ISA Transactions 65 (2016) 230–240

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

Research Article

Observer-based robust control of one-sided Lipschitz nonlinear systems

Sohaira Ahmad [a,1], Muhammad Rehan [a,*,1], Keum-Shik Hong [b,2]

[a] Department of Electrical Engineering, Pakistan Institute of Engineering and Applied Sciences (PIEAS), Islamabad, Pakistan

[b] Department of Cogno-Mechatronics Engineering and School of Mechanical Engineering, Pusan National University, 2 Busandaehak-ro, Geumjeong-gu, Busan 609-735, Republic of Korea

ARTICLE INFO

Article history:
Received 18 November 2015
Received in revised form
18 June 2016
Accepted 13 August 2016
Available online 2 September 2016
This paper was recommended for publication by Y. Chen.

Keywords:
Observer-based control
One-sided Lipschitz nonlinearity
Quadratic inner-boundedness
Parametric uncertainty
L_2 gain

ABSTRACT

This paper presents an observer-based controller design for the class of nonlinear systems with time-varying parametric uncertainties and norm-bounded disturbances. The design methodology, for the less conservative one-sided Lipschitz nonlinear systems, involves astute utilization of Young's inequality and several matrix decompositions. A sufficient condition for simultaneous extraction of observer and controller gains is stipulated by a numerically tractable set of convex optimization conditions. The constraints are handled by a nonlinear iterative cone-complementary linearization method in obtaining gain matrices. Further, an observer-based control technique for one-sided Lipschitz nonlinear systems, robust against L_2-norm-bounded perturbations, is contrived. The proposed methodology ensures robustness against parametric uncertainties and external perturbations. Simulation examples demonstrating the effectiveness of the proposed methodologies are presented.

1. Introduction

A mathematical model accurately represents the dynamics of pragmatic linear and nonlinear systems. Modeling errors and disturbances encountered due to disconcerted or lethargic parameters, data errors, environmental noises, perturbations and system age incur deviation from the original dynamics in state-estimation and control applications. Such uncertainties might lead to control-system instability, lag, mismatches or performance degradation. Consequently, robust stabilization methodologies against uncertainties in linear systems along with techniques for recouping faulty situations have been subjected to substantial experimental trials [1–7]. Still, state-feedback controller designs are crucial for dynamic real-world systems, as direct measurement of all state variables can be very, even prohibitively costly [8–10]. Thus, state-estimation-based optimal controllers have been widely adopted over the past decade for stabilization of deterministic and stochastic linear plants [7,11–13]. Also, for linear systems, observer designs and observer-based control techniques have been proposed to deal with parameter uncertainty and undesirable disturbances acting in real systems [14–18], while the counterpart nonlinear solutions are still immature. In [19], the authors studied robust H_∞ filtering for continuous-time nonlinear systems that incorporates the properties of the Lipschitz continuity under parametric uncertainties. In [20], to attain the robustness feature, the L_2-induced gain from the norm-bounded originating disturbance signals to the state-estimation errors was deduced.

For nonlinear systems, various methodologies for design of robust observers or controllers to handle exogenous disturbances have been explored [21–24]. Major examples include Kalman filtering for Gaussian measurement randomness with cognized statistics [21] as well as the H_∞ approach to arbitrary noise signals with bounded energy for ensuring the noise-evanescing level [22]. Broadly speaking, confining the difficulty of state observers to nonlinear systems, most of the studies have been carried out for particular nonlinearities. A renowned class of nonlinearity that is locally satisfied by numerous physical systems is Lipschitz nonlinearity. Several efficient observer design and state-estimation-based controller synthesis results for Lipschitzian nonlinear systems have been reported [8,25,26]. A specific state-estimation solution for a nonlinear dynamical system is provided in [25], where stability conditions, as in the algebraic Riccati equation, are explicit. In [26], an observer design for nonlinear systems and the H_∞ adaptive invariant set are presented to avoid difficulty in adaptive observer design.

* Corresponding author.
E-mail addresses: sohaahmad64@gmail.com (S. Ahmad), rehanqau@gmail.com (M. Rehan), kshong@pusan.ac.kr (K.-S. Hong).
[1] Fax: +92 51 2208070.
[2] Fax: +82 51 514 0685.

http://dx.doi.org/10.1016/j.isatra.2016.08.010
0019-0578/

Lipschitz continuity is extended to the one-sided Lipschitz condition, a more generic one that incorporates the less specific class of nonlinearities as a special case and also abridges the conservatism in observer designs (see Refs. [27–30]). State-estimation is a less pragmatic problem than observer-based control of one-sided Lipschitz nonlinear systems. Significant work has been done on observer design and observer-based control for continuous-time and discrete-time nonlinear systems in [3,31–38]. The latest work, by contrast, encompasses observer-based stabilization of nonlinear systems in the presence of noise [24,39–42]. H_∞ filtering for singular Lipschitz nonlinear systems has been addressed in [40,41]. In [43], the sufficient condition for robust H_2 fuzzy observer-based control of T–S fuzzy models that ensures the closed-loop stability of fuzzy systems has been investigated. A less conservative linear matrix inequality (LMI)-based H_∞ observer design for one-sided Lipschitz nonlinear systems in the presence of noise is presented in [42]. However, to the best of authors' knowledge, less attention has been paid to the observer-based control of one-sided Lipschitz nonlinear systems with parametric uncertainties and disturbances.

Inspired by the aforementioned intense discussion on parametric uncertainties and one-sided Lipschitz systems, the present work explores observer-based stabilization of one-sided Lipschitz nonlinear systems with parametric uncertainties and disturbances. A Lyapunov functional is adopted, the time derivative of which further involves the one-sided Lipschitz condition and quadratic inner-boundedness condition as observer-based controller design conditions. Further, Young's relation is specifically employed to resolve the bilinear matrix inequality condition for extraction of straightforward observer-based control results and, thus, enable simultaneous computation of the controller and observer gains. On this basis, robust observer-based controller synthesis strategy under parametric uncertainties is rendered against L_2-norm-bounded exogenous disturbances to ensure L_2 gain reduction from the undesired signals to an augmented vector comprising the system's state and state-estimation error. Further, a solution to the nonlinear constraint is provided by solving the optimization problem using cone-complementary linearization approach. An observer-based robust control treatment against perturbations, disturbances and parametric uncertainties has been provided for the first time to the best of our cognizance. Finally, numerical simulation examples demonstrating the effectiveness of the proposed observer-based control scheme are presented.

This paper is organized as follows. Section 2 describes the observer-based robust control system, and Section 3 summarizes its formulation and experimental results. Sections 4 and 5 provide numerical simulation results and the concluding remarks, respectively.

Standard notation is used in this paper. $\|s\|$ denotes the Euclidean norm of a vector s; the L_2 norm of the vector is given by $\|s\|_2 = \sqrt{\int_0^\infty \|s\|^2 dt}$. For the same dimension vectors r and s, the inner product of the vectors is presented as $\langle r,s\rangle$. Moreover, the quantity $\sup_{\|d\|_2 \neq 0}(\|z\|_2/\|d\|_2)$ defines the L_2 gain for a system with an input vector d and output vector z. To denote a symmetric positive (or semi-positive) matrix P, we use the matrix inequality $P>0$ (or $P\geq 0$). The representation $diag(s_1,s_2,\ldots,s_n)$ denotes a block diagonal matrix with entry s_i, for $i=1,2,\ldots,n$, at the corresponding diagonal element. Matrix A^T represents the transpose of A, and the symbol * corresponds to a term to impart symmetry to a symmetric matrix.

2. System description

Consider the continuous-time one-sided Lipschitz uncertain nonlinear system

$$\dot{x}(t) = (A+\Delta A(t))x(t) + f(t,x) + Bu(t) + d(t),$$
$$y(t) = (C+\Delta C(t))x(t), \tag{1}$$

where $x(t)\in\Re^n, y(t)\in\Re^p, d(t)\in\Re^n$ and $u(t)\in\Re^m$ represent the state, output, disturbance and control input, respectively, $f(t,x)$ represents the nonlinear dynamics associated with the state vector with $f(t,0)=0$, and A,B and C correspond to the linear constant matrices of a system of appropriate dimensions. The unknown matrices $\Delta A(t)$ and $\Delta C(t)$ are the time-varying parametric uncertainties accounted as

$$\Delta A(t) = M_1F(t)N_1,$$
$$\Delta C(t) = M_2F(t)N_2, \tag{2}$$

where M_1, M_2, N_1 and N_2 are known real constant matrices and $F(t)$ represents an unknown real-valued matrix function satisfying

$$F^T(t)F(t) \leq I, \quad \forall t\geq 0. \tag{3}$$

In order to apply the prelude condition of one-sided Lipschitz nonlinearity and quadratic inner-boundedness, define a functional region as $\wp \subseteq R^n$, and then establish the definitions (see [29,43] and references therein).

Definition 1. If there exists $\rho\in R\ \forall\ x_1,x_2\in\wp$ such that

$$\langle f(t,x_1)-f(t,x_2), x_1-x_2\rangle \leq \rho\|x_1-x_2\|^2, \tag{4}$$

the known nonlinear function $f(t,x)$ is said to be one-sided Lipschitz, where ρ is the one-sided Lipschitz constant.

Definition 2. If $\forall\ x_1,x_2\in\wp$ there exist $\alpha,\beta\in R$ such that

$$(f(t,x_1)-f(t,x_2))^T(f(t,x_1)-f(t,x_2))$$
$$\leq \beta\|x_1-x_2\|^2 + \alpha\langle x_1-x_2, f(t,x_1)-f(t,x_2)\rangle, \tag{5}$$

the nonlinear function $f(t,x)$ is said to be quadratically inner-bounded in the region $\wp$.

