

# Boundary Control of an Axially Moving Steel Strip under a Spatiotemporally Varying Tension\*

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In this paper, a vibration suppression scheme of an axially moving steel strip in the zinc galvanizing line is investigated. The moving steel strip is modeled as a moving beam, in which the tension applied to the strip is a spatiotemporally varying function. The transverse vibration of the strip is controlled by a hydraulic touch-roll actuator at the right boundary. The mathematical model of the system, which consists of a hyperbolic partial differential equation describing the dynamics of the moving beam and an ordinary differential equation describing the actuator dynamics, is derived by using the Hamilton's principle for the systems of changing mass. The Lyapunov method is employed to design a boundary control law for ensuring the vibration reduction of the system in the presence of a spatiotemporally varying tension. The exponential stability of the closed loop system under the boundary control is proved through the use of invariance principle and semigroup theory. Simulation results verify the effectiveness of the boundary control law proposed.

**Key Words:** Axially Moving System, Boundary Control, Hyperbolic Partial Differential Equation, Stability, Periodic System

## 1. Introduction

Figure 1 shows the continuous hot-dip zinc galvanizing process. The steel strips, of order of 1–1.2 m wide by 0.8–3.0 mm thick, are preheated and passed at a constant speed through a pot of molten zinc at a temperature in the region of about 450°C. A zinc film is entrained onto the strip as it emerges from the pot. In order to achieve the target deposited mass and maintain it over various process conditions, a pair of air knives, which direct a long thin wedge-shaped jet of high-velocity air onto the strip, are generally used to control the thickness of the deposited zinc by stripping out excess zinc back into the pot. The deposited film solidifies while the strip moves vertically upward, cooling as it goes, and horizontally for about 110 m, to the gauge that measures the mass of zinc deposited on the strip surfaces.

Two control objectives in the galvanizing line are to improve the uniformity of the zinc deposit on the strip surfaces and to reduce the zinc consumption. The transverse movement or vibrations of the strip is known to be the main cause for the difference between the average deposited masses on the right and left surfaces and the non-uniformity across the surfaces. As a regulation problem of deposited mass, a number of adjustment methods of the gap between the strip and air knives have been studied by several researchers: Bertin et al.<sup>(1)</sup>, McKerrow<sup>(2)</sup>, Jacobs

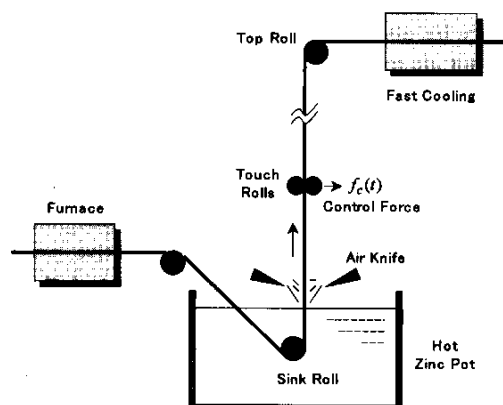


Fig. 1 An axially moving steel strip in the zinc galvanizing line

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and Chen<sup>(3)</sup>, and Chen<sup>(4)</sup>. However, a pertinacious problem was the lack of a precise knowledge on strip position due to the vibrations of the strip. Many galvanized steel manufacturers including POSCO (Korea) and U.S. Steel have attempted to measure the strip position directly by installing laser transducers near the air knives. However, no long-term success has been reported yet, because the transducers were unreliable in the high-temperature environment in the long run.

Thus, as an alternative method not using unreliable transducers, the strip vibrations need to be directly suppressed by using a more practical and reasonable method such as active boundary force control. The mathematical models describing the dynamics of a moving steel strip with a hydraulic touch-roll actuator for exerting boundary control force are represented as a coupled hyperbolic partial differential equation (PDE) and an ordinary differential equation (ODE), respectively. The coupled system of a PDE and an ODE can then be analyzed through the use of Lyapunov method and semigroup theory to ensure the stability of the closed loop system and to derive a boundary control law for reducing the vibration energy of the strip.

Axially moving systems can be found in various engineering areas: high-speed magnetic tapes, band saws, power transmission chains and belts, and paper sheets under processing. Especially, the dynamics analysis and control for axially moving beams have received a growing attention due to the entrance of new applications in flexible robotic manipulators and flexible space structures<sup>(5)–(9)</sup>. Particularly, vibration control schemes on axially moving strings include Yang et al.<sup>(10)</sup>, Chung and Tan<sup>(11)</sup>, Lee and Mote<sup>(12)</sup>, Fung et al.<sup>(13)</sup>, and Li et al.<sup>(14)</sup>, and those on axially moving beams include Choi et al.<sup>(15)</sup>, Hong et al.<sup>(16)</sup>, Lee and Mote<sup>(17)</sup>, Li and Rahn<sup>(18)</sup>, and Fung et al.<sup>(19)</sup>

The boundary force control has several advantages over control schemes acting within the spatial domain (e.g., distributed force control). A boundary control law, as apposed to a distributed control law, is not only easily implementable by active or semi-active means at the boundary, but also the dynamic model of the system equation is not altered by adding sensors and actuators. Lee and Mote<sup>(17)</sup> analyzed a boundary controller to minimize the vibration energy of a translating tensioned beam via an active or passive damping. Li and Rahn<sup>(18)</sup> reported an innovative boundary control strategy to suppress vibrations of an axially moving beam by distinguishing the controlled span from a disturbed span via an active, pivoting roller actuator. Fung et al.<sup>(19)</sup> developed an optimal boundary control strategy for an axially moving material system through a mass-spring-damper controller. In all papers above, the control laws have been designed under the assumption of a constant tension. However, in practical situations, almost every axially moving system may have

an axial tension which is a function of both time and space due to the eccentricity of a support roller, and/or external disturbances, and/or gravity etc.<sup>(20)</sup> Thus, to obtain a better performance, a boundary controller in the presence of a spatiotemporally varying tension has to be investigated.

The contributions of this paper are: The zinc galvanizing line is analyzed and a control-oriented beam model for the traveling steel strip is derived. Considering the length of the strip, the longitudinal tension of the strip is treated as a spatiotemporally varying function, which is assumed to be bounded and periodic in time. An active vibration suppression scheme in the form of a boundary control and the Lyapunov method are proposed to control the traveling strip system under a spatiotemporally varying tension using a hydraulic touch-roll actuator at the right boundary. The asymptotic and furthermore exponential stability of the closed loop system are assured through the invariance principle and semigroup theory, respectively.

The structure of this paper is organized as follows. In section 2, the governing equation and boundary conditions of an axially moving steel strip are derived by using Hamilton's principle. The boundary control problem is then formulated. In sections 3, a robust boundary force control law is proposed by using the Lyapunov method. In section 4, the asymptotic and exponential stability of the closed loop system are investigated. In section 5, computer simulations are provided. Conclusions are given in section 6.

## 2. Problem Formulation: Equations of Motion

Figure 2 shows a schematic of the axially moving steel strip for control system design purpose. The left boundary at the sink roll is assumed fixed, i.e., fixed in the sense that there is no vertical movement but it allows the strip to move in the horizontal direction. The two touch rolls located in the middle section, but rather close to the sink roll, of the strip are considered to be the right boundary, where the control input (force) is exerted with a hydraulic actuator. In the zinc galvanizing line, the moving steel strip can be modeled as a moving beam with high flexibility.

