Exponential Stabilization of an Axially Moving Tensioned Strip by Passive Damping and Boundary Control

JI-YUN CHOI
KEUM-SHIK HONG
KYUNG-JINN YANG
School of Mechanical Engineering, Pusan National University, 30 Jangjeon-dong Gumjeong-gu, Busan 609-735, Korea

(Received 28 December 2002; accepted 6 October 2003)

Abstract: In this paper, we investigate an active vibration control of a translating tensioned steel strip in the zinc galvanizing line. The dynamics of the moving strip is modeled as a Euler–Bernoulli beam with non-linear tension. The control objective is to suppress the transverse vibrations of the strip via boundary control. A right boundary control law based upon the Lyapunov second method is derived. It is revealed that a time-varying boundary force and a suitable passive damping at the right boundary can successfully suppress the transverse vibrations. The exponential stability of the closed-loop system is proved. The effectiveness of the control laws proposed is demonstrated via simulations.

Key Words: Axially moving system, exponential stability, boundary control, non-linear hyperbolic partial differential equation, Lyapunov method

1. INTRODUCTION

The control problem of axially moving systems occurs in various engineering areas: for example, the strips in thin metal-sheet production lines, the cables, belts, and chains in power transmission lines, the magnetic tapes in recorders, band saws, etc. The dynamics of these systems can be modeled differently depending on the length, flexibility, and control objectives of the system considered. For instance, the dynamics of a moving cable of an elevator can be described by a string equation, but that of a rubber belt in a traditional mill can be well represented by a belt equation. The difference between a string and a belt lies in whether the longitudinal elongation is considered or not.

In axially moving systems, the transverse vibration of the moving material often causes a serious problem in achieving good quality. It is also known that these vibrations are often caused by the eccentricity of a pulley, and/or an irregular speed of the driving motor, and/or a non-uniform material property, and/or environmental disturbances. Because the quality requirement as well as the productivity in a production line is becoming stricter, an active or a semi-active vibration control is nowadays seriously considered.

Journal of Vibration and Control, **10:** 661–682, 2004 DOI: 10.1177/1077546304038103

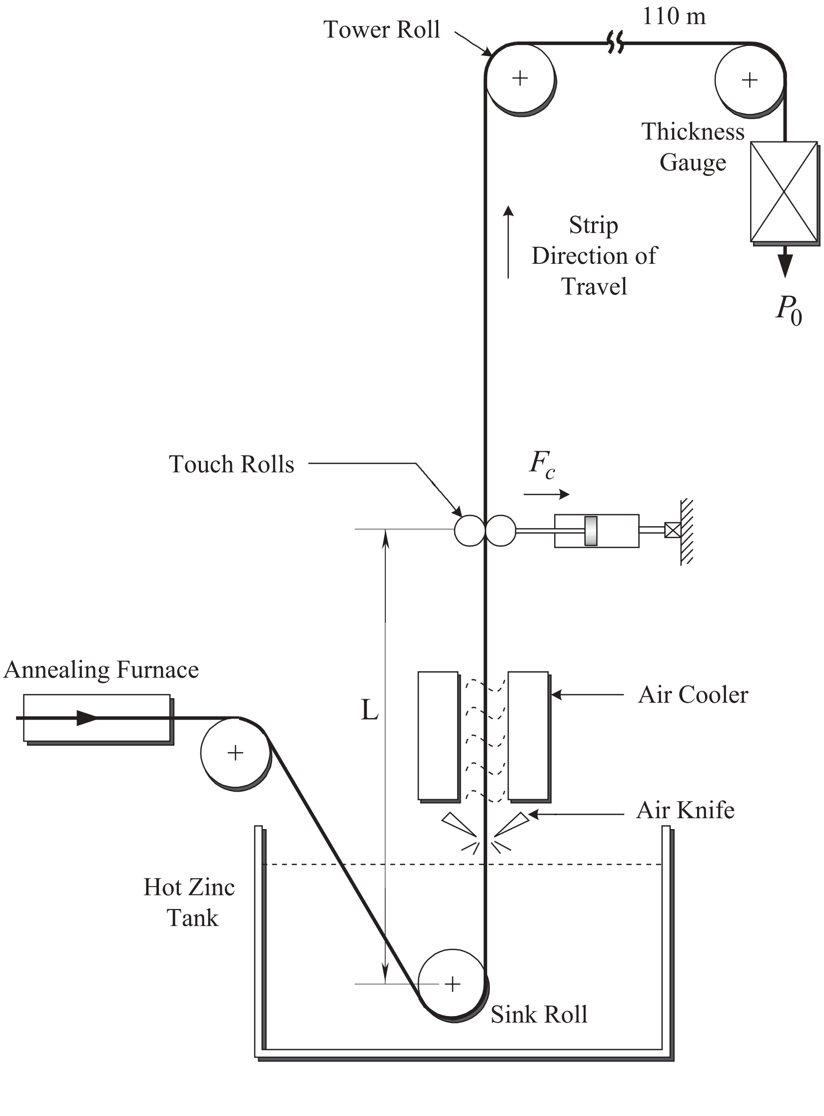

Figure 1. An axially moving steel strip in the zinc galvanizing line.

Figure 1 depicts a continuous hot-dip zinc galvanizing process with an active vibration control. The steel strip, with width varying from 800 to 1400 mm and thickness varying from 1.2 to 4.5 mm, is pre-heated in a continuous annealing furnace and then introduced at a speed around 1 m s^{-1} into the pot of molten zinc at about 450°C. The steel strip passes under the sink roll and rises vertically up, coated with a layer of zinc, from the pot. The thickness of the zinc film is controlled by a pair of air knives located about 0.5 m above the surface of the zinc tank, which direct a long thin wedge-shaped air jet toward the strip, and strip out excess zinc back to the pot. Hence, maintaining a constant gap between the strip and air knives, i.e. keeping equidistance from both air knives, is the crux to achieving the uniform thickness of zinc coating (Chen, 1995; Hong et al. 2004; Yang and Hong, 2002).

The lateral (transversal) vibration of the strip occurs during the process for various reasons. The eccentricity of the sink roll is known to be the main cause of the vibration. This lateral vibration then changes the equidistance from two air knives and therefore the thickness of the deposit will fluctuate. Various gap control methods have been applied at Pohang Steel and Iron Company, Ltd, in Korea, but a long-term successful implementation has not yet been reported. This is mainly because of the harsh and high-temperature environment near the zinc tank.

Another important aspect of the lateral vibration control is that by using vibration control the maintenance interval of the entire production line can be extended. It is also known that the eccentricity of the sink roll is caused by the wear of the copper bushing (bearing) in the sink roll. Therefore, the entire production line has to be halted frequently for the replacement of a new bushing. Hence, the necessity of an active/semi-active vibration control is fully justified from two objectives: obtaining a uniform thickness of the zinc deposit and extending the maintenance interval for increasing the line productivity.

How to model an axially moving system, i.e. as a string equation, a belt equation or a beam equation, depends on the structure of the plant and control schemes. The plant in this paper is the steel strip between the sink roll and the tower roll in Figure 1, which is 35 m in length and 1.2–4.5 mm in thickness. Therefore, it could be modeled as a string, or a belt, or a beam depending on where the actuator is actually inserted and whether the axial deformation is considered or not.