Every nonlinear function $f(t,x)$ is said to conform to the Lipschitz condition and then to satisfy the one-sided Lipschitz and quadratic inner-boundedness conditions, whereas the converse is not true [43]. It is worth adverting that the constants ρ,β and α can constitute zero, negative or positive values unlike the orthodox Lipschitz constant, which adopts the positive value only. Some significant examples in this regard include $-x^3$, which is a locally Lipschitz. This function is globally one-sided Lipschitz with the constant $\rho=0$. Similarly another function $-\mathrm{sgn}(x)\sqrt{|x|}$ is also globally one-sided Lipschitz with one-sided constant $\rho=0$, while this function is not Lipschitz in any domain containing the origin. Note that these functions also satisfy the condition in Definition 2, while the values of α and β depend on the selection of region $\wp$.

Assumption 1. The function $f(t,x)$ fulfills the one-sided Lipschitz and quadratic inner-boundedness conditions in (4) and (5).

Assumption 2. The pairs (A,B) and (A,C) are stabilizable and detectable, respectively.

In order to conclude our results, the Lemma is introduced below.

Lemma 1. *([44]). For any constant $\nu>0$ and known real matrices X,Γ and U of appropriate dimensions, the inequality.*

$$X\,\Gamma(t)U + [X\,\Gamma(t)U]^T \leq \nu^{-1}XX^T + \nu U^TU \tag{6}$$

holds, where $\Gamma(t)$ is a time-varying uncertain matrix fulfilling $\Gamma^T(t)\Gamma(t)\leq I$.

The observer dynamics, for the observer gain $L\in R^{n\times p}$, are in the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(t,\hat{x}) + Bu(t) + L(y(t)-\hat{y}(t)), \tag{7}$$

$$\hat{y}(t) = C\hat{x}(t), \tag{8}$$

where $\hat{x}(t)$ and $\hat{y}(t)$ are the estimated state and output, respectively. Defining the estimation error as $e(t)=x(t)-\hat{x}(t)$, the observer error dynamics become

$$\dot{e}(t) = (A - LC)e(t) + \Phi(x,\hat{x}) + (\Delta A - L\Delta C)x(t) + d(t),$$
$$\Phi(x,\hat{x}) = f(t,x) - f(t,\hat{x}). \tag{9}$$

Involving the controller for the system given by

$$u(t) = -K\hat{x}(t), \tag{10}$$

where $K \in R^{m\times n}$ is the controller gain, the closed-loop system is prototyped as

$$\dot{x}(t) = (A - BK + \Delta A)x(t) + f(t,x) + BKe(t) + d(t). \tag{11}$$

Hence the overall closed-loop system containing the open-loop plant, controller, and observer can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BK + \Delta A & BK \\ \Delta A - L\Delta C & A - LC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} f(t,x) \\ \Phi(x,\hat{x}) \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} d(t). \tag{12}$$

The aim of the present study is to explore observer-based robust control strategies for one-sided Lipschitz nonlinear system (1) under parametric uncertainties and disturbances by computing the appropriate values of the gain matrices K and L.

3. Observer-based control

The proposed observer-based robust control technique for the uncertain one-sided Lipschitz nonlinear systems under disturbances is demonstrated in Fig. 1. The block diagram of Fig. 1 incorporates the complete mechanism of the observer-based control, including the system dynamics followed by the state observer and the controller blocks. The output $y(t)$ of the nonlinear plant is employed for the state estimation, while the nonlinear observer uses the difference of the outputs of the system and observer, that is, $y(t) - \hat{y}(t)$ for estimating the state vector $x(t)$. The estimated state $\hat{x}(t)$, rather than the actual plant state $x(t)$, is applied for the feedback control of the nonlinear system. From the application point of view, there are three advantages of the proposed approach in Fig. 1. First, the requirement of sensors for measurement of the exact state vector of the plant is relaxed for the observer-based control case. Note that measurement of all the states of a system can be impractical in many engineering systems. Second, the proposed observer and controller can be designed by accounting the plant uncertainties and perturbations, which can deal the practical limitation of an imprecise modeling of a plant. Thirdly, the control methodology of Fig. 1 accounts the nonlinear dynamics (specifically a generalized nonlinearity in the form of one-sided Lipschitz function), while the nonlinearities are unavoidable in the practical systems.

From design point of view, there are three difficulties which have been addressed in the present work. First, there is a model

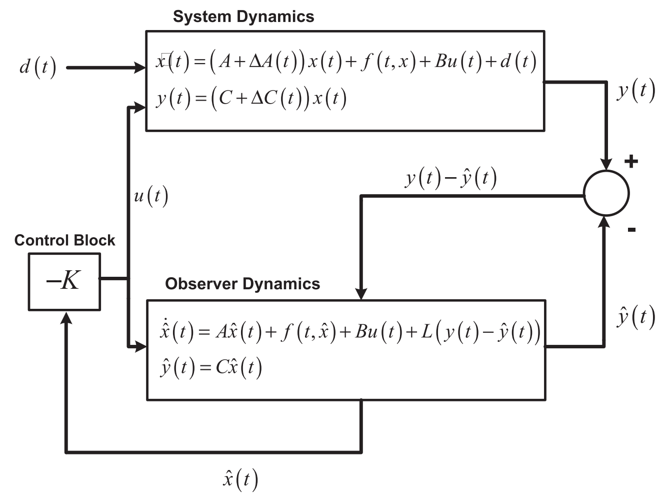

Fig. 1. Block diagram portraying the system dynamics along with the observer and the robust feedback control unit.

mismatch between the dynamics of the plant and observer due to the uncertainties $\Delta A(t)$ and $\Delta C(t)$ and disturbance $d(t)$. In fact, the observer architecture is selected by ignoring the uncertainties and disturbances, and robustness of the proposed observer-based control against these uncertainties and perturbations is a thought-provoking design task. Second, the plant considers a generalized and less conservative form of the nonlinearity described by the one-sided Lipschitz and quadratic inner-boundedness conditions, which are trickier to deal than the conventional Lipschitz counterpart for the observer-based robust control. Thirdly, two gains namely the controller gain K and the observer gain L are needed to be resolved, simultaneously, for ensuring stability of the overall closed-loop system of Fig. 1. It should be noted that existence of a decoupling approach for stability investigation of the estimation error and plant state is questionable for the case of observer-based control of various complex forms of the nonlinear systems.

The following theorem introduces the observer-based stabilization of the one-sided Lipschitz nonlinear systems in the presence of parametric uncertainties.

Theorem 1. *The states of closed-loop system (12), constituted of open-loop plant (1), observer (7) and (8) and controller (10), under Assumptions 1–2, uncertainties (2) and (3) and $d(t) = 0$, converge asymptotically to the origin, if for fixed positive scalars ε_1 and ε_2, there exist two positive definite matrices $\overline{P} \in R^{n\times n}$ and $S \in R^{n\times n}$, matrices $\hat{X} \in R^{n\times m}$ and $\hat{Y} \in R^{p\times n}$, and scalars $\varepsilon_3, \varepsilon_4, \nu_1, \nu_2, \nu_3$* and *$\nu_4$ such that the matrix inequalities.*

$$\overline{P} > 0, \quad S > 0, \quad \varepsilon_i > 0, \quad \nu_j > 0, \quad \forall i = 1,2,3,4, \;\; j = 1,2,3,4, \tag{13}$$

$$\begin{bmatrix}
\tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} & 0 & 0 & \overline{P}M_1 & N_1^T & 0 & \overline{P}N_1^T & 0 & -\overline{P}N_2^T & \overline{P} & B\hat{X}^T & 0 \\
* & -\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \tilde{\Sigma}_{13} & \tilde{\Sigma}_{14} & 0 & 0 & SM_1 & 0 & \hat{Y}M_2 & 0 & 0 & 0 & I \\
* & * & * & -\varepsilon_4 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\nu_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\overline{\nu}_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\nu_2 I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\overline{\nu}_2 I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\nu_3 I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\overline{\nu}_3 I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\kappa_1 I & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -Z_1 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -Z_2
\end{bmatrix} < 0 \tag{14}$$

are satisfied, where

$\sum_{11} = \overline{P}A^T - \hat{X}B^T + A\overline{P} - B\hat{X}^T,$

$\sum_{12} = I + \kappa_2\overline{P},$

$\sum_{13} = \sum_{13} + \kappa_3 I,$

$\sum_{13} = (A-LC)^T S + S(A-LC),$

$\sum_{14} = S + \kappa_4 I,$

$\kappa_1 = \rho\varepsilon_1 + \beta\varepsilon_2,$

$\kappa_2 = -\frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2},$

$\kappa_3 = \rho\varepsilon_3 + \beta\varepsilon_4,$

$\kappa_4 = -\frac{\varepsilon_3 I}{2} + \frac{\alpha\varepsilon_4 I}{2},$

$Z_1 = \frac{\overline{P}}{\nu_4},$

$Z_2 = \nu_4\overline{P},$

$\overline{\nu}_i = \nu_i^{-1}$ for $i = 1, 2, 3, 4.$

The observer-based control gains are given by $K = \hat{X}\overline{P}^{-1}$ *and* $L = S^{-1}\hat{Y}$.

