3764 - 2.367519.5314 - 45.5614]\,,$$

$$L = [-0.0408 - 0.2495 - 0.0446 - 0.0016]^T.$$

Figure 7 shows the responses obtained using the proposed robust adaptive consensus protocol. Figures 7(a), (b), (c) and (d) plot the states trajectories of multi-agent nonlinear robots under external disturbances; while the coupling weights of the communication topology are plotted in Figure 7(e). By application of the proposed consensus protocol (5), all of the respective states of the

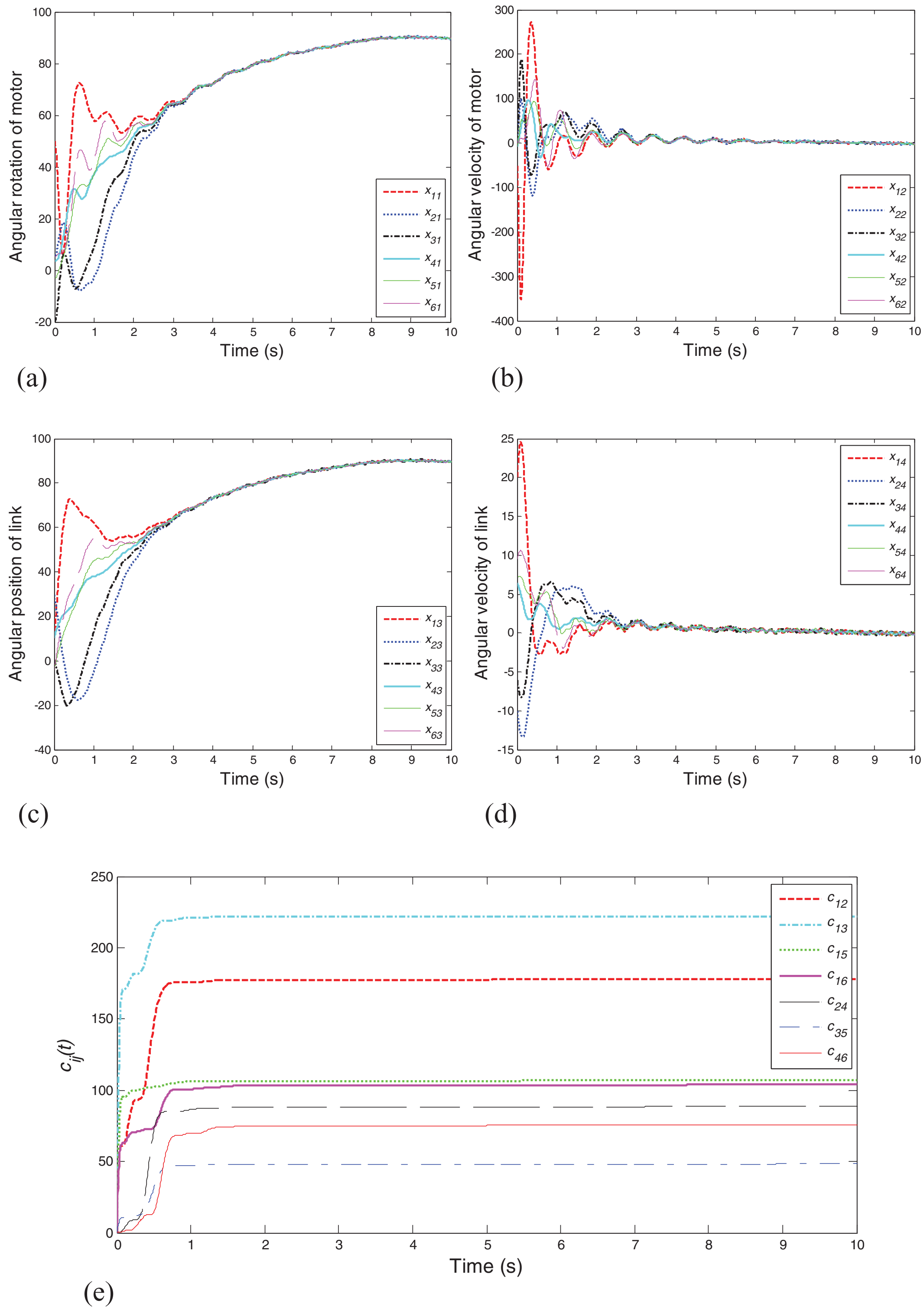

Figure 7. Distributed robust adaptive consensus of six robots under external disturbances: (a) consensus between angular rotations of motors, (b) consensus between angular velocities of motors, (c) consensus between angular positions of links, (d) consensus between angular velocities of links, and (e) adaptation of coupling weights.

multi-agent systems attain to common values. Meanwhile, it is observed that robustness against disturbances also is achieved. Moreover, the adaptive weights are converging to achieve consensus against disturbances and unknown information of the graph topology. Hence, the proposed method developed in Theorem 4.1(a) and 4.1(b) can be effectively utilised for fully distributed robust adaptive consensus protocol design for nonlinear agents under perturbations and unknown information on the connections between agents in the case of an undirected communication graph.

6. Conclusions

In this paper, the distributed adaptive consensus problem for Lipschitz nonlinear multi-agent systems was addressed. Detailed stability analysis for consensus protocol design was carried out for the cases of the absence and presence of external disturbances. Sufficient conditions were derived in the form of LMIs for dynamic adaptive controllers using output feedback to attain consensus by employment of graph theory and decoupling techniques. In contrast to the conventional work, the proposed methodologies consider dynamic protocols, fully distributed controllers due to adaptive weights, and output-feedback-based approaches for consensus control of Lipschitz nonlinear agents. Further, decoupling tactics were efficiently applied to the consensus control problem for straightforward computation of the controller gains. Simulation tests were performed for a network of single-link flexible-joint robots to illustrate the effectiveness of the proposed theoretical results. Future work is obligatory to investigate the consensus control of more complicated nonlinear multi-agent systems containing uncertainties, external disturbances, time-delays and directed or switching communication topologies.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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