In the literature, there has been diverse research on the dynamics, stability, and/or active/passive controls for axially moving systems (Carrier, 1945; Bapat and Srinivasan, 1967; Wickert and Mote, 1990; Wickert, 1992; Oshima et al., 1997; Pellicano and Zirilli, 1998; Shahruz, 1998, 2000; Oostveen and Curtain, 2000). Particularly, Mote (1965) modeled the dynamics of a band saw, as an axially moving string, and investigated its instability with respect to the moving speed and excitation frequency of the saw. Wickert and Mote (1988) reported on a passive control strategy, by changing its damping and stiffness, for axially moving continua. Morgul (1992) investigated a boundary control law that suppresses the lateral vibration of a Euler–Bernoulli beam, but in his work the beam was not axially moving. Laousy et al. (1996) investigated a boundary feedback stabilization method for a rotating body–beam system. Lee and Mote (1996) demonstrated an optimal boundary force control law that dissipates the vibration energy of an axially moving string. Fung et al. (1999a, 1999b) reported on boundary control laws for linear and non-linear strings, in which the dynamics of the actuator has been incorporated in the control law design. An optimal control (Fung et al., 2002a) and an adaptive control (Fung et al., 2002b) for an axially moving string were also investigated. For a translating linear beam, Lee and Mote (1999) investigated the wave characteristics and derived boundary control laws in terms of linear velocity, linear slope, and linear force. Li and Rahn (2000) investigated an adaptive vibration control for an axially moving linear beam by splitting the moving part into two subsystems, i.e. a controlled part and an uncontrolled part. Li et al. (2002) applied the control strategy in Li and Rahn (2000) to the linear string including experimental results. Fard and Sagatun (2001) focused on the exponential stabilization of a non-linear beam, not axially moving, by boundary control. Interesting results on energy-based control are also found in Ge et al. (2000, 2001) and Zhu and Ge (1998).

All previous works were limited either to non-linear non-axially moving systems or to linear axially moving systems. The contributions of this paper are as follows. First, an axially

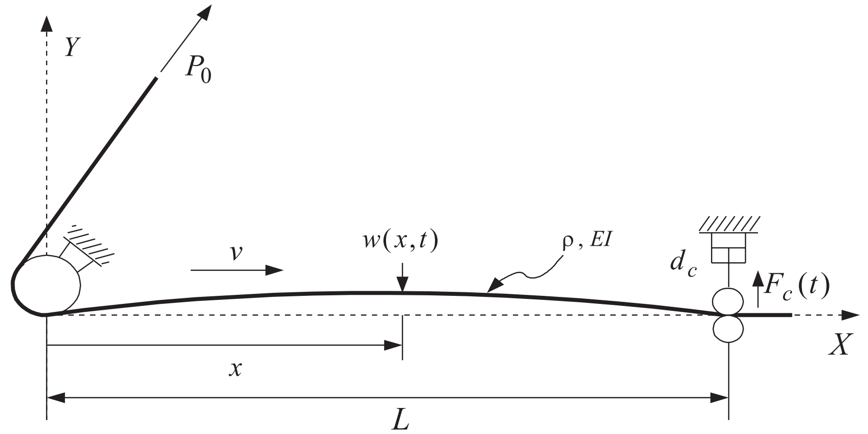

Figure 2. An axially moving strip under the right boundary control force.

moving non-linear beam equation is considered for the first time. Focusing on the vibration suppression near the air knives and assuming that the controlled part in Li and Rahn (2000) is relatively small, a non-linear beam model is adopted. Secondly, the actuator dynamics is also incorporated in the control law design. Thirdly, the derived boundary control law utilizes two pieces of information: the strip slope at the right boundary and the damping coefficient of the actuator. Hence, once the damping coefficient is properly estimated in an actuator design stage using the parameter values of the system, the final control law depends only on the slope measurement. Therefore, the use of a single slope sensor enables the implementation of the control law. Finally, the exponential stability of the closed-loop system is established.

The paper is structured as follows. In Section 2, we derive the non-linear beam equations of motion using the Hamilton principle of changing mass. In Section 3, we derive a stabilizing boundary control law that suppresses the transverse vibrations of the beam. The exponential stability of the closed-loop system is proved. In Section 4, we discuss the implementation issue of the control law derived. Simulation results are given in Section 5. Finally, Section 6 concludes the paper.

2. EQUATIONS OF MOTION

Figure 1 shows the axially moving steel strip in the zinc galvanizing line, which emerges from a hot-dip zinc pot and moves vertically upward. The distance from the sink roll to the top roll is about 35 m long. It is assumed that the controlled portion, in the sense of Li and Rahn (2000), of the strip is smaller than that of the uncontrolled portion. In other words, the touch rolls cannot be inserted too far away from the sink roll for the strip to be modeled as a beam. Figure 2 shows a schematic diagram of the plant for analyzing the dynamics and deriving a boundary control law. The strip is assumed to travel at a constant speed. The left boundary (the sink roll) is fixed, i.e. the left boundary itself does not have any vertical or longitudinal displacements, but allows the longitudinal movement of the strip. The right boundary (the touch rolls) allows the transverse displacement under a control force.

Let t be the time, let x be the spatial coordinate along the longitude of motion, let v be the axial speed of the strip, let $w(x,t)$ be the transversal displacement of the strip at spatial coordinate x and time t, and let L be the length of the strip. Then, the absolute velocity at spatial coordinate x becomes

$$\vec{v} = vi + \frac{\mathrm{d}w(x,t)}{\mathrm{d}t}j = vi + \{w_t(x,t) + vw_x(x,t)\}j, \tag{1}$$

where $(\cdot)_t = \partial(\cdot)/\partial t$ and $(\cdot)_x = \partial(\cdot)/\partial x$ denote the partial derivatives and $v = \partial x/\partial t$ has been used. Now, to derive the equations of motion, the Hamilton principle for systems of changing mass (McIver, 1973) is utilized as follows

$$\delta \int_{t_1}^{t_2} (T - U + W_{n.c.} + W_{r.b.})\,\mathrm{d}t = 0, \tag{2}$$

where T is the kinetic energy, U is the strain energy, $W_{n.c.}$ is the non-conservative work, and $W_{r.b.}$ is the virtual momentum transport at the right boundary (no variation at the left boundary). The kinetic energy is

$$T = \frac{\rho A}{2}\int_0^L \left\{v^2 + (w_t + vw_x)^2\right\}\mathrm{d}x + \frac{1}{2}mw_t^2(L,t), \tag{3}$$

where ρ is the mass per unit length, A is the cross-sectional area, and m is the mass of the actuator. The potential energy is

$$U = \int_0^L \left\{P_0\varepsilon_x + \frac{EA}{2}\varepsilon_x^2 + \frac{EI}{2}w_{xx}^2\right\}\mathrm{d}x, \tag{4}$$

where E is the coefficient of elasticity, P_0 is a constant axial tension of the strip, I is the moment of inertia of the beam cross-section, and ε_x is the strain. The first term in equation (4) is due to the strip tension, the second term reflects the strain energy due to disturbances, and the last term is from the bending moment. If the infinitesimal distance $\mathrm{d}x$ is replaced by the infinitesimal length $\mathrm{d}s$, the strain ε_x can be approximated as $\varepsilon_x \cong w_x^2/2$ (Benaroya, 1998). Then

$$U = \int_0^L \left\{\frac{P_0}{2}w_x^2 + \frac{EA}{8}w_x^4 + \frac{EI}{2}w_{xx}^2\right\}\mathrm{d}x. \tag{5}$$

The variations of equations (3) and (5) are

$$\delta T = \rho A\int_0^L (w_t + vw_x)(\delta w_t + v\delta w_x)\,\mathrm{d}x + mw_t\delta w_t(L,t), \tag{6}$$

$$\delta U = \int_0^L \left\{P_0 w_x\delta w_x + \frac{EA}{2}w_x^3\delta w_x + EIw_{xx}\delta w_{xx}\right\}\mathrm{d}x. \tag{7}$$

Also, the variations of the non-conservative work and the virtual momentum transport at the right boundary are

$$\delta W_{n.c.} = F_c \delta w(L,t) - d_c w_t(L,t)\,\delta w(L,t), \tag{8}$$

$$\delta W_{r.b.} = -\rho A v \{w_t(L,t) + v w_x(L,t)\}\,\delta w(L,t), \tag{9}$$

where d_c is the damping coefficient of the actuator and $F_c(t)$ is the control force.