Let  $t$  be the time,  $x$  be the spatial coordinate along the longitude of motion,  $v_s$  be the axial speed of the strip,  $w(x, t)$  be the transversal displacement of the strip at spa-

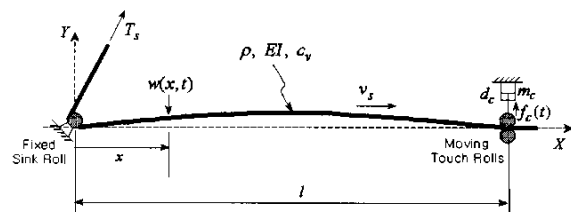


Fig. 2 A control-oriented schematic of the axially moving steel strip with a hydraulic actuator at the right boundary

tial coordinate  $x$  and time  $t$ , and  $l$  be the length of the strip from left to right boundaries. Also, let  $\rho$  be the mass per unit length,  $T_s(x, t)$  be the tension applied to the strip,  $EI$  be the flexural rigidity of the strip, and  $c_v$  be the viscous damping coefficient of the strip. Let the mass and damping coefficient of the hydraulic actuator be  $m_c$  and  $d_c$ , respectively. The control force  $f_c(t)$  is applied to the touch rolls to suppress the transverse vibrations of the strip.

Because the strip travels with a constant speed  $v_s$ , the total derivative operator (the material derivative) with respect to time should be defined as  $(\dot{\cdot}) \triangleq d(\cdot)/dt = (\cdot)_t + v_s(\cdot)_x$  <sup>(21), (22)</sup>, where  $(\cdot)_t = \partial(\cdot)/\partial t$  and  $(\cdot)_x = \partial(\cdot)/\partial x$  denote the partial derivatives.

The kinetic energy of the steel strip between  $x=0$  and  $x=l$  including the hydraulic actuator is

$$T = \frac{1}{2} \int_0^l \rho \{v_s^2 + (v_s w_x + w_t)^2\} dx + \frac{1}{2} m_c w_t^2(l, t). \quad (1)$$

For brevity,  $w_x = w_x(x, t)$  and  $w_t = w_t(x, t)$  have been used, and similar abbreviations will be used in the sequel. The potential energy of the system is

$$V = \frac{1}{2} \int_0^l T_s(x, t) w_x^2 dx + \frac{1}{2} \int_0^l EI w_{xx}^2 dx, \quad (2)$$

where  $w_{xx} = w_{xx}(x, t)$ . The virtual work by the external forces is

$$\begin{aligned} \delta W = & f_c \delta w(l, t) - \int_0^l c_v (w_t + v_s w_x) \delta w dx \\ & - d_c w_t(l, t) \delta w(l, t). \end{aligned} \quad (3)$$

Now, by using the Hamilton's principle, i.e.,  $\int_{t_0}^{t_1} (\delta T - \delta V + \delta W) dt = 0$ , the governing equation and boundary conditions are derived as follows:

$$\begin{aligned} \rho w_{tt}(x, t) + 2\rho v_s w_{xt}(x, t) + \rho v_s^2 w_{xx}(x, t) \\ - (T_s(x, t) w_x(x, t))_x + c_v (w_t(x, t) + v_s w_x(x, t)) \\ + EI w_{xxxx}(x, t) = 0, \quad 0 < x < l, \end{aligned} \quad (4)$$

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_{t0}(x), \quad (5)$$

$$w(0, t) = w_x(0, t) = 0, \quad w_x(l, t) = 0, \quad (6)$$

and

$$\begin{aligned} f_c = & m_c w_{tt}(l, t) + (d_c - \rho v_s) w_t(l, t) \\ & + (T_s(l, t) - \rho v_s^2) w_x(l, t) - EI w_{xxxx}(l, t). \end{aligned} \quad (7)$$

(4) is a linear hyperbolic PDE governing the transverse motion  $w(x, t)$  of the axially moving strip. (5) and (6)–(7) denote the initial conditions and the boundary conditions of the system, respectively. Note that the right boundary condition (7) is an ODE that describes the equation of motion of the hydraulic actuator in compliance with the transversal force at  $x=l$ . For notational brevity, all variables at boundaries will be written without explicit time in the sequel, i.e.,  $w(0) = w(0, t)$ ,  $w(l) = w(l, t)$ ,  $T_s(l) = T_s(l, t)$ , etc.

As shown in (2) and (4), the tension  $T_s(x, t)$  is a spatiotemporally, but periodic in time, varying function. Since the steel strip is moving vertically, the gravitational force  $\rho g x$ , which acts as an additional tension to the strip, cannot be neglected. Also the tension itself may be time-varying due to the eccentricity of a support roller, which causes a periodic excitation. Thus, these variations need to be incorporated in control system design. It is now assumed that  $T_s(x, t)$  is sufficiently smooth and uniformly bounded as follows:

$$T_{s, \min} \leq T_s(x, t) \leq T_{s, \max}, \quad (8)$$

$$|(T_s(x, t))_t| \leq (T_s)_{t, \max}, \quad (9)$$

$$|(T_s(x, t))_x| \leq (T_s)_{x, \max}, \quad (10)$$

for all  $x \in [0, l]$ ,  $t \geq 0$ , and some *a priori* known constants  $T_{s, \min}$ ,  $T_{s, \max}$ ,  $(T_s)_{t, \max}$ , and  $(T_s)_{x, \max}$ .

The objective of a boundary control is now to stabilize asymptotically the vibration energy of the axially moving strip with a time-varying condition at  $x=l$ . From (1)–(2), the mechanical energy  $E_m(t)$  is given by

$$\begin{aligned} E_m(t) = & \frac{1}{2} \int_0^l \rho \{v_s^2 + (v_s w_x + w_t)^2\} dx + \frac{1}{2} \int_0^l T_s w_x^2 dx \\ & + \frac{1}{2} \int_0^l EI w_{xx}^2 dx + \frac{1}{2} m_c w_t^2(l), \end{aligned} \quad (11)$$

where the traveling speed of the strip,  $v_s$ , is a constant and  $T_s = T_s(x, t)$ .