Proof. Consider the Lyapunov function.

$$V(x,e) = x^T P x + e^T S e, P > 0, \quad S > 0. \tag{15}$$

The time-derivative of the function along the trajectory of (12) yields

$$\begin{aligned}\dot{V}(x,e) &= \dot{x}^T P x + x^T P \dot{x} + \dot{e}^T S e + e^T S \dot{e},\\ &= x^T(t)\Big((A - BK + \Delta A)^T P + P(A - BK + \Delta A)\Big)x(t) + d^T(t)Px(t)\\ &\quad + x^T(t)Pd(t) + x^T(t)\Big(\big(\Delta A - L\Delta C\big)^T S + PBK\Big)e(t) + x^T(t)Pf(t,x)\\ &\quad + f^T(t,x)Px(t) + e^T(t)\Big(S\big(\Delta A - L\Delta C\big) + K^T B^T P\Big)x(t) + d^T(t)Se(t)\\ &\quad + \Phi^T(x,\hat{x})Se(t) + e^T(t)\Big((A-LC)^T S + S(A-LC)\Big)e(t)\\ &\quad + e^T(t)Sd(t) + e^T(t)S\Phi(x,\hat{x}).\end{aligned} \tag{16}$$

For $d(t) = 0$, we have

$$\dot{V}(x,e) = \lambda_1^T \sum\nolimits_1 \lambda_1, \tag{17}$$

$$\lambda_1^T = \begin{bmatrix} x^T(t) & f^T(t,x) & e^T(t) & \Phi^T(x,\hat{x}) \end{bmatrix}, \tag{18}$$

$$\sum\nolimits_1 = \begin{bmatrix} \sum_{11} + (\Delta A)^T P + P(\Delta A) & PI & (\Delta A - L\Delta C)^T S + PBK & 0 \\ * & 0 & 0 & 0 \\ * & * & \sum_{13} & SI \\ * & * & * & 0 \end{bmatrix}, \tag{19}$$

where $\sum_{11} = (A-BK)^T P + P(A-BK)$. Inferring from conditions (4) and (5) and introducing scalars $\varepsilon_i > 0$, for $i = 1, 2, 3, 4$, it follows that

$$\lambda_1^T \begin{bmatrix} \rho\varepsilon_1 I & -\frac{\varepsilon_1 I}{2} & 0 & 0 \\ -\frac{\varepsilon_1 I}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \lambda_1 \geq 0, \tag{20}$$

$$\lambda_1^T \begin{bmatrix} \beta\varepsilon_2 I & \frac{\alpha\varepsilon_2 I}{2} & 0 & 0 \\ \frac{\alpha\varepsilon_2 I}{2} & -\varepsilon_2 I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \lambda_1 \geq 0, \tag{21}$$

$$\lambda_1^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho\varepsilon_3 I & -\frac{\varepsilon_3 I}{2} \\ 0 & 0 & -\frac{\varepsilon_3 I}{2} & 0 \end{bmatrix} \lambda_1 \geq 0, \tag{22}$$

$$\lambda_1^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta\varepsilon_4 I & \frac{\alpha\varepsilon_4 I}{2} \\ 0 & 0 & \frac{\alpha\varepsilon_4 I}{2} & -\varepsilon_4 I \end{bmatrix} \lambda_1 \geq 0. \tag{23}$$

Incorporating (17) and (20)–(23), and further segregating the matrix for uncertain and known terms, yields

$$\sum\nolimits_1 = \begin{bmatrix} \sum_{11} + \kappa_1 I & PI + \kappa_2 I & PBK & 0 \\ * & -\varepsilon_2 I & 0 & 0 \\ * & * & \sum_{13} + \kappa_3 I & S + \kappa_4 I \\ * & * & * & -\varepsilon_4 I \end{bmatrix} + \begin{bmatrix} \Delta A^T P + P\Delta A & 0 & (\Delta A - L\Delta C)^T S & 0 \\ 0 & 0 & 0 & 0 \\ S(\Delta A - L\Delta C) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} < 0. \tag{24}$$

Pre- and post-multiplying $diag(\overline{P}, I, I, I)$ for $\overline{P} = P^{-1}$ with (24) results in

$$\tilde{\sum}_1 = \begin{bmatrix} \sum_{11} + \kappa_1 \overline{P}\overline{P} & \tilde{\sum}_{12} & BK & 0 \\ * & -\varepsilon_2 I & 0 & 0 \\ * & * & \tilde{\sum}_{13} & \tilde{\sum}_{14} \\ * & * & * & -\varepsilon_4 I \end{bmatrix} + \begin{bmatrix} \overline{P}\Delta A^T + \Delta A\overline{P} & 0 & \overline{P}(\Delta A - L\Delta C)^T S & 0 \\ 0 & 0 & 0 & 0 \\ S(\Delta A - L\Delta C)\overline{P} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} < 0, \tag{25}$$

$$\tilde{\sum}_{11} = \overline{P}A^T - \hat{X}B^T + A\overline{P} - B\hat{X}^T,$$

which further implies

$$\begin{aligned}\tilde{\sum}_1 &= \overline{\sum}_1 + \begin{bmatrix} \overline{P}M_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} F\begin{bmatrix} N_1 & 0 & 0 & 0 \end{bmatrix} + \left\{ \begin{bmatrix} \overline{P}M_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} F\begin{bmatrix} N_1 & 0 & 0 & 0 \end{bmatrix} \right\}^T \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ SM_1 \\ 0 \end{bmatrix} F\begin{bmatrix} N_1\overline{P} & 0 & 0 & 0 \end{bmatrix} + \left\{ \begin{bmatrix} 0 \\ 0 \\ SM_1 \\ 0 \end{bmatrix} F\begin{bmatrix} N_1\overline{P} & 0 & 0 & 0 \end{bmatrix} \right\}^T \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ SLM_2 \\ 0 \end{bmatrix} F\begin{bmatrix} -N_2\overline{P} & 0 & 0 & 0 \end{bmatrix} \\ &\quad + \left\{ \begin{bmatrix} 0 \\ 0 \\ SLM_2 \\ 0 \end{bmatrix} F\begin{bmatrix} -N_2\overline{P} & 0 & 0 & 0 \end{bmatrix} \right\}^T\end{aligned}$$

$$+\kappa_1\begin{bmatrix}\overline{P}\\0\\0\\0\end{bmatrix}\begin{bmatrix}\overline{P}\\0\\0\\0\end{bmatrix}^T+\begin{bmatrix}BK\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\I\\0\end{bmatrix}^T+\begin{bmatrix}0\\0\\I\\0\end{bmatrix}\begin{bmatrix}BK\\0\\0\\0\end{bmatrix}^T<0, \tag{26}$$

$$\overline{\sum}_1=\begin{bmatrix}\tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} & BK & 0\\ * & -\varepsilon_2 I & 0 & 0\\ * & * & \tilde{\Sigma}_{13} & \tilde{\Sigma}_{14}\\ * & * & * & -\varepsilon_4 I\end{bmatrix}. \tag{27}$$

Following Lemma 1 and Young's relation employed in [45], defining $\hat{Y}^T=L^TS$ and incorporating the scalars ν_1,ν_2,ν_3 and ν_4, we have

$$\tilde{\sum}_1\le\overline{\sum}_1+\frac{1}{\nu_1}\begin{bmatrix}\overline{P}M_1\\0\\0\\0\end{bmatrix}\begin{bmatrix}\overline{P}M_1\\0\\0\\0\end{bmatrix}^T+\frac{1}{\nu_2}\begin{bmatrix}0\\0\\SM_1\\0\end{bmatrix}\begin{bmatrix}0\\0\\SM_1\\0\end{bmatrix}^T$$

$$+\nu_1\begin{bmatrix}N_1 & 0 & 0 & 0\end{bmatrix}^T\begin{bmatrix}N_1 & 0 & 0 & 0\end{bmatrix}$$
$$+\nu_2\begin{bmatrix}N_1\overline{P} & 0 & 0 & 0\end{bmatrix}^T\begin{bmatrix}N_1\overline{P} & 0 & 0 & 0\end{bmatrix}$$
$$+\nu_3\begin{bmatrix}-N_2\overline{P} & 0 & 0 & 0\end{bmatrix}^T\begin{bmatrix}-N_2\overline{P} & 0 & 0 & 0\end{bmatrix}$$
$$+\frac{1}{\nu_3}\begin{bmatrix}0\\0\\\hat{Y}M_2\\0\end{bmatrix}\begin{bmatrix}0\\0\\\hat{Y}M_2\\0\end{bmatrix}^T+\kappa_1\begin{bmatrix}\overline{P}\\0\\0\\0\end{bmatrix}\begin{bmatrix}\overline{P}\\0\\0\\0\end{bmatrix}^T+\begin{bmatrix}BK\\0\\0\\0\end{bmatrix}\nu_4\overline{P}\begin{bmatrix}BK\\0\\0\\0\end{bmatrix}^T$$
$$+\begin{bmatrix}0\\0\\I\\0\end{bmatrix}\left(\frac{\overline{P}^{-1}}{\nu_4}\right)\begin{bmatrix}0\\0\\I\\0\end{bmatrix}^T<0. \tag{28}$$

Encouraging transformations and rearranging as

$$\tilde{\sum}_1\le\overline{\sum}_1-\left(-\frac{1}{\nu_1}\right)\begin{bmatrix}\overline{P}M_1\\0\\0\\0\end{bmatrix}\begin{bmatrix}\overline{P}M_1\\0\\0\\0\end{bmatrix}^T-\left(-\frac{1}{\nu_2}\right)\begin{bmatrix}0\\0\\SM_1\\0\end{bmatrix}\begin{bmatrix}0\\0\\SM_1\\0\end{bmatrix}^T$$
$$-\left(-\frac{1}{\nu_3}\right)\begin{bmatrix}0\\0\\\hat{Y}M_2\\0\end{bmatrix}\begin{bmatrix}0\\0\\\hat{Y}M_2\\0\end{bmatrix}^T$$
$$-\begin{bmatrix}\nu_1N_1 & 0 & 0 & 0\end{bmatrix}^T(-\nu_1)^{-1}\begin{bmatrix}\nu_1N_1 & 0 & 0 & 0\end{bmatrix}$$
$$-\begin{bmatrix}\nu_2N_1\overline{P} & 0 & 0 & 0\end{bmatrix}^T(-\nu_2)^{-1}\begin{bmatrix}\nu_2N_1\overline{P} & 0 & 0 & 0\end{bmatrix}$$
$$-\begin{bmatrix}-\nu_3N_2\overline{P} & 0 & 0 & 0\end{bmatrix}^T(-\nu_3)^{-1}\begin{bmatrix}-\nu_3N_2\overline{P} & 0 & 0 & 0\end{bmatrix}$$
$$-(-\kappa_1)\begin{bmatrix}\overline{P}\\0\\0\\0\end{bmatrix}\begin{bmatrix}\overline{P}\\0\\0\\0\end{bmatrix}^T-\begin{bmatrix}B\hat{X}^T\\0\\0\\0\end{bmatrix}\left(-\nu_4\overline{P}^{-1}\right)\begin{bmatrix}B\hat{X}^T\\0\\0\\0\end{bmatrix}^T$$
$$-\begin{bmatrix}0\\0\\I\\0\end{bmatrix}\left(\frac{-\overline{P}^{-1}}{\nu_4}\right)\begin{bmatrix}0\\0\\I\\0\end{bmatrix}^T<0, \tag{29}$$