The substitution of equations (6)–(9) into equation (2) yields the non-linear equation of motion as follows:

$$\rho A\left(w_{tt} + 2v w_{xt} + v^2 w_{xx}\right) - \left(P_0 + \frac{3EA}{2} w_x^2\right) w_{xx} + EI w_{xxxx} = 0. \tag{10}$$

The boundary conditions are

$$w(0,t) = 0, \quad w_x(0,t) = 0, \quad w_{xx}(L,t) = 0, \quad \text{and} \tag{11}$$

$$m w_{tt}(L,t) + d_c w_t(L,t) + P_0 w_x(L,t) + \frac{EA}{2} w_x^3(L,t) - EI w_{xxx}(L,t) = F_c(t). \tag{12}$$

Equation (10) is a non-linear partial differential equation representing the transverse motion, where $3EAw_x^2/2$ is the non-linearity, which is again due to $EAw_x^4/8$ in equation (5). Note that $(P_0 + 3EAw_x^2/2)$ is often called as a non-linear tension (Qu, 2002). Note also that equation (12) is an ordinary differential equation relating the strip motion at the right boundary and the control force.

Remark 1. Without $EA\varepsilon_x^2/2$ in equation (4), the following linear beam equation would have been derived (Lee and Mote, 1999):

$$\rho A\left(w_{tt} + 2v w_{xt} + v^2 w_{xx}\right) - P_0 w_{xx} + EI w_{xxxx} = 0. \tag{13}$$

Lee and Mote (1999) revealed that the strip moving speed v, to avoid a divergence of the solution, should be smaller than some critical speed given by

$$0 < v < v_{cr} = \sqrt{\frac{P_0}{\rho A}}. \tag{14}$$

Hence, the satisfaction of equation (14) is also assumed in this paper.

3. BOUNDARY CONTROL LAW

In this section, a right boundary control law that suppresses the transverse vibration of the strip governed by equations (10)–(12) is derived.

Let L^2 and $H_{0,l}^k$ be defined as

$$L^2 \triangleq \left\{ f : [0,L] \longrightarrow R \mid \int_0^L f^2 \mathrm{d}x < \infty \right\}, \tag{15}$$

$$H^k_{0,l} \triangleq \left\{ f \in L^2 \left| f', f'', \cdots, f^{(k)} \in L^2, \text{ and } f(0) = 0 \right. \right\}, \tag{16}$$

where the subscript l in H denotes that functions have the left support. Now, the state space Λ, whose first component is the displacement and the second is the velocity, is introduced as follows:

$$\Lambda \triangleq H^2_{0,l} \times L^2 = \left\{ z(t) \triangleq [w(x,t)\ \dot{w}(x,t)]^{\mathrm{T}} \mid w \in H^2_{0,l}, \dot{w} \in L^2 \right\}. \tag{17}$$

From equation (17), the following energy inner product is defined in Λ:

$$\begin{aligned} < z, \tilde{z} >_\Lambda &= < (w, \dot{w}), (\tilde{w}, \dot{\tilde{w}}) >_\Lambda \\ &= \frac{\rho A}{2} \int_0^L \dot{w} \dot{\tilde{w}} \mathrm{d}x + \frac{P_0}{2} \int_0^L w_x \tilde{w}_x \mathrm{d}x \\ &+ \frac{EA}{8} \int_0^L w_x^2 \tilde{w}_x^2 \mathrm{d}x + \frac{EI}{2} \int_0^L w_{xx} \tilde{w}_{xx} \mathrm{d}x. \end{aligned} \tag{18}$$

Note that the Λ space equipped with the energy inner product (18) becomes a Hilbert space (Chen and Zhou, 1993; Matsuno et al., 2002). The energy norm induced by the energy inner product (18) denotes the mechanical energy of the strip as follows:

$$\begin{aligned} E_{strip} &= < z, \tilde{z} >_\Lambda = \|z(t)\|^2_\Lambda \\ &= \frac{\rho A}{2} \int_0^L \dot{w}^2 \mathrm{d}x + \frac{P_0}{2} \int_0^L w_x^2 \mathrm{d}x + \frac{EA}{8} \int_0^L w_x^4 \mathrm{d}x + \frac{EI}{2} \int_0^L w_{xx}^2 \mathrm{d}x \\ &= \frac{\rho A}{2} \int_0^L (w_t + v w_x)^2 \mathrm{d}x + \frac{P_0}{2} \int_0^L w_x^2 \mathrm{d}x \\ &+ \frac{EA}{8} \int_0^L w_x^4 \mathrm{d}x + \frac{EI}{2} \int_0^L w_{xx}^2 \mathrm{d}x. \end{aligned} \tag{19}$$

The following lemmas are then stated.

Lemma 1. The mechanical energy E_{strip} of equation (19) and the following function are equivalent:

$$V_{strip} = \alpha E_{strip} + \beta \rho A \int_0^L x w_x (w_t + v w_x) \mathrm{d}x. \tag{20}$$

That is, there exist constants $\alpha, \beta, \beta_1 > 0$ such that

$$(\alpha - \beta \beta_1) E_{strip} \leq V_{strip} \leq (\alpha + \beta \beta_1) E_{strip}, \tag{21}$$

where

$$\beta < \alpha / \beta_1. \tag{22}$$

Table 1. The plant parameters used for simulations.

Symbol	Definition	Value
A	Cross-section area	1.4×0.0045 m^2
E	Elastic modulus of the steel	2×10^{11} N m^{-2}
L	Length of the controlled part	17.5 m
P_0	Tension of the strip	9800 kN
m	Mass of the actuator	5 kg
v_0	Strip moving speed	1.67 m s^{-1}
ρ	Mass per unit area	7850 kg m^{-2}

Proof. If using $2ab \leq a^2 + b^2$, the following inequalities for equation (20) hold:

$$\begin{aligned} \rho A \int_0^L x w_x \left(w_t + v w_x\right) \mathrm{d}x &\leq \frac{\rho AL}{2}\left[\int_0^L w_x^2 \mathrm{d}x + \int_0^L \left(w_t + v w_x\right)^2 \mathrm{d}x\right] \\ &\leq L\left[\frac{\rho A}{P_0}\cdot\frac{P_0}{2}\int_0^L w_x^2 \mathrm{d}x + \frac{\rho A}{2}\int_0^L \left(w_t + v w_x\right)^2 \mathrm{d}x\right] \\ &\leq L \cdot \max\left\{1, \frac{\rho A}{P_0}\right\} \cdot E_{strip} = \beta_1 \cdot E_{strip}, \end{aligned} \tag{23}$$

where

$$\beta_1 = L \cdot \max\left\{1, \frac{\rho A}{P_0}\right\}. \tag{24}$$

Using the parameters in Table 1, for example, $\beta_1 = 17.5 \times \max\{1,\ 0.00005\} = 17.5$ is obtained. The substitution of equation (23) into equation (20) yields

$$V_{strip}\ (t) \leq \alpha \cdot E_{strip}\ + \beta\beta_1 \cdot E_{strip} = (\alpha + \beta\beta_1)\, E_{strip}. \tag{25}$$

By the same token, the left inequality in equation (21) is achieved.