In the remainder of this section, to order to give a general idea regarding how the boundary control action will work, the dynamics of the moving strip with both fixed boundaries is first analyzed. In this case, the boundary conditions in (6) and (7) become  $w(0) = w_x(0) = w(l) = w_x(l) = 0$ , and henceforth  $m_c w_t^2(l) = 0$  in (11). The time derivative of  $E_m(t)$  in (11) is now evaluated by applying the one-dimensional transport theorem of moving material <sup>(23)</sup> as follows:

$$\begin{aligned} \dot{E}_m(t) = & \frac{1}{2} \int_0^l 2\rho (w_t + v_s w_x) (w_{tt} + 2v_s w_{xt} + v_s^2 w_{xx}) dx \\ & + \int_0^l T_s w_x (w_{xt} + v_s w_{xx}) dx \\ & + \frac{1}{2} \int_0^l \{(T_s)_t + v_s (T_s)_x\} w_x^2 dx \\ & + \int_0^l EI w_{xx} (w_t + v_s w_x)_{xx} dx \\ = & \int_0^l (w_t + v_s w_x) ((T_s w_x)_x \\ & - c_v (w_x + v_s w_x) - EI w_{xxxx}) dx \\ & + \int_0^l T_s w_x (w_{xt} + v_s w_{xx}) dx \\ & + \frac{1}{2} \int_0^l \{(T_s)_t + v_s (T_s)_x\} w_x^2 dx \\ & + \int_0^l EI w_{xx} (w_t + v_s w_x)_{xx} dx, \end{aligned} \quad (12)$$

where (4) has been used in deriving the second equality. Integrating by parts, the terms in (12) are further simplified as follows:

$$\int_0^l \{w_t(T_s w_x)_x + T_s w_x w_{xt}\} dx = [w_t(T_s w_x)]_0^l, \quad (13)$$

$$\int_0^l w_x(T_s w_x)_x dx = \int_0^l \frac{d}{dx}(T_s w_x^2) dx - \int_0^l T_s w_x w_{xx} dx = [T_s w_x^2]_0^l - \int_0^l T_s w_x w_{xx} dx, \quad (14)$$

$$\int_0^l w_{xxx}(w_t + v_s w_x) dx = [w_{xxx}(w_t + v_s w_x)]_0^l - [w_{xx}(w_t + v_s w_x)]_0^l + \int_0^l w_{xx}(w_t + v_s w_x)_{xx} dx. \quad (15)$$

The substitution of (13)–(15), together with the fixed boundary conditions, into (12) yields:

$$\dot{E}_m(t) = -c_v \int_0^l (w_t + v_s w_x)^2 dx - v_s EI w_{xx}^2(0) + v_s EI w_{xx}^2(l) + \frac{1}{2} \int_0^l \{(T_s)_t + v_s(T_s)_x\} w_x^2 dx. \quad (16)$$

Now, the following observations from (16) are made: (i) The viscous damping reduces the mechanical energy of the axially moving strip, which can be seen from the existence of  $-c_v \int_0^l (w_t + v_s w_x)^2 dx$ . (ii) Even though the transverse velocities and the slopes at both boundaries are zero, i.e.,  $w_t(0) = w_t(l) = w_x(0) = w_x(l) = 0$ , the instantaneous bending moment,  $v_s EI w_{xx}$ , can still deform the strip. The bending moment at  $x = 0$  decreases the mechanical energy, while that at  $x = l$  increases it. (iii) The time derivative of  $T_s(x, t)$ ,  $(T_s)_t + v_s(T_s)_x$ , increases the energy flux by the factor of  $w_x^2$ . The last observation indicates that the time rate of the change of  $T_s(x, t)$  cannot be simply neglected.

Thus, it is concluded that for the traveling strip with fixed boundary conditions, the bending moment occurring at the right boundary  $x = l$  and the time rate of the change of tension  $T_s(x, t)$  should be treated properly to decrease  $E_m(t)$ .

### 3. Design of Boundary Control Law

The selection of a suitable Lyapunov function and the construction of an effective control input are most important issues in the Lyapunov method. To obtain the asymptotic stability of system (4)–(7), the convergence of the hydraulic actuator displacement  $w(l)$  to zero should also be satisfied. But the mechanical energy  $E_m(t)$  in (11) does not involve a state variable of the hydraulic actuator,  $w(l)$ . Thus, a modification of (11) is pursued.

Note that by boundary conditions (6) and Poincaré's inequality<sup>(24)</sup>, there exists a positive constant  $C_1$  such that

$$\int_0^l w_0^2 dx \leq C_1 \int_0^l w_x^2 dx \quad \text{for } x \in [0, l]. \quad (17)$$

From (17), it is seen that the convergence of the transverse displacement  $w(x, t)$  for  $x \in [0, l]$  to zero can be obtained by considering the convergence of the slope  $w_x(x, t)$  for  $x \in [0, l]$ . That is, if  $\lim_{t \rightarrow \infty} \int_0^l w_x^2 dx = 0$  is satisfied, then

$\lim_{t \rightarrow \infty} \int_0^l w_0^2 dx = 0$  is also satisfied. Thus, the asymptotic stability of the hydraulic actuator system can be analyzed by adding the slope term at  $x = l$  in the mechanical energy.

The following positive definite functional  $V_m(t)$ , replacing  $E_m(t)$  in (11), is now considered.

$$V_m(t) = \frac{1}{2} \int_0^l \rho(v_s w_x + w_t)^2 dx + \frac{1}{2} \int_0^l T_s w_x^2 dx + \frac{1}{2} \int_0^l EI w_{xx}^2 dx + \frac{1}{2} m_c \{w_t^2(l) + w_x^2(l)\}. \quad (18)$$

Note that the constant term  $v_s^2$  has been omitted in (18) because it doesn't make any difference in deriving the final control law. It is also noted that there exist positive constants  $C_2$  and  $C_3$  such that the following holds:

$$C_2 \left\{ w_t^2(l) + w_x^2(l) + \int_0^l w_x^2 dx \right\} \leq \{w_t(l) + w_x(l)\}^2 + \int_0^l w_x^2 dx \leq C_3 \left\{ w_t^2(l) + w_x^2(l) + \int_0^l w_x^2 dx \right\}. \quad (19)$$

Using inequality (19), some positive constants  $C_4$  and  $C_5$  can be chosen, so that the following holds:

$$C_4 \left[ \frac{1}{2} m_c \{w_t^2(l) + w_x^2(l)\} + \frac{1}{2} \int_0^l T_s w_x^2 dx \right] \leq \frac{1}{2\alpha} m_c \{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\}^2 + \frac{1}{2} \int_0^l T_s w_x^2 dx \leq C_5 \left[ \frac{1}{2} m_c \{w_t^2(l) + w_x^2(l)\} + \frac{1}{2} \int_0^l T_s w_x^2 dx \right], \quad (20)$$

where  $\alpha$  and  $\beta$  are positive constants.

Now, a further modification of the positive definite functional,  $V_0(t)$ , is proposed as follows:

$$V_0(t) = \frac{1}{2} \int_0^l \rho(v_s w_x + w_t)^2 dx + \frac{1}{2} \int_0^l T_s w_x^2 dx + \frac{1}{2} \int_0^l EI w_{xx}^2 dx + \frac{1}{2\alpha} m_c \{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\}^2. \quad (21)$$

From (18), (20) and (21), the following holds:

$$C_4 V_m(t) \leq V_0(t) \leq C_5 V_m(t), \quad (22)$$

which means that  $V_0(t)$  is equivalent to  $V_m(t)$ .