which, after applying the Shur complement and substituting $Z_1=\overline{P}/\nu_4$ and $Z_2=\nu_4\overline{P}$, yields

$$\begin{bmatrix}
\tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} & 0 & 0 & \overline{P}M_1 & \nu_1N_1^T & 0 & \nu_2\overline{P}N_1^T & 0 & -\nu_3\overline{P}N_2^T & \overline{P} & B\hat{X}^T & 0\\
* & -\varepsilon_2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & \tilde{\Sigma}_{13} & \tilde{\Sigma}_{14} & 0 & 0 & SM_1 & 0 & \hat{Y}M_2 & 0 & 0 & 0 & I\\
* & * & * & -\varepsilon_4I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -\nu_1I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & -\nu_1I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -\nu_2I & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & -\nu_2I & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -\nu_3I & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -\nu_3I & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & -\kappa_1I & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & -Z_1 & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & -Z_2
\end{bmatrix}<0 \tag{30}$$

By application of the congruence transform $diag(I,I,I,I,\nu_1^{-1},I,\nu_2^{-1},I,\nu_3^{-1},I,I,I,I)$, the constraint (30) results in (14). This completes the proof of Theorem 1. □

Remark 1. Linear and nonlinear state-feedback control approaches for different forms of one-sided Lipschitz nonlinear system have been investigated in recent works [46–48]. These techniques cannot be applied to the stabilization of nonlinear systems if the full state vector is not known. The present work, contrastingly, employs observers for state-estimation of nonlinear systems, the proposed control strategy utilizing the estimated states rather than the true values. A stability condition for the overall closed-loop system is derived that guarantees simultaneous stability of the system's state vector and the estimation error.

Remark 2. Recently, observer-based control strategies for one-sided Lipschitz nonlinear systems have been explored for the discrete- and continuous-time domains, respectively [49,50]. These control strategies, however, cannot be employed for stabilization of nonlinear systems under unknown parametric variations and uncertainties. The present technique introduced in Theorem 1 fills this research gap. Note that incorporation of the parametric uncertainties for observer-based control is a non-trivial research problem, because these uncertainties appear in both the system and the estimation error dynamics (as seen in (12)) and requires the tedious uncertainty separation and handling tactics employed in (24).

Remark 3. Lately, an effective finite-time robust state-feedback controller design scheme has been developed for one-sided Lipschitz nonlinear systems with uncertain parameters [51] and, further, has been extended to output feedback stabilization of continuous-time nonlinear systems. Whereas the controller using output feedback [51] can be treated as a static controller, the proposed observer-based control is a dynamic controller. Dynamic controllers, difficult to design owing to computation of the multiple gains (like K and L), can be employed to attain multiple design objectives. Further, the output feedback approach in [51] is restrictive, as it is based on the conventional state-feedback control strategy. This approach demands an invertible matrix B^TB for computation of the controller gain matrix, while the proposed scheme determines the controller gain matrix without having to involve such a constraint.

The design procedure for observer-based control outlined in Theorem 1 does not incorporate external noises and exogenous disturbances, because they can result in small perturbations in the estimated state, which can cause the system parameters to drift into instability. Consequently, in the worst case, the performance of the observer-based controller can be affected, or the state-estimation error and system's state vector might even diverge to infinity. Defining an augmented vector $z(t)=\left[x^T(t) \quad e^T(t)\right]^T$, the results for the L_2 gain reduction from detrimental signal $d(t)$ to vector $z(t)$ are proposed in the following theorem to ensure robustness against perturbations.

Theorem 2. *Consider the closed-loop system (12), constituted of open-loop plant (1), observer (7) and (8) and controller (10), under Assumptions 1–2 and uncertainties (2) and (3). Suppose that for fixed positive scalars ε_1 and ε_2, there exist two positive definite matrices $\overline{P} \in R^{n\times n}$ and $S \in R^{n\times n}$, matrices $\hat{X} \in R^{n\times m}$ and $\hat{Y} \in R^{p\times n}$, and scalars ε_3, ε_4, ν_1, ν_2, ν_3, ν_4, and γ, such that the matrix inequalities.*

$$\overline{P}>0, S>0, \gamma>0, \varepsilon_i>0, \nu_j>0, \quad \forall i=1,2,3,4, j=1,2,3,4, \tag{31}$$

$$\begin{bmatrix} \overline{\sum}_1 & \Theta_1 \\ * & \Theta_2 \end{bmatrix} < 0, \tag{32}$$

$$\Theta_1 = \begin{bmatrix} \overline{P} & \overline{P}M_1 & N_1^T & 0 & \overline{P}N_1^T & 0 & -\overline{P}N_2^T & \overline{P} & B\hat{X}^T & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S & 0 & 0 & SM_1 & 0 & \hat{Y}M_2 & 0 & 0 & 0 & I & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Theta_2 = diag(-\gamma I, -\nu_1 I, -\overline{\nu}_1 I, -\nu_2 I, -\overline{\nu}_2 I, -\nu_3 I, -\overline{\nu}_3 I, -\kappa_1 I, -Z_1, -Z_2, -\gamma I - \gamma I).$$

are feasible. The observer-based control gains given by $K=\hat{X}\overline{P}^{-1}$ and $L=S^{-1}\hat{Y}$ ensure that the vector $z(t)$ converges to the origin for $d(t)=0$,and the L_2 gain of $z(t)$ with respect to disturbance $d(t)$ is less than γ, for $d(t)\neq 0$.

Proof. Choosing Lyapunov function (15) and incorporating the inequality.

$$J(z,d)=\dot{V}(z,t)+\gamma^{-1}z^T(t)z(t)-\gamma d^T(t)d(t)<0 \tag{33}$$

for L_2 gain reduction entails

$$\int_0^t \dot{V}(z,t)dt+\int_0^t \left(\gamma^{-1}z^T(t)z(t)\right) dt-\int_0^t \left(\gamma d^T(t)d(t)\right) dt<0,$$

$$V(z,t)-V(z,0)+\int_0^t \left(\gamma^{-1}z^T(t)z(t)\right) dt-\int_0^t \left(\gamma d^T(t)d(t)\right) dt<0.$$

It furthers yields

$$\sqrt{\int_0^t \left(\gamma^{-1}z^T(t)z(t)\right) dt}<\gamma\sqrt{\int_0^t \left(d^T(t)d(t)\right) dt},$$

which is to say, $\|z(t)\|_2<\gamma\|d(t)\|_2$. Under $d(t)=0$, (33) guarantees $\dot{V}(z,t)<0$; that is, convergence of $z(t)$ to the origin. Incorporating (16) into (33), employing one-sided Lipschitz condition and applying the quadratic inner-boundedness condition, we obtain $J(e,x,d)=\bar{\lambda}_2{}^T\sum_2\bar{\lambda}_2$, where

$$\bar{\lambda}_2{}^T=\left[x^T(t) \quad f^T(t,x) \quad e^T(t) \quad \Phi^T(x,\hat{x}) \quad d^T(t)\right], \tag{34}$$

$$\sum_2=\begin{bmatrix} \tilde{\sum}_{11}+\gamma^{-1}I+\kappa\overline{P}\overline{P} & I+\kappa_2\overline{P} & 0 & 0 & \overline{P} \\ * & -\varepsilon_2 I & 0 & 0 & 0 \\ * & * & \sum_{13}+\kappa_3 I+\gamma^{-1}I & SI+\kappa_4 I & S \\ * & * & * & -\varepsilon_4 I & 0 \\ * & * & * & * & -\gamma I \end{bmatrix}$$

$$+\begin{bmatrix} \overline{P}\Delta A^T+\Delta A\overline{P} & 0 & \overline{P}(\Delta A-L\Delta C)^T S & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ S(\Delta A-L\Delta C)\overline{P} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}<0. \tag{35}$$

Substituting the uncertainties ΔA and ΔC according to (6), taking $\sum_2<0$, and, further, dealing with the bilinear terms using Lemma 1 and Young's relation, yields

$$\tilde{\sum}_2 \le \overline{\sum}_2+\frac{1}{\nu_1}\begin{bmatrix} \overline{P}M_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \overline{P}M_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T+\frac{1}{\nu_2}\begin{bmatrix} 0 \\ 0 \\ SM_1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ SM_1 \\ 0 \\ 0 \end{bmatrix}^T$$

$$+\nu_1\left[N_1 \quad 0 \quad 0 \quad 0 \quad 0\right]^T\left[N_1 \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$$+\nu_2\left[N_1\overline{P} \quad 0 \quad 0 \quad 0 \quad 0\right]^T\left[N_1\overline{P} \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$$+\nu_3\left[-N_2\overline{P} \quad 0 \quad 0 \quad 0 \quad 0\right]^T\left[-N_2\overline{P} \quad 0 \quad 0 \quad 0 \quad 0\right]$$

$$+\frac{1}{\nu_3}\begin{bmatrix} 0 \\ 0 \\ \hat{Y}M_2 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \hat{Y}M_2 \\ 0 \\ 0 \end{bmatrix}^T+\kappa_1\begin{bmatrix} \overline{P} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \overline{P} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T+\begin{bmatrix} BK \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\nu_4\overline{P}\begin{bmatrix} BK \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$+\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix}\left(\frac{\overline{P}^{-1}}{\nu_4}\right)\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix}^T<0. \tag{36}$$

$$\overline{\sum}_2=\begin{bmatrix} \tilde{\sum}_{11}+\gamma^{-1}I & I+\kappa_2\overline{P} & 0 & 0 & \overline{P} \\ * & -\varepsilon_2 I & 0 & 0 & 0 \\ * & * & \sum_{13}+\kappa_3 I+\gamma^{-1}I & SI+\kappa_4 I & S \\ * & * & * & -\varepsilon_4 I & 0 \\ * & * & * & * & -\gamma I \end{bmatrix}.$$