Now, with Lemma 1, the following Lyapunov function candidate is proposed

$$V(t) = V_{strip}\ + V_{actuator}\ , \tag{26}$$

where

$$\begin{aligned} V_{actuator} &= \frac{m}{\xi}\left\{\xi\, w_t(L,t) + \tau w_x(L,t)\right\}^2, \\ \xi &= \alpha/2 > 0,\ \ \tau = (\alpha v + \beta L)/2 > 0. \end{aligned} \tag{27}$$

The reason for choosing such ξ and τ above will become clear in the following.

Lemma 2. Equation (26) satisfies the following inequalities

$$k_1\left[E_{strip}+\frac{m}{\xi}\left\{\xi\, w_t(L,t)+\tau w_x(L,t)\right\}^2\right]\le V(t)$$
$$\le\ k_2\left[E_{strip}+\frac{m}{\xi}\left\{\xi\, w_t(L,t)+\tau w_x(L,t)\right\}^2\right], \tag{28}$$

where

$$k_1=\min\{\alpha-\beta\beta_1,\ 1\}>0\ \text{ and }\ k_2=\max\{\alpha+\beta\beta_1,\ 1\}>0. \tag{29}$$

Proof. From equations (21) and (27), the following holds:

$$(\alpha-\beta\beta_1)\,E_{strip}+\frac{m}{\xi}\left\{\xi\, w_t(L,t)+\tau w_x(L,t)\right\}^2\le V_{strip}\ +V_{actuator}$$
$$\le\ (\alpha+\beta\beta_1)\,E_{strip}+\frac{m}{\xi}\left\{\xi\, w_t(L,t)+\tau w_x(L,t)\right\}^2.$$

Therefore,

$$\min\{\alpha-\beta\beta_1,\ 1\}\left[E_{strip}+\frac{m}{\xi}\left\{\xi\, w_t(L,t)+\tau w_x(L,t)\right\}^2\right]\le V(t)$$
$$\le\ \max\{\alpha+\beta\beta_1,\ 1\}\left[E_{strip}+\frac{m}{\xi}\left\{\xi\, w_t(L,t)+\tau w_x(L,t)\right\}^2\right] \tag{30}$$

is achieved.

Now, the total derivative (or the material derivative) of equation (26) is evaluated. First, the time derivative of V_{strip} becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_{strip}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \tilde{V}_{strip}(x,t)\mathrm{d}x=\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}\tilde{V}_{strip}(x,t)\mathrm{d}x\\ &= \int_0^L\left[\frac{\partial}{\partial t}\tilde{V}_{strip}(x,t)+\frac{\partial x}{\partial t}\frac{\partial}{\partial x}\tilde{V}_{strip}\ (x,t)\right]\mathrm{d}x\\ &= \frac{\partial}{\partial t}\int_0^L \tilde{V}_{strip}(x,t)\mathrm{d}x+v\tilde{V}_{strip}(x,t)\Big|_0^L\\ &= \frac{\partial}{\partial t}V_{strip}\ +v\frac{\partial}{\partial x}V_{strip},\end{aligned} \tag{31}$$

where

$$\begin{aligned}
\tilde{V}_{strip}(x,t) &= \alpha\left[\frac{\rho A}{2}\left\{v^2+(w_t+vw_x)^2\right\}+\frac{P_0}{2}w_x^2+\frac{EA}{8}w_x^4+\frac{EI}{2}w_{xx}^2\right], \quad (32)\\
\frac{\partial}{\partial t}V_{strip} &= \frac{\partial}{\partial t}\int_0^L \tilde{V}_{strip}(x,t)\mathrm{d}x, \quad \text{and}\\
\frac{\partial}{\partial x}V_{strip} &= \int_0^L \frac{\partial}{\partial x}\tilde{V}_{strip}(x,t)\mathrm{d}x.
\end{aligned}$$

Because the system involves a mass flow entering in and out at the boundaries, the net change of the total energy is the sum of the change in the control volume, i.e. $\frac{\partial}{\partial t}V_{strip}$, and the energy flux at the boundaries, i.e. $v\,\tilde{V}_{strip}\big|_0^L$. Now, equation (31) is calculated as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_{strip} &= \frac{\partial}{\partial t}V_{strip}+v\frac{\partial}{\partial x}V_{strip}\\
&= \alpha\left[\rho A\int_0^L (w_t+vw_x)(w_{tt}+vw_{xt})\,\mathrm{d}x\right.\\
&+ EI\int_0^L w_{xx}w_{xxt}\,\mathrm{d}x+P_0\int_0^L w_xw_{xt}\,\mathrm{d}x\\
&+ \left.\frac{EA}{2}\int_0^L w_x^3w_{xt}\,\mathrm{d}x\right]+\beta\rho A\int_0^L [xw_{xt}(w_t+vw_x)\\
&+ xw_x(w_{tt}+vw_{xt})]\,\mathrm{d}x+\alpha\left[\rho A\int_0^L (w_t+vw_x)(w_{tx}+vw_{xx})\,\mathrm{d}x\right.\\
&+ \left.EI\int_0^L w_{xx}w_{xxx}\,\mathrm{d}x+P_0\int_0^L w_xw_{xx}\,\mathrm{d}x+\frac{EA}{2}\int_0^L w_x^3w_{xx}\,\mathrm{d}x\right]\\
&+ \beta\rho A\int_0^L \left[w_xw_t+vw_x^2+xw_{xx}(w_t+vw_x)\right.\\
&+ xw_x(w_{xt}+vw_{xx})]\,\mathrm{d}x. \qquad (33)
\end{aligned}$$

Using equation (10), equation (33) can be written as

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}V_{strip}\right|_{A104} &= \alpha\left[\rho A\int_0^L (w_t+vw_x)\left(w_{tt}+2vw_{xt}+v^2w_{xx}\right)\mathrm{d}x\right.\\
&+ EI\int_0^L (w_{xx}w_{xxt}+vw_{xx}w_{xxx})\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+ \quad P_0 \int_0^L \left(w_x w_{xt} + v w_x w_{xx}\right) \mathrm{d}x + \frac{EA}{2} \int_0^L \left(w_x^3 w_{xt} + v w_x^3 w_{xx}\right) \Bigg] \\
&+ \quad \beta \rho A \int_0^L \left(x w_{xt} w_t + v x w_{xt} w_x + v^2 x w_{xx} w_x + v x w_{xx} w_t\right) \mathrm{d}x \\
&+ \quad \beta \rho A \int_0^L \left(w_x w_t + v^2 w_x^2\right) \mathrm{d}x \\
&+ \quad \beta \rho A \int_0^L x w_x \left(w_{tt} + 2v w_{xt} + v^2 w_{xx}\right) \mathrm{d}x \\
&= \quad \alpha \left[\int_0^L \left(w_t + v w_x\right) \left\{\left(P_0 + \frac{3EA}{2} w_x^2\right) w_{xx} - EI w_{xxxx}\right\} \mathrm{d}x\right. \\
&+ \quad EI \int_0^L w_{xx} w_{xxt} \, \mathrm{d}x + EIv \int_0^L w_{xx} w_{xxx} \, \mathrm{d}x \\
&+ \quad P_0 \int_0^L w_x w_{xt} \, \mathrm{d}x + P_0 v \int_0^L w_x w_{xx} \, \mathrm{d}x + \frac{EA}{2} \int_0^L w_x^3 w_{xt} \, \mathrm{d}x \\
&+ \quad \left.\frac{EAv}{2} \int_0^L w_x^3 w_{xx} \, \mathrm{d}x\right] + \beta \rho A \int_0^L x w_{xt} w_t \mathrm{d}x \\
&+ \quad \beta \rho A \int_0^L x v^2 w_x w_{xx} \, \mathrm{d}x + \beta \rho A \int_0^L v^2 w_x^2 \mathrm{d}x \\
&+ \quad \beta \rho A \int_0^L \left(x v w_{xt} w_x + x v w_{xx} w_t + v w_x w_t\right) \mathrm{d}x \\
&+ \quad \beta P_0 \int_0^L x w_x w_{xx} \, \mathrm{d}x + \frac{3\beta EA}{2} \int_0^L x w_x^3 w_{xx} \, \mathrm{d}x \\
&- \quad \beta EI \int_0^L x w_x w_{xxxx} \, \mathrm{d}x. \qquad (34)
\end{aligned}
$$