Using  $V_0(t)$ , define a functional  $V(t)$  such that

$$V(t) = \alpha V_0(t) + 2\beta \int_0^l \rho x w_x (w_t + v_s w_x) dx \quad (23)$$

$$= \alpha V_0(t) + \beta \int_0^l \rho x \{w_x + (w_t + v_s w_x)\}^2 dx - \beta \int_0^l \rho x w_x^2 dx - \beta \int_0^l \rho x (w_t + v_s w_x)^2 dx, \quad (24)$$

where

$$\frac{\alpha}{2} \int_0^l \rho (w_t + v_s w_x)^2 dx \geq \beta \int_0^l \rho l (w_t + v_s w_x)^2 dx \geq \beta \int_0^l \rho x (w_t + v_s w_x)^2 dx, \quad (25)$$

$$\frac{\alpha}{2} \int_0^l T_s(x,t) w_x^2 dx \geq \frac{\alpha}{2} T_{s,\min} \int_0^l w_x^2 dx \geq \beta \int_0^l \rho l w_x^2 dx \geq \beta \int_0^l \rho x w_x^2 dx. \quad (26)$$

From (25) and (26), it is seen that  $V(t)$  is a positive definite functional if  $\alpha > 2\beta l$  and  $T_{s,\min} > \rho$ .

By using the Cauchy-Schwarz inequality, it can also be shown that there exists a positive constant  $C_6$  such that

$$2\beta \int_0^l \rho x w_x (v_s w_x + w_t) dx \leq \beta \rho l \left\{ \int_0^l w_x^2 dx + \int_0^l (w_t + v_s w_x)^2 dx \right\} \leq C_6 \left\{ \frac{1}{2} T_{s,\min} \int_0^l w_x^2 dx + \frac{1}{2} \int_0^l \rho (w_t + v_s w_x)^2 dx \right\} \leq C_6 \left\{ \frac{1}{2} \int_0^l T_s w_x^2 dx + \frac{1}{2} \int_0^l \rho (w_t + v_s w_x)^2 dx \right\} \leq C_6 V_0(t), \quad (27)$$

where  $C_6 \geq \max\left(\frac{2\beta\rho l}{T_{s,\min}}, 2\beta l\right)$ . From (21), (23) and (27), the following holds:

$$(\alpha - C_6)V_0(t) \leq V(t) \leq (\alpha + C_6)V_0(t), \quad (28)$$

where  $\alpha > \max\left(\frac{2\beta\rho l}{T_{s,\min}}, 2\beta l\right)$ . From (28), it is concluded that  $V(t)$  in (23) is equivalent to the positive definite functional  $V_0(t)$  in (21).

In this paper, the positive definite functional  $V(t)$  in (23) is considered as a Lyapunov function candidate for system (4)–(7) to determine a boundary force control law for the stabilization of the vibration energy  $E_m(t)$  in (11) and to obtain the asymptotic stability of the closed loop system.

The time derivative of  $V(t)$  along (4) yields:

$$\begin{aligned} \dot{V}(t) &= \alpha \dot{V}_0 + \frac{d}{dt} \left[ 2\beta \int_0^l \rho x w_x (w_t + v_s w_x) dx \right] \\ &= \alpha \dot{V}_0(t) + 2\beta \rho \int_0^l x (w_{xt} + v_s w_{xx}) (w_t + v_s w_x) dx \\ &\quad + 2\beta \rho \int_0^l x w_x (w_{tt} + 2v_s w_{xt} + v_s^2 w_{xx}) dx \\ &\quad + 2\beta \rho v_s \int_0^l w_x (w_t + v_s w_x) dx \end{aligned}$$

$$\begin{aligned} &= \alpha \dot{V}_0(t) + \beta \rho \left[ x (w_t + v_s w_x)^2 \right]_0^l \\ &\quad - \beta \rho \int_0^l (w_t + v_s w_x)^2 dx \\ &\quad + 2\beta \int_0^l x w_x ((T_s w_x)_x - c_0 (w_x + v_s w_x) - E l w_{xxxx}) dx \\ &\quad + 2\beta \rho v_s \int_0^l w_x (w_t + v_s w_x) dx. \end{aligned} \quad (29)$$

The integration by parts yields:

$$\begin{aligned} 2 \int_0^l x w_x (T_s w_x)_x dx &= 2 \int_0^l x \frac{d}{dx} (T_s w_x^2) dx \\ &\quad - 2 \int_0^l x T_s w_x w_{xx} dx = \left[ x (T_s w_x^2) \right]_0^l \\ &\quad - \int_0^l T_s w_x^2 dx + \int_0^l x (T_s)_x w_x^2 dx, \end{aligned} \quad (30)$$

$$\begin{aligned} - \int_0^l x w_x w_{xxxx} dx &= - \left[ x w_x w_{xxx} \right]_0^l + \left[ w_x w_{xx} \right]_0^l \\ &\quad - \int_0^l w_{xx}^2 dx + \int_0^l x w_{xx} w_{xxx} dx. \end{aligned} \quad (31)$$

Since  $\left[ x w_{xx}^2 \right]_0^l = \int_0^l w_{xx}^2 dx + 2 \int_0^l x w_{xx} w_{xxx} dx$ , (31) can be rewritten as

$$\begin{aligned} - \int_0^l x w_x w_{xxxx} dx &= - \left[ x w_x w_{xxx} \right]_0^l + \left[ w_x w_{xx} \right]_0^l \\ &\quad + \left[ x w_{xx}^2 \right]_0^l - 2 \int_0^l w_{xx}^2 dx - \int_0^l x w_{xx} w_{xxx} dx. \end{aligned} \quad (32)$$

From (31) and (32), the following is derived:

$$\begin{aligned} - \int_0^l x w_x w_{xxxx} dx &= -2 \left[ x w_x w_{xxx} \right]_0^l \\ &\quad + 2 \left[ w_x w_{xx} \right]_0^l + \left[ x w_{xx}^2 \right]_0^l - 3 \int_0^l w_{xx}^2 dx. \end{aligned} \quad (33)$$

The following inequality is also utilized.

$$uw \leq \gamma u^2 + \frac{1}{\gamma} v^2, \text{ for any } \gamma > 0. \quad (34)$$

Using (34), the followings are derived.

$$\begin{aligned} &\int_0^l x w_x (w_t + v_s w_x) dx \\ &\leq l \gamma_1 \int_0^l w_x^2 dx + \frac{l}{\gamma_1} \int_0^l (w_t + v_s w_x)^2 dx, \end{aligned} \quad (35)$$

$$\begin{aligned} &\int_0^l w_x (w_t + v_s w_x) dx \\ &\leq \gamma_2 \int_0^l w_x^2 dx + \frac{1}{\gamma_2} \int_0^l (w_t + v_s w_x)^2 dx, \end{aligned} \quad (36)$$

where  $\gamma_i > 0, i = 1, 2$ .