Employing the Shur complement results in

$$\sum_2=\begin{bmatrix} \tilde{\Pi}_1 & \tilde{\Pi}_2 \\ * & \tilde{\Pi}_4 \end{bmatrix}<0, \tag{37}$$

$$\tilde{\Pi}_1 = \begin{bmatrix} \tilde{\sum}_{11} + \gamma^{-1} I & I + \kappa_2 \overline{P} & 0 & 0 \\ * & -\varepsilon_2 I & 0 & 0 \\ * & * & \sum_{13} + \kappa_3 I + \gamma^{-1} I & SI + \kappa_4 I \\ * & * & * & -\varepsilon_4 I \end{bmatrix},$$

$$\tilde{\Pi}_2 = \begin{bmatrix} \overline{P} & \overline{P}M_1 & N_1^T & 0 & \overline{P}N_1^T & 0 & -\overline{P}N_2^T & \overline{P} & B\hat{X}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S & 0 & 0 & SM_1 & 0 & \hat{Y}M_2 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\Pi}_4 = diag(-\gamma I, -\nu_1 I, -\overline{\nu}_1 I, -\nu_2 I, -\overline{\nu}_2 I, -\nu_3 I, -\overline{\nu}_3 I, -\kappa_1 I, -\overline{P}/\nu_4, -\nu_4 \overline{P}).$$

Using the congruence transformation for inequality (37) by means of $diag(I, I, I, I, I, \nu_1^{-1}, I, \nu_2^{-1}, I, \nu_3^{-1}, I, I, I, I, I, I)$ and substituting $Z_1 = \overline{P}/\nu_4$ and $Z_2 = \nu_4 \overline{P}$ yields (32), which completes the proof of Theorem 2.

Remark 4. Compared with Theorem 1, the controller design proposed in Theorem 2, intriguingly, is robust against disturbances and (thus) reduced instability in the presence of perturbations. This design, in contrast to the existing methodologies [46–51], introduces a robust observer-based control scheme to handle the parametric uncertainties and L_2-norm-bounded disturbances. Positive results for observer-based control of uncertain one-sided Lipschitz nonlinear systems under parametric variations and disturbances are elusive to the efforts to achieve robust design, which makes the proposed method of stabilization viable. Moreover, the present work employs one-sided Lipschitz quadratic inner-boundedness conditions efficiently for nonlinear dynamical processes in the perturbation environment.

The constraints in Theorems 1 and 2 integrate the nonlinear terms in $\xi = diag(-\nu_1 I, -\overline{\nu}_1 I, -\nu_2 I, -\overline{\nu}_2 I, -\nu_3 I, -\overline{\nu}_3 I, -\overline{P}/\nu_4, -\nu_4 \overline{P})$, which are further worked out by engaging the cone-complementary linearization approach (see [29] and references therein). These terms introduce difficulty into the process of determining the observer and controller gain matrices, and so the diagonal structure can be replaced with $\xi = diag(-\nu_1 I, -\overline{\nu}_1 I, -\nu_2 I, -\overline{\nu}_2 I, -\nu_3 I, -\overline{\nu}_3 I, -Z_1, -Z_2)$. Further, we assign $\mu = \sqrt{\nu_4}$ and $\overline{\mu} = \sqrt{\nu_4^{-1}} = \sqrt{\overline{\nu}_4}$, which implies that $\nu_4 = \mu^2$ and $\overline{\nu}_4 = \overline{\mu}^2$. The entities in the matrix inequalities are additionally manipulated as $\overline{P}/\nu_4 = \overline{\mu}\overline{P}\overline{\mu} = Z_1$ and $\nu_4 \overline{P} = \mu \overline{P} \mu = Z_2$. The original nonlinear constraints can be solved by minimizing $Trace(\overline{\nu}_1 \nu_1 I + \overline{\nu}_2 \nu_2 I + \overline{\nu}_3 \nu_3 I + \overline{\nu}_4 \nu_4 I + Z_1 \overline{Z}_1 + Z_2 \overline{Z}_2 + \overline{P}P + \overline{\mu}\overline{P}\overline{\mu}\overline{Z}_1 + \mu \overline{P} \mu \overline{Z}_2 + \mu\overline{\mu} I + \mu P \mu \overline{Z}_1 + \overline{\mu} P \overline{\mu} \overline{Z}_2)$ subject to constraints in Theorem 1 or 2 and the following matrix inequalities (as observed in [29])

$$\begin{bmatrix} \nu_j & I \\ * & \overline{\nu}_j \end{bmatrix} \geq 0, \quad \begin{bmatrix} \overline{P} & I \\ * & P \end{bmatrix} \geq 0, \quad \begin{bmatrix} \mu & I \\ * & \overline{\mu} \end{bmatrix} \geq 0, \tag{38}$$

$$\begin{bmatrix} Z_i & I \\ * & \overline{Z}_i \end{bmatrix} \geq 0, \tag{39}$$

where $\overline{\nu}_j, P, \overline{\mu}$ and $\overline{Z}_i$ are engaged to correspond to the inverses of $\nu_j, \overline{P}, \mu$ and Z_i, respectively, for $i = 1, 2$ and $j = 1, 2, 3, 4$. Subsequently, as $Z_1 = \overline{\mu}\overline{P}\overline{\mu}$ and $Z_2 = \mu \overline{P} \mu$, from (39), further results are deduced as

$$\begin{bmatrix} \overline{\mu}\overline{P}\overline{\mu} & I \\ * & \overline{Z}_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \mu \overline{P} \mu & I \\ * & \overline{Z}_2 \end{bmatrix} \geq 0. \tag{40}$$

Applying congruence transformation by $diag(\mu, I)$ and $diag(\overline{\mu}, I)$ to the first and the second inequalities in (40), respectively, the inequalities

$$\begin{bmatrix} \overline{P} & \mu \\ * & \overline{Z}_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \overline{P} & \overline{\mu} \\ * & \overline{Z}_2 \end{bmatrix} \geq 0 \tag{41}$$

are obtained. Similarly, for $\overline{Z}_1 = \mu P \mu$ and $\overline{Z}_2 = \overline{\mu} P \overline{\mu}$, we have

$$\begin{bmatrix} P & \overline{\mu} \\ * & Z_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} P & \mu \\ * & Z_2 \end{bmatrix} \geq 0. \tag{42}$$

Therefore, a more proper optimization problem for solving the conditions in Theorems 1 and 2 and ensuring $Z_1 = \overline{\mu}\overline{P}\overline{\mu}$, $Z_2 = \mu \overline{P} \mu$, $\overline{Z}_1 = \mu P \mu$ and $\overline{Z}_2 = \overline{\mu} P \overline{\mu}$ is given by

$$\begin{cases} \min \quad Trace\left(\sum_{j=1}^{4} (\overline{\nu}_j \nu_j I) + \sum_{i=1}^{2} (0.5 Z_i \overline{Z}_i + 0.5 \overline{\mu}\overline{P}\overline{\mu}\overline{Z}_i + 0.5 \mu \overline{P} \mu \overline{Z}_i) + \overline{P}P + \mu\overline{\mu} I \right), \\ \text{subject to (38), (39), (41), (42) and the inequalities in} \\ \text{Theorem 1 or 2.} \end{cases} \tag{43}$$

This nonlinear optimization problem can be solved using a recursive cone-complementary linearization algorithm and LMI-tools.

Remark 5. A comprehensive study of parameter optimization is presented in [3], yet there is an evolved cumbersome set of scalars needed in the selection, as observed in the inequalities of Theorem 1 or 2. The existing methods (for instance see [46]) provide approximate results for state feedback, but the scalars excerption is not addressed. In the proposed method, for application of the cone-complementary linearization technique, the scalars $\mu = \sqrt{\nu_4}$ and $\overline{\mu} = \sqrt{\overline{\nu}_4}$ are employed to handle the nonlinear terms $\overline{P}/\nu_4$ and $\nu_4 \overline{P}$, respectively, and the inequalities in Theorems 1 and 2 are derived by using scalars ν_i and their inverses $\overline{\nu}_i$ for $i = 1, 2, 3, 4$. It is worth noting that the constraints in Theorems 1 and 2 containing several variables to ensure robustness against parametric uncertainties and disturbances can be straightforwardly solved via the cone-complementary linearization algorithm for selection of only two scalars ε_1 and ε_2. Also note that selection of these parameters is required for control of one-sided Lipschitz nonlinear systems [47–50], even in the absence of disturbances and parametric variations.

The conditions given by (14), (31) and (32) are nonlinear, which can be converted into linear constrains with nonlinear optimization as seen in (43). The resultant inequalities can be solved using cone-complementary linearization algorithm and LMI-tools. Intensive research for solving the control problems developed LMI solving algorithms and tools, which dominantly adopt the semi-definitive programming to provide a solution focusing the local convergence rate, computations, and worst-case complexity. The linear and nonlinear convex problems are solved by various programming algorithms, which render direct solutions of the variables involved in the problem. Different algorithms are adopted for obtainment of the solution, which include linear programming, quadratic programming, linear complimentary programming, and interior point methods [52]. Smaller to medium-sized problems are easily handled from any of these algorithms; while the major concerns arise for the larger and complex problems. The interior point methods have shown rapid solutions for the complex constraints like (14), (31) and (32). However, the selection of algorithm plays a vital role in this regard. Among the interior point algorithms, primal-dual algorithms are the most useful in this class.