Lemma 3. Because $w(x,t)$ satisfies equation (11), the following equalities hold:

$$
P_0 \int_0^L \left(w_t w_{xx} + w_x w_{xt}\right) \mathrm{d}x = P_0 \left[w_t w_x\right]_0^L = P_0 w_t(L,t)\, w_x(L,t), \qquad (35a)
$$

$$
\begin{aligned}
&-EI \int_0^L w_t w_{xxxx} \, \mathrm{d}x + EI \int_0^L w_{xx} w_{xxt} \, \mathrm{d}x = -EI \left[w_{xxx} w_t\right]_0^L + EI \left[w_{xx} w_{xt}\right]_0^L \\
= \quad &-EI w_{xxx}(L,t) w_t(L,t), \qquad (35b)
\end{aligned}
$$

$$
\frac{EA}{2} \int_0^L \left(w_x^3 w_{xt} + 3 w_t w_x^2 w_{xx}\right) \mathrm{d}x = \frac{EA}{2} \left[w_x^3 w_t\right]_0^L = \frac{EA}{2} w_x^3(L,t) w_t(L,t), \quad (35c)
$$

$$2EAv\int_0^L w_x^3 w_{xx}\,\mathrm{d}x = 2EAv\left[\frac{1}{4}w_x^4\right]_0^L = \frac{EAv}{2}w_x^4(L,t), \tag{35d}$$

$$2P_0 v\int_0^L w_x w_{xx}\,\mathrm{d}x = P_0 v\left[w_x^2\right]_0^L = P_0 v w_x^2(L,t), \tag{35e}$$

$$-EIv\int_0^L w_x w_{xxxx}\,\mathrm{d}x = -EIv w_x(L,t) w_{xxx}(L,t) - \frac{EIv}{2}w_{xx}^2(0,t), \tag{35f}$$

$$EIv\int_0^L w_{xx} w_{xxx}\,\mathrm{d}x = \frac{EIv}{2}\left[w_{xx}^2\right]_0^L = -\frac{EIv}{2}w_{xx}^2(0,t), \tag{35g}$$

$$\beta\rho A\int_0^L x w_{xt} w_t\,\mathrm{d}x = \frac{\beta\rho AL}{2}w_t^2(L,t) - \frac{\beta\rho A}{2}\int_0^L w_t^2\mathrm{d}x, \tag{35h}$$

$$\beta\rho A\int_0^L x v^2 w_x w_{xx}\,\mathrm{d}x = \frac{\beta\rho ALv^2}{2}w_x^2(L,t) - \frac{\beta\rho Av^2}{2}\int_0^L w_x^2\mathrm{d}x, \tag{35i}$$

$$\beta P_0\int_0^L x w_x w_{xx}\,\mathrm{d}x = \frac{\beta P_0 L}{2}w_x^2(L,t) - \frac{\beta P_0}{2}\int_0^L w_x^2\mathrm{d}x, \tag{35j}$$

$$\frac{3\beta EA}{2}\int_0^L x w_x^3 w_{xx}\,\mathrm{d}x = \frac{3\beta EAL}{8}w_x^4(L,t) - \frac{3\beta EA}{8}\int_0^L w_x^4\mathrm{d}x, \tag{35k}$$

$$-\beta EI\int_0^L x w_x w_{xxxx}\,\mathrm{d}x = -\beta EIL w_x(L,t) w_{xxx}(L,t) - \frac{3\beta EI}{2}\int_0^L w_{xx}^2\,\mathrm{d}x, \tag{35l}$$

$$\begin{aligned}\beta\rho A\int_0^L (xvw_{xt} w_x + xvw_{xx} w_t + vw_x w_t)\,\mathrm{d}x &= \beta\rho Av\left[xw_x w_t\right]_0^L \\ &= \beta\rho ALvw_x(L,t) w_t(L,t).\end{aligned} \tag{35m}$$

Proof. The integration by parts yields all above equalities.
Now, by using Lemma 3, equation (34) is modified as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_{strip}\bigg|_{A104} &= \alpha\left[P_0 w_t(L,t) w_x(L,t) + \frac{EA}{2}w_x^3(L,t) w_t(L,t) + \frac{EAv}{2}w_x^4(L,t)\right.\\ &+ P_0 v w_x^2(L,t) - EIv w_{xx}^2(0,t) - EI w_{xxx}(L,t)\left\{w_t(L,t) + v w_x(L,t)\right\}]\\ &- \frac{\beta\rho A}{2}\int_0^L w_t^2\mathrm{d}x - \frac{\beta}{2}\left(P_0 + \rho Av^2\right)\int_0^L w_x^2\mathrm{d}x - \frac{3\beta EA}{8}\int_0^L w_x^4\mathrm{d}x\\ &- \frac{3\beta EI}{2}\int_0^L w_{xx}^2\,\mathrm{d}x + \beta\rho ALv w_x(L,t) w_t(L,t) + \frac{\beta\rho AL}{2}w_t^2(L,t)\\ &+ \frac{\beta L}{2}\left(P_0 + \rho Av^2\right) w_x^2(L,t) + \frac{3\beta EAL}{8}w_x^4(L,t)\end{aligned}$$

$$- \quad \beta EILw_x(L,t)w_{xxx}(L,t) + \beta\rho Av^2 \int_0^L w_x^2 \mathrm{d}x. \tag{36}$$

On the other hand, the time derivative of equation (27) becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V_{actuator} &= \frac{2m}{\xi}\{\xi\, w_t(L,t) + \tau w_x(L,t)\} \cdot \{\xi\, w_{tt}(L,t) + \tau w_{xt}(L,t)\} \\ &= 2m\xi\, w_t(L,t) \cdot w_{tt}(L,t) + 2m\tau w_t(L,t) \cdot w_{xt}(L,t) \\ &+ \quad 2m\tau w_x(L,t) \cdot w_{tt}(L,t) + \frac{2m\tau^2}{\xi} w_x(L,t) \cdot w_{xt}(L,t). \end{aligned} \tag{37}$$