Thus, by substituting (4), (6), (8)–(10), (13)–(15), (30), (33), and (35)–(36) into (29), the time derivative of

the Lyapunov function candidate  $V(t)$ , with a right boundary actuator at  $x = l$  and the fixed condition at left boundary, becomes:

$$\begin{aligned} \dot{V}(t) \leq & -\left(\alpha c_v + \beta \rho - \frac{2\beta c_v l}{\gamma_1} - \frac{2\beta \rho v_s}{\gamma_2}\right) \int_0^l (w_t + v_s w_x)^2 dx \\ & - \left\{ \beta T_{s,\min} - \left(\beta l + \frac{\alpha v_s}{2}\right) (T_s)_{x,\max} - \frac{\alpha}{2} (T_s)_{t,\max} \right. \\ & \left. - 2\beta(c_v \gamma_1 l + \rho v_s \gamma_2) \right\} \int_0^l w_x^2 dx - 3\beta EI \int_0^l w_{xx}^2 dx \\ & + \alpha T_s(l) w_x(l) (w_t(l) + v_s w_x(l)) - \alpha v_s T_s(0) w_x^2(0) \\ & + \beta l T_s(l) w_x^2(l) - \alpha EI w_{xxx}(l) (w_t(l) + v_s w_x(l)) \\ & + \alpha EI v_s w_{xxx}(0) w_x(0) + \alpha EI w_{xx}(l) (w_{xt}(l) + v_s w_{xx}(l)) \\ & - \alpha EI v_s w_{xx}^2(0) - \alpha EI w_{xx}(0) w_{xt}(0) \\ & - 2\beta EI w_x(l) w_{xxx}(l) + 2\beta EI w_x(l) w_{xx}(l) \\ & - 2\beta EI w_x(0) w_{xx}(0) + \beta EI w_{xx}^2(l) + \beta \rho l (w_t(l) + v_s w_x(l))^2 \\ & + \frac{d}{dt} \left[ \frac{1}{2\alpha} m_c \{ \alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l) \}^2 \right] \\ = & -\left(\alpha c_v + \beta \rho - \frac{2\beta c_v l}{\gamma_1} - \frac{2\beta \rho v_s}{\gamma_2}\right) \int_0^l (w_t + v_s w_x)^2 dx \\ & - \left\{ \beta T_{s,\min} - \left(\beta l + \frac{\alpha v_s}{2}\right) (T_s)_{x,\max} - \frac{\alpha}{2} (T_s)_{t,\max} \right. \\ & \left. - 2\beta(c_v \gamma_1 l + \rho v_s \gamma_2) \right\} \int_0^l w_x^2 dx \\ & - 3\beta EI \int_0^l w_{xx}^2 dx - \alpha EI v_s w_{xx}^2(0) \\ & + (\alpha T_s(l) + 2v_s \beta \rho l) w_t(l) w_x(l) + \beta \rho l w_t^2(l) \\ & + (\alpha v_s + \beta l) T_s(l) + \beta \rho l v_s^2 w_x^2(l) \\ & - EI w_{xxx}(l) \{ \alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l) \} \\ & + \frac{d}{dt} \left[ \frac{1}{2\alpha} m_c \{ \alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l) \}^2 \right]. \quad (37) \end{aligned}$$

The time derivative, along (7), of the last term on the right in (37) yields:

$$\begin{aligned} & \frac{d}{dt} \left[ \frac{1}{2\alpha} m_c \{ \alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l) \}^2 \right] \\ & = \frac{1}{\alpha} m_c \{ \alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l) \} \{ \alpha w_{tt}(l) \\ & \quad + (\alpha v_s + 2\beta l) w_{xt}(l) \} \\ & = \{ \alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l) \} \left\{ f_c + \frac{m_c}{\alpha} (\alpha v_s + 2\beta l) w_{xt}(l) \right\} \\ & \quad - \alpha (d_c - \rho v_s) w_t^2(l) \\ & \quad - \{ (\alpha v_s + 2\beta l) (d_c - \rho v_s) + \alpha (T_s(l) - \rho v_s^2) \} w_t(l) w_x(l) \\ & \quad - (\alpha v_s + 2\beta l) (T_s(l) - \rho v_s^2) w_x^2(l) \\ & \quad + \{ \alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l) \} EI w_{xxx}(l). \quad (38) \end{aligned}$$

A boundary force control law, which makes the time derivative of  $V(t)$  negative definite in the presence of the spatiotemporally varying tension, is then proposed as follows:

$$f_c = (d_c - 2\rho v_s) w_t(l) - \frac{m_c}{\alpha} (\alpha v_s + 2\beta l) w_{xt}(l). \quad (39)$$

The substitution of (38) and boundary force control law (39) into (37) yields:

$$\begin{aligned} \dot{V}(t) \leq & -\left(\alpha c_v + \beta \rho - \frac{2\beta c_v l}{\gamma_1} - \frac{2\beta \rho v_s}{\gamma_2}\right) \int_0^l (w_t + v_s w_x)^2 dx \\ & - \left\{ \beta T_{s,\min} - \left(\beta l + \frac{\alpha v_s}{2}\right) (T_s)_{x,\max} - \frac{\alpha}{2} (T_s)_{t,\max} \right. \\ & \left. - 2\beta(c_v \gamma_1 l + \rho v_s \gamma_2) \right\} \int_0^l w_x^2 dx \\ & - 3\beta EI \int_0^l w_{xx}^2 dx - \alpha EI v_s w_{xx}^2(0) \\ & - \rho (3\alpha v_s - \beta l) w_t^2(l) \\ & - \left\{ \beta l T_s(l) - (\alpha v_s + 3\beta l) \rho v_s^2 \right\} w_x^2(l), \quad (40) \end{aligned}$$

where  $(3\alpha v_s - \beta l) > 0$  and  $\left\{ \beta l T_s(l) - (\alpha v_s + 3\beta l) \rho v_s^2 \right\} > 0$ .

If  $T_{s,\min}$  is sufficiently large, the positive values  $\alpha$ ,  $\beta$ , and  $\gamma_i$ ,  $i = 1, 2$ , can be chosen to satisfy

$$\left(\alpha c_v + \beta \rho - \frac{2\beta c_v l}{\gamma_1} - \frac{2\beta \rho v_s}{\gamma_2}\right) > 0, \quad (41)$$

$$\begin{aligned} & \left\{ \beta T_{s,\min} - \left(\beta l + \frac{\alpha v_s}{2}\right) (T_s)_{x,\max} - \frac{\alpha}{2} (T_s)_{t,\max} \right. \\ & \left. - 2\beta(c_v \gamma_1 l + \rho v_s \gamma_2) \right\} > 0. \quad (42) \end{aligned}$$

Thus, the following is obtained:

$$\begin{aligned} \dot{V}(t) \leq & -c_0 \left( \int_0^l (w_t + v_s w_x)^2 dx + \int_0^l w_x^2 dx \right. \\ & \left. + \int_0^l w_{xx}^2 dx + w_t^2(l) + w_x^2(l) \right), \quad (43) \end{aligned}$$

where

$$\begin{aligned} c_0 = \min \left\{ \left( \alpha c_v + \beta \rho - \frac{2\beta c_v l}{\gamma_1} - \frac{2\beta \rho v_s}{\gamma_2} \right), \right. \\ \left. 3\beta EI, \rho (3\alpha v_s - \beta l), \beta l T_s(l) - (\alpha v_s + 3\beta l) \rho v_s^2, \right. \\ \left. \beta T_{s,\min} - \left(\beta l + \frac{\alpha v_s}{2}\right) (T_s)_{x,\max} - \frac{\alpha}{2} (T_s)_{t,\max} \right. \\ \left. - 2\beta(c_v \gamma_1 l + \rho v_s \gamma_2) \right\}. \end{aligned}$$

From (43), it is concluded that the positive definite functional  $V(t)$  in (23) is non-increasing and is a Lyapunov function, because  $\dot{V}(t)$  is negative definite. Hence, all the signals in the closed loop system are bounded.