4. Simulation results

In this section the proposed methodologies are demonstrated using two practical examples.

Example 1. Consider a moving object in Cartesian co-ordinates ([53,54]) presented as

$$f(t,x) = -(x_1^2+x_2^2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$
$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T. \tag{44}$$

The uncertainty matrices are taken to be

$$M_1 = \begin{bmatrix} 0.3 & 0.03 \end{bmatrix}^T,\ M_2 = 0.1,$$
$$N_1 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, N_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}. \tag{45}$$

By solving optimization problem (43) and by the appropriate choice of the one-sided Lipschitz constant and quadratic inner-roundedness constants, feasible results are obtained. The convex optimization problem incorporates iterations that, further, by minimization of the objective function, deduce the appropriate set of LMI variables. For $\rho = 0$, $\alpha = -0.3$, $\beta = 0.1$, and $\varepsilon_1 = \varepsilon_2 = 1$, the resultant LMI variables in the first iteration are obtained as

$$\overline{P} = \begin{bmatrix} 3.191 & 0.004 \\ 0.004 & 4.858 \end{bmatrix},\ S = \begin{bmatrix} 162.032 & -53.281 \\ -53.281 & 21.471 \end{bmatrix}, \tag{46}$$

by solving the constraints in Theorem 1. Further, after eight iterations of the convex optimization problem, the results for the LMI variables and nominal gains for the observer and controller are improved as

$$\overline{P} = \begin{bmatrix} 10.675 & 0.133 \\ 0.133 & 47.084 \end{bmatrix},\ S = \begin{bmatrix} 17.744 & -4.050 \\ -4.050 & 0.960 \end{bmatrix}, \tag{47}$$

$$L = \begin{bmatrix} 151.961 \\ 640.790 \end{bmatrix},\ K = \begin{bmatrix} 14.240 & 0.608 \\ 2.657 & 16.268 \end{bmatrix}. \tag{48}$$

Fig. 2(a) plots the simulation results depicting the asymptotic convergence of the original states x_1 and x_2 to the origin in the absence of disturbances, while Fig. 2(b) demonstrates the fast convergence of the estimation errors to the origin.

Further solving optimization problem (43) for Theorem 2 using the one-sided Lipschitz constant and quadratic inner-boundedness constants in the presence of unknown disturbances; we obtain, in the first iteration, the LMI variables

$$\overline{P} = \begin{bmatrix} 2.150 & -1.085 \\ -1.085 & 114.114 \end{bmatrix},\ S = 10^4 \times \begin{bmatrix} 2.6082 & -0.0002 \\ -0.0002 & 0.0000 \end{bmatrix}. \tag{49}$$

Proceeding with additional iterations, a set of feasible LMI variables and, consequently improved observer and controller gains are obtained as

$$\overline{P} = \begin{bmatrix} 111.371 & -6.913 \\ -6.913 & 139.889 \end{bmatrix},\ S = \begin{bmatrix} 17.019 & -4.360 \\ -4.360 & 1.356 \end{bmatrix}, \tag{50}$$

$$L = \begin{bmatrix} 33.480 \\ 107.586 \end{bmatrix},\ K = \begin{bmatrix} 100.049 & 0.0004 \\ 0.0019 & 100.072 \end{bmatrix}, \tag{51}$$

for $\gamma = 0.001$.

In order to test the obtained solution the unknown exogenous disturbance is supposed as

$$d = \begin{bmatrix} 2.5 \sin 100t \\ 7.5 \cos 80t \end{bmatrix}. \tag{52}$$

Simulation results for the states and estimation errors against disturbances are plotted in Fig. 3. The perturbations caused by the exogenous disturbances and parametric uncertainties, it can be seen, are efficiently overtaken. Hence, the proposed methodologies can be employed in an effective robust observer-based control scheme for uncertain and perturbed one-sided Lipschitz nonlinear systems.

Example 2. Consider a nonlinear dynamical model of a flexible link robot (as seen in [8,55]). The flexible joint brings non-linearities which are modeled as a stiffening torsional spring under the effect of gravitational force as shown in Fig. 4. By introducing (θ_m, θ_l) and (ω_m, ω_l) as the position and velocities, respectively, for the motor with subscript m and for the link of robot with subscript l, the dynamic of the robotic system is presented by four differential equations, given by

$$\begin{aligned} \dot{\theta}_m &= \omega_m, \\ \dot{\omega}_m &= \tfrac{\tau}{J_m}(\theta_l - \theta_m) - \tfrac{b}{J_m}\omega_m + \tfrac{K_\tau}{J_m}u, \\ \dot{\theta}_l &= \omega_l, \\ \dot{\omega}_l &= -\tfrac{\tau}{J_l}(\theta_l - \theta_m) - \tfrac{Mgh}{J_l}\sin(\theta_l), \end{aligned} \tag{53}$$

where $2h$ and M are the length and mass of the link, J_m and J_l are the motor and link inertia, K_τ denotes the amplifier gain and the

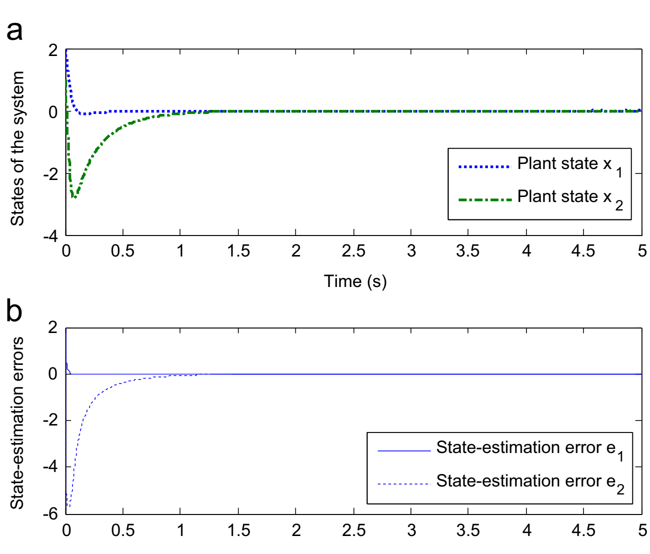

Fig. 2. States and estimation errors converging to the origin under parametric uncertainties: (a) states of the system and (b) state-estimation errors.

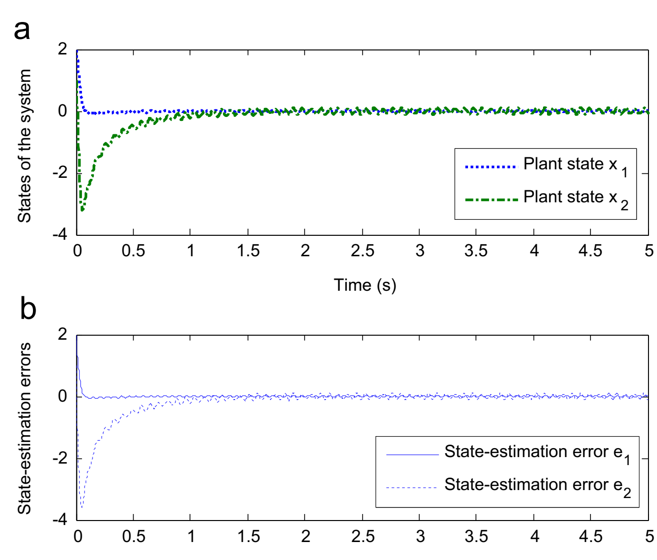

Fig. 3. Convergence of states and estimation errors under parametric uncertainties and disturbance: (a) states of the system and (b) state-estimation errors.

viscous friction is represented as b. From the values of positive constants provided in Table 1, the robotic system in (53) can be reformulated in the form of (1) by selecting

$$f(t,x)=\begin{bmatrix}0\\0\\0\\-3.33\sin x_3\end{bmatrix}, A=\begin{bmatrix}0 & 1 & 0 & 0\\-48.6 & -1.25 & 48.6 & 0\\0 & 0 & 0 & 1\\19.5 & 0 & -19.5 & 0\end{bmatrix}, \tag{54}$$

$$B=\begin{bmatrix}0\\21.6\\0\\0\end{bmatrix}, C=\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}. \tag{55}$$

From (2) and (3), the uncertainty matrices for the above system are taken to be

$$M_1=\begin{bmatrix}0.01 & 0.01 & 0.01 & 0.01\end{bmatrix}^T, M_2=\begin{bmatrix}0.01 & 0.01\end{bmatrix}^T,$$
$$N_1=\begin{bmatrix}0.01 & 0.01 & 0.01 & 0.01\end{bmatrix}, N_2=\begin{bmatrix}0.1 & 0.1 & 0.1 & 0.1\end{bmatrix}.$$

By involving the optimization in (43) and iteratively solving the convex problem for Theorem 1, a feasible set of variables and the observer and controller gains are obtained as

$$\overline{P}=\begin{bmatrix}3.752 & -6.509 & 2.983 & -4.192\\-6.509 & 519.82 & -0.281 & -0.737\\2.983 & -0.281 & 3.560 & -2.489\\-4.192 & -0.737 & -2.489 & 19.763\end{bmatrix},$$

$$S=10^5\times\begin{bmatrix}2.677 & -2.406 & -0.291 & 0.021\\-2.406 & 4.712 & -2.224 & 0.021\\-0.291 & -2.224 & 2.467 & -0.059\\0.021 & 0.021 & -0.059 & 0.057\end{bmatrix},$$

$$L=\begin{bmatrix}744.142 & -726.248\\589.861 & -572.132\\620.292 & -602.061\\142.247 & -136.763\end{bmatrix}, K=\begin{bmatrix}27.888\\4.6250\\-20.253\\3.475\end{bmatrix}^T. \tag{56}$$

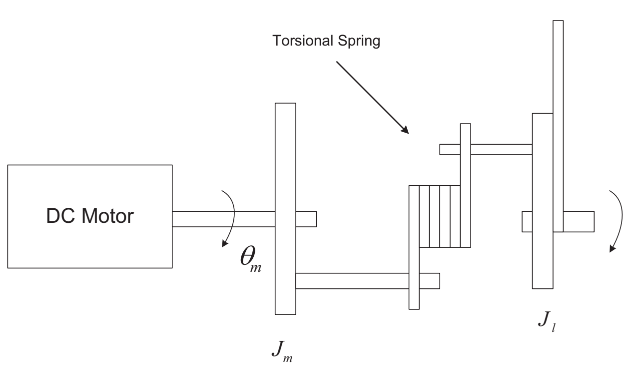

Fig. 4. The flexible joint link robot model by representing the joint flexibility by stiffening torsional spring.