From equations (36) and (37), the total derivative of equation (26) becomes

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}V(t)\right|_{\Lambda 104} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(V_{strip} + V_{actuator}\right) \\ &= -\alpha EIvw_{xx}^2(0,t) - \frac{\beta\rho A}{2}\int_0^L w_t^2 \mathrm{d}x - \frac{\beta\left(P_0 - \rho Av^2\right)}{2}\int_0^L w_x^2 \mathrm{d}x \\ &- \quad \frac{3\beta EA}{8}\int_0^L w_x^4 \mathrm{d}x - \frac{3\beta EI}{2}\int_0^L w_{xx}^2\, \mathrm{d}x + \frac{4\alpha EAv + 3\beta EAL}{8} w_x^4(L,t) \\ &+ \quad \frac{\alpha EA}{2} w_x^3(L,t)w_t(L,t) + \left(P_0\alpha v + \frac{\beta P_0 L}{2} + \frac{\beta\rho ALv^2}{2}\right) w_x^2(L,t) \\ &+ \quad \frac{\beta\rho AL}{2} w_t^2(L,t) + \left(P_0\alpha + \beta\rho ALv\right)\, w_x(L,t)w_t(L,t) \\ &- \quad EIw_{xxx}(L,t)\left[\alpha w_t(L,t) + (\alpha v + \beta L)\, w_x(L,t)\right] \\ &+ \quad mw_{tt}(L,t)\left\{\, 2\xi\, w_t(L,t) + 2\tau w_x(L,t)\right\} \\ &+ \quad 2m\tau w_t(L,t)w_{xt}(L,t) + \frac{2m\tau^2}{\xi} w_x(L,t)w_{xt}(L,t). \end{aligned} \tag{38}$$

The substitution of equation (12) into equation (38) yields

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}V(t)\right|_{\Lambda 104, \Lambda 124} &= -\alpha EIvw_{xx}^2(0,t) - \frac{\beta\rho A}{2}\int_0^L w_t^2 \mathrm{d}x - \frac{\beta\left(P_0 - \rho Av^2\right)}{2}\int_0^L w_x^2 \mathrm{d}x \\ &- \quad \frac{3\beta EA}{8}\int_0^L w_x^4 \mathrm{d}x - \frac{3\beta EI}{2}\int_0^L w_{xx}^2\, \mathrm{d}x \\ &+ \quad \frac{4\alpha EAv + 3\beta EAL}{8} w_x^4(L,t) + \frac{\alpha EA}{2} w_x^3(L,t)w_t(L,t) \end{aligned}$$

$$
\begin{aligned}
&+ \left(P_0\alpha v + \frac{\beta P_0 L}{2} + \frac{\beta\rho A L v^2}{2}\right) w_x^2(L,t) + \frac{\beta\rho A L}{2} w_t^2(L,t) \\
&+ (P_0\alpha + \beta\rho A L v)\, w_x(L,t) w_t(L,t) \\
&- EI w_{xxx}(L,t)\,[\alpha w_t(L,t) + (\alpha v + \beta L)\, w_x(L,t)] \\
&+ \{2\xi\, w_t(L,t) + 2\tau w_x(L,t)\} \cdot [F_c(t) - d_c w_t(L,t) - P_0 w_x(L,t) \\
&- \frac{EA}{2} w_x^3(L,t) + EI w_{xxx}(L,t)] \\
&+ 2m\left\{\tau w_t(L,t) w_{xt}(L,t) + \frac{\tau^2}{\xi} w_x(L,t) \cdot w_{xt}(L,t)\right\}. \qquad (39)
\end{aligned}
$$

Finally, the main theorem of this paper is stated as follows.

Theorem. Consider the system (10)–(12). Let the right boundary control force $F_c(t)$ and the damping coefficient of the actuator d_c in equation (12) be given, respectively, by

$$
\begin{aligned}
F_c(t) &= -K w_{xt}(L,t), \qquad (40a) \\
d_c &= \frac{\beta\rho A L v}{\alpha v + \beta L} = \frac{\beta\rho A L}{\alpha + \beta L/v}, \qquad (40b)
\end{aligned}
$$

where

$$
K = \frac{m(\alpha v + \beta L)}{\alpha}, \quad \alpha > 0, \quad \text{and} \quad 0 < \beta < \min\left\{\frac{v}{L}\alpha, \frac{\alpha}{\beta_1}\right\}. \qquad (41)
$$

Then, the dynamics of the closed-loop system is exponentially stable, i.e.

$$
V(t) \le V(0)\,\mathrm{e}^{-\lambda t} \qquad (42)
$$

where

$$
\begin{aligned}
\lambda = \min\Bigg\{ &\frac{3\beta}{\alpha(\alpha + \beta\beta_1)}, \frac{\beta(P_0 - \rho A v^2)}{2\alpha P_0(\alpha + \beta\beta_1)}, \frac{\beta(P_0 - \rho A v^2)}{4\alpha\rho A v^2(\alpha + \beta\beta_1)}, \frac{\alpha\beta L(P_0 - \rho A v^2)}{2m(\alpha v + \beta L)^2}, \\
&\frac{\beta\rho A L\xi}{m\alpha}\left[\alpha\frac{\beta\rho A L v}{\alpha v + \beta L} - \frac{1}{2}\right]\Bigg\}. \qquad (43)
\end{aligned}
$$

Proof. Let $\xi = \alpha/2$ and $\tau = (\alpha v + \beta L)/2$, then equation (39) becomes

$$
\left.\frac{\mathrm{d}}{\mathrm{d}t}V(t)\right|_{\Lambda 104, \Lambda 124} = -\alpha EI v w_{xx}^2(0,t) - \frac{\beta\rho A}{2}\int_0^L w_t^2 \mathrm{d}x - \frac{\beta(P_0 - \rho A v^2)}{2}\int_0^L w_x^2 \mathrm{d}x
$$

$$
\begin{aligned}
&- \frac{3\beta EA}{8} \int_0^L w_x^4 \mathrm{d}x - \frac{3\beta EI}{2} \int_0^L w_{xx}^2 \, \mathrm{d}x \\
&+ \frac{4\alpha EAv + 3\beta EAL}{8} w_x^4(L,t) + \frac{\alpha EA}{2} w_x^3(L,t) w_t(L,t) \\
&+ \left(P_0 \alpha v + \frac{\beta P_0 L}{2} + \frac{\beta \rho A L v^2}{2} \right) w_x^2(L,t) + \frac{\beta \rho A L}{2} w_t^2(L,t) \\
&+ (P_0 \alpha + \beta \rho A L v) \; w_x(L,t) w_t(L,t) \\
&+ \{ \alpha \, w_t(L,t) + (\alpha v + \beta L) \; w_x(L,t) \} \times [F_c(t) - d_c w_t(L,t) \\
&- P_0 w_x(L,t) - \frac{EA}{2} w_x^3(L,t) \Big] + m \left\{ (\alpha v + \beta L) \; w_t(L,t) w_{xt}(L,t) \right. \\
&\left. + \frac{(\alpha v + \beta L)^2}{\alpha} w_x(L,t) w_{xt}(L,t) \right\} . \qquad (44)
\end{aligned}
$$

In equation (44), if we set the control input $F_c(t)$ to be $0 < \beta \frac{L}{v} < \alpha$ and the gain K to be $m(\alpha v + \beta L)/\alpha$, then all terms involving $w_{xt}(L,t)$ can be eliminated, i.e.

$$
\begin{aligned}
\left. \frac{\mathrm{d}}{\mathrm{d}t} V(t) \right|_{\Lambda 104, \Lambda 124, \Lambda 40a4} &= -\alpha EIv w_{xx}^2(0,t) - \frac{\beta \rho A}{2} \int_0^L w_t^2 \mathrm{d}x \\
&- \frac{\beta (P_0 - \rho A v^2)}{2} \int_0^L w_x^2 \mathrm{d}x \\
&- \frac{3\beta EA}{8} \int_0^L w_x^4 \mathrm{d}x - \frac{3\beta EI}{2} \int_0^L w_{xx}^2 \, \mathrm{d}x \\
&- \frac{\beta EAL}{8} w_x^4(L,t) - \frac{\beta L}{2} \left(P_0 - \rho A v^2 \right) w_x^2(L,t) \\
&+ \left(\frac{\beta \rho A L}{2} - \alpha \, d_c \right) w_t^2(L,t) \\
&+ \{ \beta \rho A L v - (\alpha v + \beta L) \, d_c \} \, w_x(L,t) w_t(L,t). \qquad (45)
\end{aligned}
$$