In the boundary force control law (39), the velocity  $w_t(l)$  and slope rate  $w_{xt}(l)$  at  $x = l$  are involved. The slope rate  $w_{xt}(l)$  can be implemented by backward differentiation of the signal from an encoder, which measures the slope  $w_x(l)$ <sup>(16)</sup>.

#### 4. Stability Analysis

In this section, the asymptotic and, furthermore, exponential stability of the axially moving strip under

boundary control law (39) are proven. The substitution of (39) into (7) yields:

$$m_c w_{tt}(l) + \rho v_s w_t(l) + (T_s(l) - \rho v_s^2) w_x(l) - EI w_{xxx}(l) + m_c \alpha^{-1} (\alpha v_s + 2\beta l) w_{xt}(l) = 0. \quad (44)$$

In order to analyze the asymptotic stability of the closed loop system (4)–(6) and (44), the state space  $\mathfrak{J}$  is defined as follows:

$$\mathfrak{J} \triangleq \{(w, \dot{w}, w(l), w_t(l))^T \mid w \in H_L^2, \dot{w} \in L^2, \text{ and } w(l), w_t(l) \in R\}, \quad (45)$$

where the superscript  $T$  stands for transpose. The spaces  $L_2$  and  $H_L^k$  are defined as follows:

$$L^2 \triangleq \left\{ f : [0, l] \rightarrow R \mid \int_0^l f^2 dx < \infty \right\}, \quad (46)$$

$$H_L^k \triangleq \{f \in L^2 \mid f', f'', \dots, f^{(k)} \in L^2, \text{ and } f(0) = 0\}. \quad (47)$$

In the space  $\mathfrak{J}$ , the inner-product is defined as follows:

$$\begin{aligned} \langle z, \hat{z} \rangle_{\mathfrak{J}} &\triangleq \frac{\alpha}{2} \int_0^l (T_s w_x \hat{w}_x + EI w_{xx} \hat{w}_{xx}) dx + \frac{\alpha}{2} \int_0^l \rho \dot{w} \hat{w} dx \\ &+ \beta \int_0^l \rho x (w_x + \dot{w})(\hat{w}_x + \hat{w}) dx \\ &- \beta \int_0^l \rho x w_x \hat{w}_x dx - \beta \int_0^l \rho x \dot{w} \hat{w} dx \\ &+ \frac{1}{2} m_c \{ (\alpha v_s + 2\beta l) w_x(l) + \alpha w_t(l) \\ &\times [(\alpha v_s + 2\beta l) \hat{w}_x(l) + \alpha \hat{w}_t(l)] \}, \\ &= \frac{\alpha}{2} \int_0^l (T_s w_x \hat{w}_x + EI w_{xx} \hat{w}_{xx}) dx + \frac{\alpha}{2} \int_0^l \rho \dot{w} \hat{w} dx \end{aligned}$$

$$\begin{aligned} &+ \beta \int_0^l \rho x w_x \hat{w} dx + \beta \int_0^l \rho x \dot{w} \hat{w}_x dx \\ &+ \frac{1}{2} m_c \{ (\alpha v_s + 2\beta l) w_x(l) + \alpha w_t(l) \} \{ (\alpha v_s \\ &+ 2\beta l) \hat{w}_x(l) + \alpha \hat{w}_t(l) \}, \end{aligned} \quad (48)$$

where  $z = (w, \dot{w}, w(l), w_t(l))^T$ ,  $\hat{z} = (\hat{w}, \hat{w}, \hat{w}(l), \hat{w}_t(l))^T \in \mathfrak{J}$ , and  $\alpha, \beta > 0$ . The norm induced by inner-product (48) is equivalent to  $V(t)$  in (23), i.e.,

$$\begin{aligned} V(t) &= \langle z, z \rangle_{\mathfrak{J}} = \|z(t)\|_{\mathfrak{J}}^2 \\ &= \frac{\alpha}{2} \int_0^l \rho \dot{w}^2 dx + \frac{\alpha}{2} \int_0^l T_s w_x^2 dx + \frac{\alpha}{2} \int_0^l EI w_{xx}^2 dx \\ &+ 2\beta \int_0^l \rho x w_x \dot{w} dx + \frac{1}{2} m_c \{ \alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l) \}^2. \end{aligned} \quad (49)$$

By using

$$\ddot{w} = \frac{d^2}{dt^2} w = \frac{d}{dt} (w_t + v_s w_x) = w_{tt} + 2v_s w_{xt} + v_s^2 w_{xx},$$

the closed loop system (4)–(6) and (44) can be rewritten in the following abstract form.

$$\dot{z} = A(t)z, \quad z(0) \in \mathfrak{J}, \quad (50)$$

where  $z = (w, \dot{w}, w(l), w_t(l))^T \in \mathfrak{J}$ . Let

$$D \triangleq \{(w, \dot{w}, w(l), w_t(l))^T \mid w \in H_L^4, \dot{w} \in H_L^2, \text{ and } w(l), w_t(l) \in R\}, \quad (51)$$

and define the family of operators  $\{A(t)\}_{t \geq 0}$ ,  $A : D \subset \mathfrak{J} \rightarrow \mathfrak{J}$  by

$$A(t) \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ \rho^{-1} \left\{ \frac{\partial}{\partial x} \left( T_s \frac{\partial}{\partial x} \right) - EI \frac{\partial^4}{\partial x^4} \right\} & -\rho^{-1} c_v & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -m_c^{-1} \left\{ (T_s - \rho v_s^2) \frac{\partial}{\partial x} - EI \frac{\partial^3}{\partial x^3} \right\}_{x=l} & -\frac{1}{\alpha} (\alpha v_s + 2\beta l) \frac{\partial}{\partial x} \Big|_{x=l} & 0 & -m_c^{-1} \rho v_s \end{bmatrix}. \quad (52)$$

Since the time derivative of  $V(t)$  is negative definite as shown in (43),  $A(t)$  given by (51) is dissipative in the space  $\mathfrak{J}$  because of  $\dot{V}(t) = \langle \dot{z}, z \rangle_{\mathfrak{J}} = \langle A(t)z, z \rangle_{\mathfrak{J}} \leq 0$  for  $V(t) = \|z(t)\|_{\mathfrak{J}}^2$  using (49) and  $\dot{z} = A(t)z$ . This implies that  $\{A(t)\}_{t \geq 0}$  is a stable family of infinitesimal generators of  $C_0$  semigroups on  $\mathfrak{J}$ . Further, the map  $t \rightarrow A(t)z$  for  $z \in D$  is strongly continuously differentiable in  $\mathfrak{J}$ . Thus, it follows from Theorem 4.8 in Pazy<sup>(25)</sup> that there exists a unique evolution system,  $\{S(t, s) : 0 \leq s \leq t\}$ , on  $\mathfrak{J}$  associated with the homogeneous system corresponding to (50). That is, the system (50) admits a unique solution  $z(t)$  such that

$$z(t) = S(t, 0)z_0, \quad t \geq 0. \quad (53)$$

#### 4.1 Asymptotic stability

The closed loop system (50) is shown to be asymptotically stable by applying the invariance principle<sup>(26)</sup> in the state space  $\mathfrak{J}$ .