Table 1
Robot parameters for testing the proposed methodology.

System parameter (Units)	Values
Motor inertia J_m (kg m^2)	3.7×10^{-3}
Link inertia J_l (kg^2)	9.3×10^{-3}
Pointer mass m (kg)	2.1×10^{-1}
Link length $2b$ (m)	3.0×10^{-1}
Torsional spring constant k (N m rad^{-1})	1.8×10^{-1}
Viscous friction coefficient B (N m V^{-1})	4.6×10^{-2}
Amplifier gain K_τ (Nm V^{-1})	8.0×10^{-2}

The system states and their estimation errors are shown in Fig. 5, which depicts the efficiency of the proposed control technique in the presence of uncertainty. The corresponding control signal for the observer-based stabilization of the robotic arm is plotted in Fig. 6. The observer states and the estimation errors converge asymptotically to the origin. Moreover, in order to incur the execution against perturbations, from Theorem 2, unbridling the optimization problem (43) results into LMI variables and the gain matrices as

$$\overline{P}=\begin{bmatrix}3.465 & -5.662 & 3.235 & -3.578\\-5.662 & 338.21 & 0.878 & -0.007\\3.235 & 0.878 & 3.283 & -3.361\\-3.578 & -0.007 & -3.361 & 5.344\end{bmatrix},$$

$$S=10^6\times\begin{bmatrix}0.809 & -0.840 & 0.027 & 0.002\\-0.840 & 1.133 & -0.282 & 0.002\\0.027 & -0.282 & 0.249 & -0.005\\0.002 & 0.002 & -0.005 & 0.005\end{bmatrix},$$

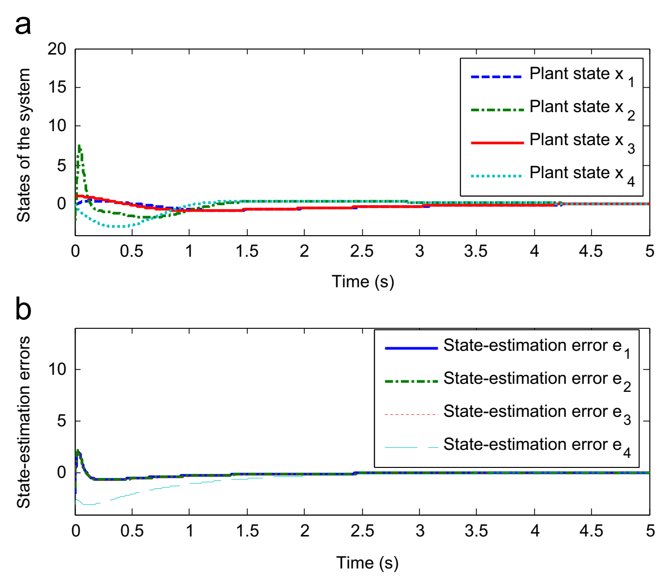

Fig. 5. States and estimation errors for robotic arm converging to the origin under parametric uncertainties: (a) states of the system and (b) state-estimation errors.

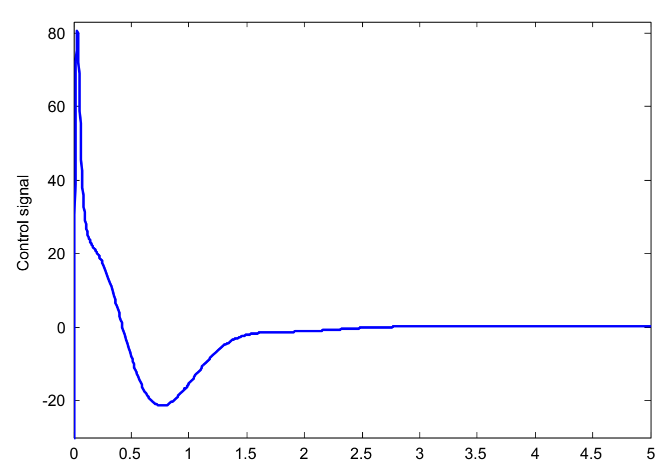

Fig. 6. Control signal $u(t)$ for the robot arm in the absence of disturbance.

$$L = 10^3 \times \begin{bmatrix} 2.626 & -2.607 \\ 2.213 & -2.194 \\ 2.232 & -2.213 \\ 0.488 & -0.482 \end{bmatrix}, K = \begin{bmatrix} 122.856 \\ 4.628 \\ -107.855 \\ 13.3775 \end{bmatrix}^T, \quad (57)$$

having $\gamma = 1.382$.

For simulation infix the disturbance vector is selected as

$$d = \begin{bmatrix} 2.5 \sin 100t \\ 7.5 \cos 80t \\ 3.5 \sin 100t \\ 5.5 \cos 80t \end{bmatrix}. \quad (58)$$

Fig. 7 plots the states and state estimation errors against time in the presence of perturbations and parametric uncertainties. The robustness of the proposed observer-based control approach for the uncertain nonlinear system is demonstrated through convergence of the estimation errors and the robotic arm states in the neighborhood of the origin. Fig. 8 presents the proposed control signal in the presence of parametric uncertainties and disturbances. It is observed that the proposed control signal is changing to maintain the behavior of the overall closed-loop system against the perturbations and uncertainties.

It can be concluded from the simulation studies that the proposed methodologies can be effectively applied to synthesize a state-estimation-based controller for the one-sided Lipschitz nonlinear systems against uncertainties and perturbations.

5. Conclusions

In this paper, an observer-based control scheme was proposed for a class of uncertain continuous-time one-sided Lipschitz nonlinear systems. Time-varying norm-bounded uncertainties are imparted to the system's state and output matrices in the observer-based controller design. The proposed methodology adapts the techniques provided in the literature for the one-sided Lipschitz nonlinearity, quadratic inner-boundedness, parametric uncertainties, disturbances, matrix inequalities, observer design, and state-feedback control. The proposed nonlinear matrix inequalities were later resolved by adopting suitable variables, convex optimization, and the iterative cone-complementary linearization technique. Further, the L_2 stability theory was employed to attain robustness against perturbations for observer-based controller synthesis purposes. Numerical simulation examples were provided to show the proficiency of the proposed designs.

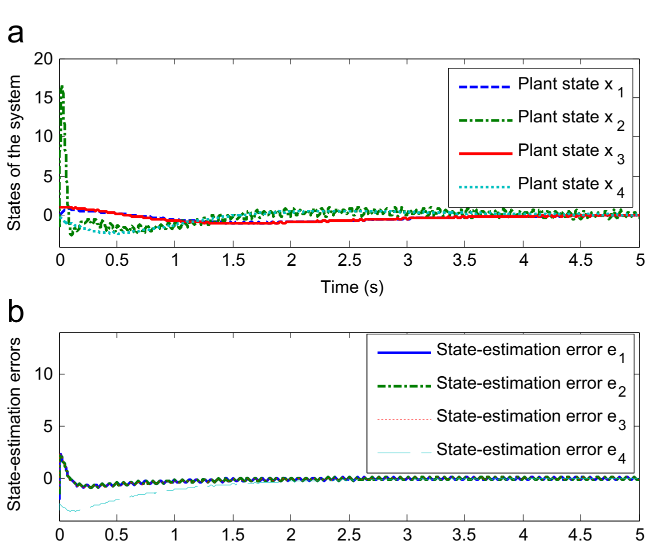

Fig. 7. States and estimation errors for robotic arm converging asymptotically to origin under parametric uncertainties and disturbance: (a) states of the system and (b) state-estimation errors.

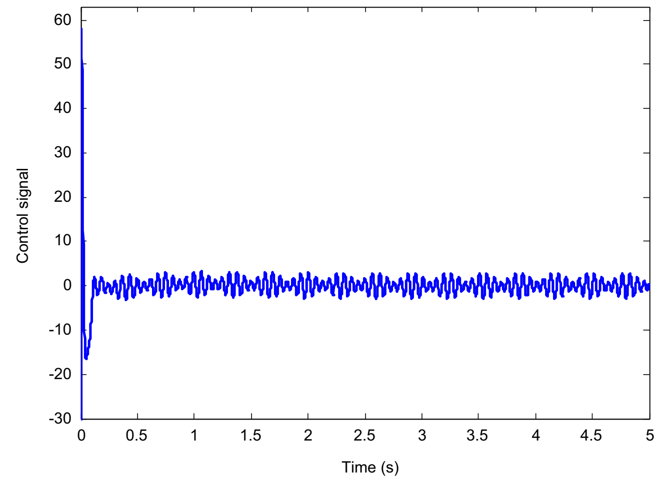

Fig. 8. Control signal $u(t)$ in the presence of exogenous disturbances and uncertainties.

Acknowledgement

This work was supported by the Higher Education Commission (HEC) of Pakistan under grant agreement PIN no. 213-54309-2EG2-007 through indigenous Ph.D. scholarship program (phase II, batch II, 2013) and the National Research Foundation of Korea under the auspices of the Ministry of Science, ICT and Future Planning, Korea (grant no. NRF-2014-R1A2A1A10049727).