Because $P_0 > \rho A v^2$ is assumed, see equation (14), all terms except the last two terms in equation (45) are negative. Therefore, by establishing the relationship between α, β, and d_c such that

$$
\frac{\beta \rho A L}{2} - \alpha \, d_c < 0, \qquad (46)
$$

$$
\beta \rho A L v - (\alpha v + \beta L) \; d_c = 0, \qquad (47)
$$

the negative value of the last two terms can be achieved. It is noted that to satisfy equations (46) and (47) the following inequality is also needed:

$$0<\beta \frac{L}{v}<\alpha . \tag{48}$$

To satisfy equation (22), β should further satisfy

$$0<\beta<\min \left\{\frac{v}{L} \alpha, \frac{\alpha}{\beta_{1}}\right\} .$$

Hence, the total derivative of the Lyapunov function candidate becomes negative as follows

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d} t} V(t)\right|_{\Lambda 104, \Lambda 124, \Lambda 40 a, b 4} = & -\alpha E I v w_{x x}^{2}(0, t)-\frac{\beta \rho A}{2} \int_{0}^{L} w_{t}^{2} \mathrm{~d} x \\
& -\frac{\beta\left(P_{0}-\rho A v^{2}\right)}{2} \int_{0}^{L} w_{x}^{2} \mathrm{~d} x \\
& -\frac{3 \beta E A}{8} \int_{0}^{L} w_{x}^{4} \mathrm{~d} x-\frac{3 \beta E I}{2} \int_{0}^{L} w_{x x}^{2} \mathrm{~d} x \\
& -\frac{\beta E A L}{8} w_{x}^{4}(L, t)-\frac{\beta L}{2}\left(P_{0}-\rho A v^{2}\right) w_{x}^{2}(L, t) \\
& -\beta \rho A L\left(\frac{\alpha v}{\alpha v+\beta L}-\frac{1}{2}\right) w_{t}^{2}(L, t)<0
\end{aligned} \tag{49}$$

where $\alpha v /(\alpha v+\beta L)>1 / 2$. Equation (49) can be rewritten as

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d} t} V(t)\right|_{\Lambda 104, \Lambda 124, \Lambda 40 a, b 4} = & -\alpha E I v w_{x x}^{2}(0, t)-\frac{\beta \rho A}{2} \int_{0}^{L} w_{t}^{2} \mathrm{~d} x \\
& -\frac{\beta\left(P_{0}-\rho A v^{2}\right)}{4} \int_{0}^{L} w_{x}^{2} \mathrm{~d} x \\
& -\frac{\beta\left(P_{0}-\rho A v^{2}\right)}{4 v^{2}} \int_{0}^{L}\left(v w_{x}\right)^{2} \mathrm{~d} x-\frac{3 \beta E A}{8} \int_{0}^{L} w_{x}^{4} \mathrm{~d} x \\
& -\frac{3 \beta E I}{2} \int_{0}^{L} w_{x x}^{2} \mathrm{~d} x-\frac{\beta E A L}{8} w_{x}^{4}(L, t) \\
& -\frac{\beta L}{2}\left(P_{0}-\rho A v^{2}\right) w_{x}^{2}(L, t)-\beta \rho A L\left(\frac{\alpha v}{\alpha v+\beta L}-\frac{1}{2}\right) w_{t}^{2}(L, t) \\
\leq & -\alpha E I v w_{x x}^{2}(0, t)-\frac{\beta\left(P_{0}-\rho A v^{2}\right)}{4} \int_{0}^{L} w_{x}^{2} \mathrm{~d} x
\end{aligned}$$

$$
\begin{aligned}
&- \frac{3\beta EA}{8}\int_0^L w_x^4 \mathrm{d}x - \frac{3\beta EI}{2}\int_0^L w_{xx}^2\,\mathrm{d}x \\
&- \min\left\{\frac{\beta\rho A}{2}, \frac{\beta\left(P_0-\rho Av^2\right)}{4v^2}\right\}\left[\int_0^L w_t^2 \mathrm{d}x + \int_0^L \left(vw_x\right)^2 \mathrm{d}x\right] \\
&- \frac{\beta EAL}{8} w_x^4(L,t) - \frac{\beta L}{2}\left(P_0-\rho Av^2\right) w_x^2(L,t) \\
&- \beta\rho AL\left(\frac{\alpha v}{\alpha v+\beta L}-\frac{1}{2}\right) w_t^2(L,t). \qquad (50)
\end{aligned}
$$

Now, the application of $-\int_0^L w_t^2 \mathrm{d}x - \int_0^L \left(vw_x\right)^2 \mathrm{d}x \le -\frac{1}{2}\int_0^L \left(w_t+vw_x\right)^2 \mathrm{d}x$ to equation (50) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}V(t)\bigg|_{\Lambda 104,\Lambda 124,\Lambda 40a,b4} \le -\alpha EIvw_{xx}^2(0,t) - \frac{\beta\left(P_0-\rho Av^2\right)}{4}\int_0^L w_x^2 \mathrm{d}x \\
&- \frac{3\beta EA}{8}\int_0^L w_x^4 \mathrm{d}x - \frac{3\beta EI}{2}\int_0^L w_{xx}^2\,\mathrm{d}x - \min\left\{\frac{\beta\rho A}{4}, \frac{\beta\left(P_0-\rho Av^2\right)}{8v^2}\right\} \\
&\times \left[\int_0^L \left(w_t+vw_x\right)^2 \mathrm{d}x\right] - \frac{\beta EAL}{8} w_x^4(L,t) - \frac{\beta L}{2}\left(P_0-\rho Av^2\right) w_x^2(L,t) \\
&- \beta\rho AL\left(\frac{\alpha v}{\alpha v+\beta L}-\frac{1}{2}\right) w_t^2(L,t) \;\le\; -\frac{3\beta}{\alpha}\alpha\frac{EA}{8}\int_0^L w_x^4 \mathrm{d}x - \frac{3\beta}{\alpha}\alpha\frac{EI}{2}\int_0^L w_{xx}^2\,\mathrm{d}x \\
&- \frac{\beta\left(P_0-\rho Av^2\right)}{2\alpha P_0}\alpha\frac{P_0}{2}\int_0^L w_x^2 \mathrm{d}x \\
&- \min\left\{\frac{\beta}{2\alpha}, \frac{\beta\left(P_0-\rho Av^2\right)}{4\alpha\rho Av^2}\right\}\alpha\frac{\rho A}{2}\int_0^L \left(w_t+vw_x\right)^2 \mathrm{d}x \\
&- \frac{2\beta L\left(P_0-\rho Av^2\right)}{\left(av+\beta L\right)^2}\left(\frac{\alpha v+\beta L}{2} w_x(L,t)\right)^2 \\
&- \frac{4\beta\rho AL}{\alpha^2}\left\{\frac{\alpha v}{\alpha v+\beta L}-\frac{1}{2}\right\}\left(\frac{\alpha}{2} w_t(L,t)\right)^2 \\
&\le -\min\left\{\frac{3\beta}{\alpha}, \frac{\beta\left(P_0-\rho Av^2\right)}{2\alpha P_0}, \frac{\beta\left(P_0-\rho Av^2\right)}{4\alpha\rho Av^2}\right\} \\
&\times \alpha\left[\frac{EA}{8}\int_0^L w_x^4 \mathrm{d}x + \frac{EI}{2}\int_0^L w_{xx}^2\,\mathrm{d}x + \frac{P_0}{2}\int_0^L w_x^2 \mathrm{d}x + \frac{\rho A}{2}\int_0^L \left(w_t+vw_x\right)^2 \mathrm{d}x\right] \\
&- \min\left\{\frac{\alpha\beta L\left(P_0-\rho Av^2\right)}{2m\left(\alpha v+\beta L\right)^2}, \frac{\beta\rho AL}{m\alpha}\left[\frac{\alpha v}{\alpha v+\beta L}-\frac{1}{2}\right]\right\}
\end{aligned}
$$

$$\times \quad \frac{2m}{\alpha}\left\{\frac{\alpha}{2}w_t(L,t)+\frac{\alpha v+\beta L}{2}w_x(L,t)\right\}^2. \tag{51}$$