**Theorem 1.** Consider the system (50) representing (4)–(6) and (44). Suppose that the following hold:

$$\alpha > \max \left( \frac{2\beta \rho l}{T_{s, \min}}, 2\beta l \right), \quad T_{s, \min} > \rho, \quad (3\alpha v_s - \beta l) > 0,$$

$$\begin{aligned} & \{\beta l T_s(l) - (\alpha v_s + 3\beta l) \rho v_s^2\} > 0, \\ & \left( \alpha c_v + \beta \rho - \frac{2\beta c_v l}{\gamma_1} - \frac{2\beta \rho v_s}{\gamma_2} \right) > 0, \text{ and} \\ & \left\{ \beta T_{s,\min} - \left( \beta l + \frac{\alpha v_s}{2} \right) (T_s)_{x,\max} - \frac{\alpha}{2} (T_s)_{t,\max} \right. \\ & \quad \left. - 2\beta (c_v \gamma_1 l + \rho v_s \gamma_2) \right\} > 0. \end{aligned}$$

Then, the solution  $z(t)$  of system (50) is asymptotically stable, i.e.,  $z(t) \rightarrow 0$  as  $t \rightarrow \infty$ , by boundary force control law (39).

**Proof.**  $V(t)$  in (49) is a Lyapunov function on  $\mathfrak{J}$  because  $\dot{V}(t) \leq 0$  for all  $z \in \mathfrak{J}$  as shown in (43). Since the tension  $T_s(x, t)$  is assumed to be spatiotemporally varying but periodic in time, the system (50) is a periodic system. Thus, a set  $B = \{z \in \mathfrak{J} | \dot{V}(t) = 0\}$  with  $B_m$ , where  $B_m$  is the maximal invariant subset of  $B$ , can be defined<sup>(26)</sup>. Then  $z(t) = S(t, 0)z_0 \rightarrow B_m$  as  $t \rightarrow \infty$  for all  $z_0 \in \mathfrak{J}$ , i.e., any solution of system (50) asymptotically approaches to the maximal invariant set  $B_m$ . Any element of  $B$ , except zero solution, cannot be invariant, because waves of a nonzero solution propagate to  $x = l$  with the speed  $v_s$ , by the characteristics of wave equation. Therefore, the maximal invariant set includes only the origin,  $B_m = \{0\}$ , and  $z(t)$  is asymptotically stabilized by boundary control law (39) at  $x = l$ .

**4.2 Exponential stability**

To verify that right boundary force control law (39) induces a rate of exponential decay of  $V(t)$ , the following theorem is proved using a semigroup of closed-loop system (50).

**Theorem 2.** Consider system (50) as Theorem 1. Suppose that there exist

$$\begin{aligned} \alpha & > \max \left( \frac{2\beta \rho l}{T_{s,\min}}, 2\beta l \right), \quad T_{s,\min} > \rho, \quad (3\alpha v_s - \beta l) > 0, \\ & \{\beta l T_s(l) - (\alpha v_s + 3\beta l) \rho v_s^2\} > 0, \\ & \left( \alpha c_v + \beta \rho - \frac{2\beta c_v l}{\gamma_1} - \frac{2\beta \rho v_s}{\gamma_2} \right) > 0, \text{ and} \\ & \left\{ \beta T_{s,\min} - \left( \beta l + \frac{\alpha v_s}{2} \right) (T_s)_{x,\max} \right. \\ & \quad \left. - \frac{\alpha}{2} (T_s)_{t,\max} - 2\beta (c_v \gamma_1 l + \rho v_s \gamma_2) \right\} > 0. \end{aligned}$$

Then, the Lyapunov function  $V(t)$  given by (23) decays exponentially to zero along the solution of system (50). That is, there exist constants  $\mu > 0$  and  $M > 0$  such that

$$V(t) \leq M e^{-\mu t}, \quad t \geq 0.$$

**Proof.** There exists a constant  $\sigma > 0$  such that

$$\begin{aligned} & \int_0^l (w_t + v_s w_x)^2 dx + \int_0^l w_x^2 dx \\ & + \int_0^l w_{xx}^2 dx + w_t^2(l) + w_x^2(l) \end{aligned}$$

$$\begin{aligned} & \geq \sigma \left[ \frac{\alpha}{2} \int_0^l \rho (v_s w_x + w_t)^2 dx + \frac{\alpha}{2} \int_0^l T_s w_x^2 dx \right. \\ & \left. + \frac{\alpha}{2} \int_0^l E I w_{xx}^2 dx + \frac{1}{2} m_c (\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l))^2 \right] \\ & = \sigma \alpha V_0(t). \end{aligned} \tag{54}$$

From (24) and (54), (43) can be rewritten as

$$\begin{aligned} \dot{V}(t) & \leq -c_0 \sigma \alpha V_0(t) \\ & \leq -c_0 \sigma \left[ \alpha V_0(t) - \beta \int_0^l \rho x w_x^2 dx \right. \\ & \quad \left. - \beta \int_0^l \rho x (w_t + v_s w_x)^2 dx \right. \\ & \quad \left. + \beta \int_0^l \rho x \{w_x + (w_t + v_s w_x)\}^2 dx \right] \\ & + c_0 \sigma \beta \int_0^l \rho x \{w_x + (w_t + v_s w_x)\}^2 dx \\ & \leq -c_0 \sigma V(t) + c_0 \sigma \beta \int_0^l \rho x \{w_x + (w_t + v_s w_x)\}^2 dx. \end{aligned} \tag{55}$$

Note that from Theorem 1, the closed loop system (50) is asymptotically stable, i.e.,  $\lim_{t \rightarrow \infty} \int_0^l \{w_x + (w_t + v_s w_x)\}^2 dx =$

0, and then  $\int_0^\infty \int_0^l x \{w_x + (w_t + v_s w_x)\}^2 dx dt < \infty$ .

Thus, from (49) and (53), the integration of both sides of (55) from 0 to  $\infty$  yields:

$$\begin{aligned} \int_0^\infty V(t) dt & = \int_0^\infty \|z(t)\|_{\mathfrak{J}}^2 dt = \int_0^\infty \|S(t, 0)z_0\|_{\mathfrak{J}}^2 dt \\ & \leq \frac{1}{c_0 \sigma} (V(0) - V(\infty)) \\ & + \beta \int_0^\infty \int_0^l \rho x \{w_x + (w_t + v_s w_x)\}^2 dx dt < \infty. \end{aligned} \tag{56}$$

By (56) and Theorem 4.1 in Pazy<sup>(25)</sup>, there exist  $M_1 \geq 1$  and  $\mu_1 > 0$  such that

$$\|S(t, 0)\|_{\mathfrak{J}} \leq M_1 e^{-\mu_1 t}. \tag{57}$$

The application of (57) to (53) gives

$$\|z(t)\|_{\mathfrak{J}} \leq M_1 \|z_0\|_{\mathfrak{J}} e^{-\mu_1 t}, \tag{58}$$

proving the exponential stability of the solution of system (50). Therefore, the Lyapunov function  $V(t)$  also decays exponentially such that