References

[1] Ibrir S, Diopt S. Novel LMI conditions for observer-based stabilization of Lipschitzian nonlinear systems and uncertain linear systems in discrete-time. Appl Math Comput 2008;206(2):579–88.
[2] Lien C. Robust observer-based control of systems with state perturbations via LMI approach. IEEE Trans Autom Control 2004;49(8):1365–70.
[3] Heemels W, Daafouz J, Millerioux G. Observer-based control of discrete-time LPV systems with uncertain parameters. IEEE Trans Autom Control 2010;55 (9):2130–5.
[4] Li L, Zhao J, Dimirovski G. Observer-based reliable exponential stabilization and H_∞ control for switched systems with faulty actuators: an average dwell time approach. Nonlinear Anal: Hybrid Syst 2010;5(3):479–91.
[5] Lee HJ, Park JB, Chen G. Robust fuzzy control of nonlinear systems with parametric uncertainties. IEEE Trans Fuzzy Syst 1996;9(2):369–79.
[6] Tong S, Li HH. Observer based robust fuzzy control of nonlinear systems with parametric uncertainties. Fuzzy Sets Syst 2002;131(2):165–84.
[7] Arcak M, Kokotovic P. Observer-based control of systems with slope-restricted nonlinearities. IEEE Trans Autom Control 2001;46(7):1146–50.
[8] Ibrir S, Xie WF, Su C-Y. Observer-based control of discrete-time Lipschitzian non-linear systems: application to one-link flexible joint robot. Int J Control 2005;78:385–95.
[9] Atassi A, Khalil H. Separation results for the stabilization of nonlinear systems using different high-gain observer designs. Syst Control Lett 2000;39(3):183–91.
[10] Hendricks E, Luther J. Model and observer-based control of internal combustion engines. In: Proceedings of international workshop on modeling, emissions and control in automotive engines (MECA). Fisciano, Italy; 2001. p. 9–21.
[11] Kalman RE. A new approach to linear filtering and prediction problems. J Basic Eng 1960;82(1):35–45.
[12] Luenberger DG. An introduction to observers. IEEE Trans Autom Control 1971;16(6):596–602.
[13] Shokouhi-Nejad H, Rikhtehgar-Ghiasi A. Robust H_∞ observer-based controller for stochastic genetic regulatory networks. Math Biosci 2014;250:41–53.
[14] Kheloufi H, Zemouche A, Bedouhene F, et al. On LMI conditions to design observer-based controllers for linear systems with parameter uncertainties. Automatica 2013;49(12):3700–4.

[15] Lin Z, Guan X, Liu Y, Shi P. Observer-based robust control for uncertain systems with time-varying delay. IMA J Math Control Inf 2001;18(3):439–50.
[16] Ibrir S. Stability and robust stabilization of discrete-time switched systems with time-delays: LMI approach. Appl Math Comput 2008;206(2):570–8.
[17] Kalsi K, Lian J, Zak SH. Decentralized dynamic output feedback control of nonlinear interconnected systems. IEEE Trans Autom Control 2001;55(8):1964–70.
[18] Mahmoud MS. Resilient control of uncertain dynamical systems (Lecture notes in control and information sciences). Berlin Heidelberg: Springer; 2004.
[19] De Souza CE, Xie L, Wang Y. H_∞ filtering for a class of uncertain nonlinear systems. Syst Control Lett 1993;20(6):419–26.
[20] Wang Y, Xie L, De Souza CE. Robust control of a class of uncertain nonlinear systems. Syst Control Lett 1992;19(2):139–49.
[21] Xie L, Soh YC, De Souza CE. Robust kalman filtering for uncertain discrete-time systems. IEEE Trans Autom Control 1994;39(6):1310–4.
[22] Xu S, Dooren PV. Robust H_∞ filtering for a class of non-linear systems with state delay and parameter uncertainty. Int J Control 2002;75(10):766–74.
[23] Pertew A, Marquez H, Zhao Q. H_∞ synthesis of unknown input observers for a non-linear Lipschitz systems. Int J Control 2005;78:1155–65.
[24] Xu S, Lam J, Mao X. Delay-dependent H_∞ control and filtering for uncertain Markovian jump systems with time-varying delays. IEEE Trans Circuits Syst I 2007;54(9):2070–7.
[25] Zemouche A, Boutayeb M, Bara GI. Observers for a class of Lipschitz systems with extension to H_∞ performance analysis. Syst Control Lett 2008;57:18–27.
[26] Zemouche A, Boutayeb M. A unified H_∞ adaptive observer synthesis method for a class of systems with both Lipschitz and monotone nonlinearities. Syst Control Lett 2009;58:282–8.
[27] Hu G. Observers for one-sided Lipschitz nonlinear systems. IMA J Math Control Inf 2006;23(4):395–401.
[28] Zhao Y, Tao J, Shi NZ. A note on observer design for one-sided Lipschitz nonlinear systems. Syst Control Lett 2010;59(1):66–71.
[29] Ahmad S, Majeed R, Hong K-S, Rehan M. Observer design for one-sided Lipschitz nonlinear systems subject to measurement delays. Math Probl Eng 2015;2015(879492):13. http://dx.doi.org/10.1155/2015/879492.
[30] Barbata A, Zasadzinski M, Souley-Ali H, Messaoud H. Exponential observer for a class of one-sided Lipschitz stochastic nonlinear systems. IEEE Trans Autom Control 2015;60(1):259–64.
[31] Ibrir S. Static output feedback and guaranteed cost control of a class of discrete-time nonlinear systems with partial state measurements. Nonlinear Anal: Theory Methods Appl 2008;68(7):1784–92.
[32] Grandvallet B, Zemouche A, Souley-Ali H, Bautayed M. New LMI condition for observer-based H_∞ stabilization of a class of nonlinear discrete-time systems. SIAM J Control Optim 2013;51(1):784–800.
[33] Kheloufi H, Zemouche A, Bedouhene F, Souley-Ali H. Robust H_∞ Observer-based controller for Lipschitz nonlinear discrete-time systems with parameter uncertainties. In: Proceedings of IEEE conference on decision and control. Los Angeles, California, USA; 2014. p. 15–17.
[34] Song B, Hedrick J. Observer-based dynamic surface control for a class of nonlinear systems: an LMI approach. IEEE Trans Autom Control 2004;49(11):1995–2001.
[35] Karafyllis I, Kravaris C. Robust output feedback stabilization and nonlinear observer design. Syst Control Lett 2005;54(10):925–38.
[36] Kheloufi H, Bedouhene F, Zemouche A, Alessandri A. Observer-based stabilisation of linear systems with parameter uncertainties by using enhanced LMI conditions. Int J Control 2015;88(6):1189–200.
[37] Huang D, Xu J-X, Yang S, Jin X. Observer based repetitive learning control for a class of nonlinear systems with non-parametric uncertainties. Int J Robust Nonlinear Control 2015;25:1214–29.
[38] Ginoya D, Shendge PD, Phadke SB. Disturbance observer based sliding mode control of nonlinear mismatched uncertain systems. Commun Nonlinear Sci Numer Simul 2015;26:98–107.
[39] Pertew A, Marquez H, Zhao Q. H_∞ observer design for Lipschitz nonlinear systems. IEEE Trans Autom Control 2006;51(7):1211–6.
[40] Abbaszadeh M, Marquez H. A generalized framework for robust nonlinear H_∞ filtering of Lipschitz descriptor systems with parametric and nonlinear uncertainties. Automatica 2012;48(5):894–900.
[41] Darouach M, Baddas LB-, Zerrougui M. H_∞ observers design for a class of nonlinear singular systems. Automatica 2011;47(11):2517–25.
[42] Derakhshan SF, Fatehi A. Non-monotonic robust H_2 fuzzy observer-based control for discrete time nonlinear systems with parametric uncertainties. Int J Syst Sci 2015;46(12):2134–49.
[43] Zhang W, Su H, Su S, Wang D. Nonlinear H_∞ observer design for one-sided Lipschitz systems. Neurocomputing 2014;145:505–11.
[44] Chang X, Yang G. New results on output feedback H_∞ control for linear discrete-time systems. IEEE Trans Autom Control 2014;59(5):1355–9.
[45] Boyd S, Ghaoui LE, Feron E, Balakrishnan V. Linear matrix inequalities in system and control theory. Philadelphia: SIAM; 1994.
[46] Fu F, Hou M, Duan G. Stabilization of quasi-one-sided Lipschitz nonlinear systems. IMA J Math Control Inf 2013;30:169–84.
[47] Cai X, Gao H, Liu L, Zhang W. Control design for one-sided Lipschitz nonlinear differential inclusions. ISA Trans 2014;53:298–304.
[48] Cai X, Wang Z, Liu L. Control design for one-side Lipschitz nonlinear differential inclusion systems with time-delay. Neurocomputing 2015;165:182–9.
[49] Benallouch M, Boutayeb M, Trinh H. H_∞ observer-based control for discrete-time one-sided Lipschitz systems with unknown inputs. SIAM J Control Optim 2014;52(6):3751–75.
[50] Wu R. Observer-based control of one-sided Lipschitz nonlinear systems. In: Proceedings of IEEE chinese guidance, navigation and control conference; 2014. p. 2375–80.
[51] Song J, He S. Robust finite-time H_∞ control for one-sided Lipschitz nonlinear systems via state feedback and output feedback. J Frankl Inst 2015;352(8):3250–66.
[52] Gopal V, Biegler LT. Large scale inequality constrained optimization and control. IEEE Control Syst 1998:59–68.
[53] Abbaszadeh M, Marquez H. Nonlinear observer design for one-sided Lipschitz systems. In: Proceedings of the American control conference. Baltimore, USA; 2010. p. 5284–89.
[54] Zhang W, Su H, Liang Y, Han Z. Nonlinear observer design for one-sided Lipschitz systems: a linear matrix inequality approach. IET Control Theory Appl 2012;6:1297–303.
[55] Spong M. Modeling and control of elastic joint robots. J Dyn Syst Meas Control-Trans ASME 1987;109:310–9.