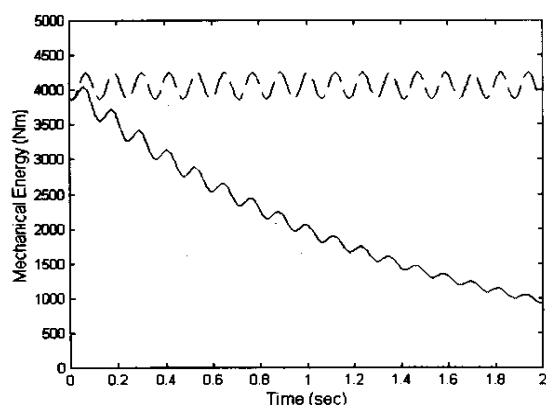
$$V(t) = \|z(t)\|_{\mathfrak{J}}^2 \leq M_1^2 \|z_0\|_{\mathfrak{J}}^2 e^{-2\mu_1 t} \leq M e^{-\mu t}, \tag{59}$$

where  $M = M_1^2 \|z_0\|_{\mathfrak{J}}^2$  and  $\mu = 2\mu_1$ .

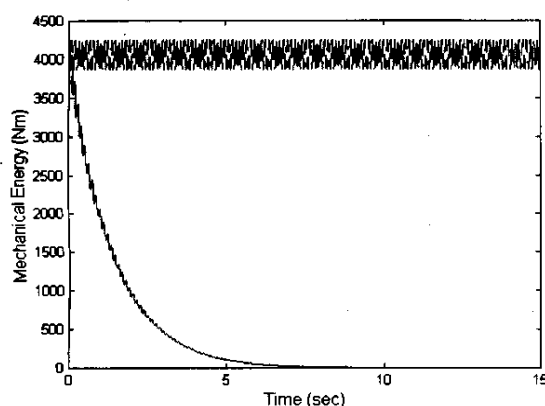
**5. Numerical Simulations**

The effectiveness of the proposed boundary control law is illustrated by numerical simulations using a finite difference method. An implicit difference operator is used because the fourth order term  $w_{xxxx}(x, t)$  requires a small time increment in an explicit method<sup>(17)</sup>.





(a) During 1 to 2 seconds



(b) During 1 to 15 seconds

Fig. 3 Comparison of the decays of mechanical energy (11) of the system with boundary force controller (39) (solid line) and the system with no control (dashed line):  $\alpha = 1$ ,  $\beta = 0.01$

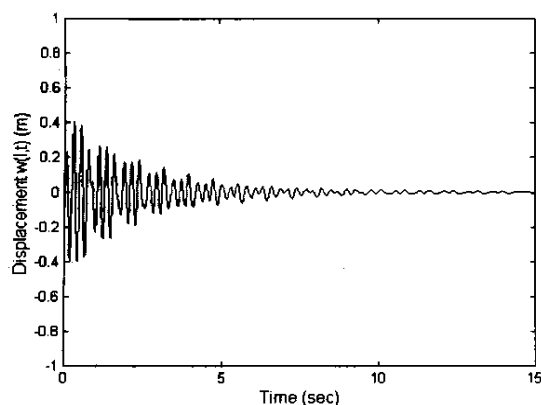


Fig. 4 The transverse displacement at the right boundary,  $w(l,t)$

The parameters of the axially moving beam used for numerical simulations are:  $\rho = 2.7 \text{ kg/m}$ ,  $EI = 146 \text{ Nm}^2$ ,  $c_v = 0.001 \text{ Nm}^2\text{sec}$ ,  $l = 35 \text{ m}$ ,  $v_s = 1 \text{ m/sec}$ ,  $m_c = 10 \text{ kg}$ ,  $d_c = 0.25 \text{ N/m/sec}$ , and  $T_s = 26000 + 9.8\rho x + \sin t \text{ N}$ . The initial conditions are  $w_0 = 0.5 \sin 3\pi x$  and  $w_{t0} = 0$ .

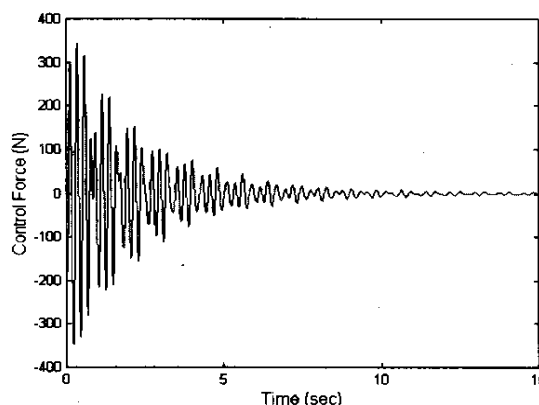


Fig. 5 The control input force used:  $f_c(t)$

The controller parameters are selected as  $\alpha = 1$ ,  $\beta = 0.01$ , and  $\gamma_1 = \gamma_2 = 10$ , from which the conditions in Theorem 1 and Theorem 2 are satisfied, i.e.,

$$\alpha > \max\left(\frac{2\beta\rho l}{T_{s,\min}}, 2\beta l\right) = 0.7,$$

$$\left(\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1} - \frac{2\beta\rho v_s}{\gamma_2}\right) = 0.0225 > 0,$$

$$T_{s,\min} > \rho, (3\alpha v_s - \beta l) = 2.65 > 0,$$

$$\{\beta l T_s(l) - (\alpha v_s + 3\beta l)\rho v_s^2\} > 9400 > 0, \text{ and}$$

$$\left\{\beta T_{s,\min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,\max}\right.$$

$$\left. - \frac{\alpha}{2}(T_s)_{r,\max} - 2\beta(c_v\gamma_1 l + \rho v_s\gamma_2)\right\} = 236.4 > 0,$$

where  $T_{s,\min} = 25999 \text{ N}$ ,  $(T_s)_{x,\max} = 9.8\rho \text{ N}$ , and  $(T_s)_{r,\max} = 1 \text{ N}$ .

A comparison of the mechanical energies,  $E_m(t)$  of (11), with boundary force control and without any control action is shown in Fig. 3. As analyzed in section 4, the initial vibration energy dissipates exponentially with the control action. However, the strip vibrations without any control remain almost at the same level. Figure 4 shows the displacement at the right boundary,  $w(l,t)$ . Figure 5 depicts the control force used in getting Fig. 4.

## 6. Conclusions

In this paper, a boundary control scheme to suppress the transverse vibration of an axially moving steel strip with a spatiotemporally varying tension in the hot-dip galvanizing line has been investigated. The following findings are concluded:

(i) In the case of a traveling strip, even if the displacements, velocities, and slopes at both boundaries are zero, the bending moment occurring at the right boundary,  $x = l$ , and the time rate of the change of tension  $T_s(x,t)$  can cause an increase of the total mechanical energy.

(ii) A boundary force control can achieve the suppression of the vibration of the entire strip in the presence of a spatiotemporally varying tension. The exponen-

tial stability of the closed loop system with the boundary control law proposed has been proved with the Lyapunov method and semi-group theory.

(iii) Since the feedback terms in the boundary control law are the velocity and slope rate at the right boundary, the vibration suppression of the axially moving strip can be successfully implemented by the boundary control law.

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