Using equations (21), (26), and (27), equation (51) becomes

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}V(t)\right|_{\Lambda 104,\Lambda 124,\Lambda 40a,b4} \leq -\min\left\{\frac{3\beta}{\alpha},\frac{\beta\left(P_0-\rho Av^2\right)}{2\alpha P_0},\frac{\beta\left(P_0-\rho Av^2\right)}{4\alpha\rho Av^2}\right\} \\
&\times \quad \frac{1}{\alpha+\beta\beta_1}V_{strip}-\min\left\{\frac{\alpha\beta L\left(P_0-\rho Av^2\right)}{2m\left(\alpha v+\beta L\right)^2},\frac{\beta\rho AL}{m\alpha}\left[\frac{\alpha v}{\alpha v+\beta L}-\frac{1}{2}\right]\right\}\times V_{actuator} \\
&\leq \quad -\min\left\{\frac{3\beta}{\alpha\left(\alpha+\beta\beta_1\right)},\frac{\beta\left(P_0-\rho Av^2\right)}{2\alpha P_0\left(\alpha+\beta\beta_1\right)},\frac{\beta\left(P_0-\rho Av^2\right)}{4\alpha\rho Av^2\left(\alpha+\beta\beta_1\right)},\frac{\alpha\beta L\left(P_0-\rho Av^2\right)}{2m\left(\alpha v+\beta L\right)^2},\right. \\
&\left.\frac{\beta\rho AL\xi}{m\alpha}\left[\frac{\alpha v}{\alpha v+\beta L}-\frac{1}{2}\right]\right\}\times\left(V_{strip}+V_{actuator}\right)=-\lambda\, V(t).
\end{aligned} \tag{52}$$

The theorem is now proved.

Equations (42), (28), and (29) imply that

$$E_{strip} \;= \|z(t)\|_\Lambda^2 \leq \frac{1}{\min\{\alpha-\beta\beta_1,1\}}V(t)\leq V(0)\mathrm{e}^{-\lambda t}.$$

Hence, it is seen that the mechanical energy (19) of the strip decays exponentially, which again implies that all state variables decay exponentially in time.

4. IMPLEMENTATION OF THE CONTROL LAW

The implementation of equations (40a) and (40b) requires two things: the design of control force $F_c(t)$ and the satisfaction of a damping coefficient d_c. Because the satisfaction of a damping coefficient is related to the design of an actuator, it must be answered beforehand. Note that β should satisfy both equations (24) and (41). Hence, β is selected as follows:

$$\beta = k_3\alpha, \quad \text{where} \;\; 0<k_3<\min\{v/L, 1/\beta_1\}. \tag{54}$$

The substitution of equation (54) into d_c in equation (40b) yields

$$d_c = k_3\rho ALv/(v+k_3L). \tag{55}$$

Note that equation (55) is an increasing function in k_3. Hence, if k_3 satisfies equation (54), d_c will assume the following

$$0<d_c<M\rho ALv/(v+ML), \tag{56}$$

where $M = \min\{v/L, 1/\beta_1\}$ Because all terms on the right-hand side of equation (56) are already known, the range of the damping coefficient can be achieved. Once d_c is determined, α can take an appropriate constant value and β is chosen as explained above, which determines the gains in equation (41). The implementation of $w_{xt}\,(L, t)$ in equation (40a) can be achieved by backwards differencing of $w_x\,(L, t)$ measured at each step.

5. SIMULATIONS

To demonstrate the performance of the closed-loop system, computer simulations using the finite difference scheme have been performed. The values used in simulations are listed in Table 1.

Let x be chosen to be 1. The plausible range of d_c, equation (56), can be estimated using the plant parameter values in Table 1 as follows:

$$0 < d_c < 30.93. \tag{57}$$

Let $d_c = 15$ (see Rao, 1990). Then, from equations (54) and (55), β is calculated as follows:

$$\begin{aligned} \beta &= k_3\alpha = v\alpha/(\rho ALv/d_c - L) \\ &= 1.67 \times 1/(7850 \times 1.4 \times 0.0045 \times 17.5 \times 1.67/15 - 17.5) \\ &= 0.02. \end{aligned} \tag{58}$$

Thus, the control gain in equation (41) becomes

$$K = 5(1.67 + 0.021 \times 17.5)/1 = 10.19. \tag{59}$$

Let the initial conditions be

$$w(x, 0) = 2\sin(3\pi)\ \text{cm} \quad \text{and} \quad w_t\,(x, 0) = 0\ \text{m s}^{-1} \tag{60}$$

Now, simulations using equations (57)–(60) have been performed for 20 s. Figures 3 and 4 show the transverse displacement at $x = L/2$ and the control force at $x = L$, respectively. As shown in Figure 3, the lateral vibration has been suppressed within 4 s. Figure 5 shows the decay of the total mechanical energy of the strip in time. It shows that the total energy with control decays exponentially, while the energy without control is sustained in time.

6. CONCLUSIONS

In this paper we investigate a boundary control law for suppressing the transverse vibration of an axially moving steel strip in the zinc galvanizing line. Because the strip was modeled as a Euler–Bernoulli beam equation with a non-linear tension, the method developed is general in the sense that it can be applied to any system in a similar form. Once the range of damping coefficient is established, an appropriate value for β can be selected for given system parameters. Achieving the exponential stability by using one sensor and one actuator is the main contribution of the algorithm proposed.

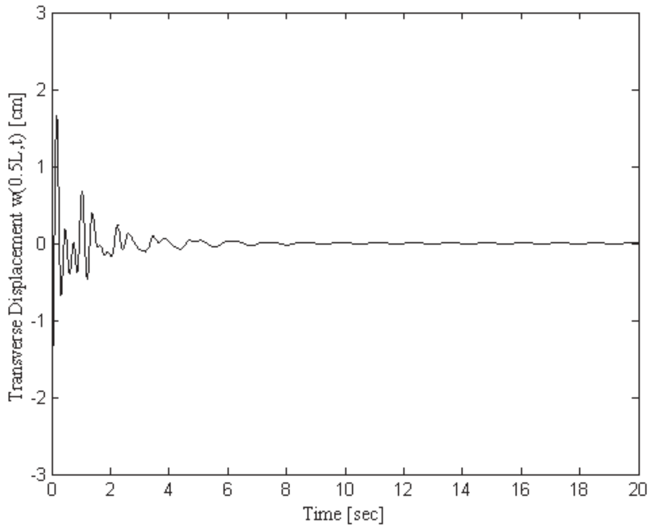

Figure 3. Transversal displacement $w(L/2, t)$ with control gain $K = 10.19$ and damping coefficient $d_c = 15$.

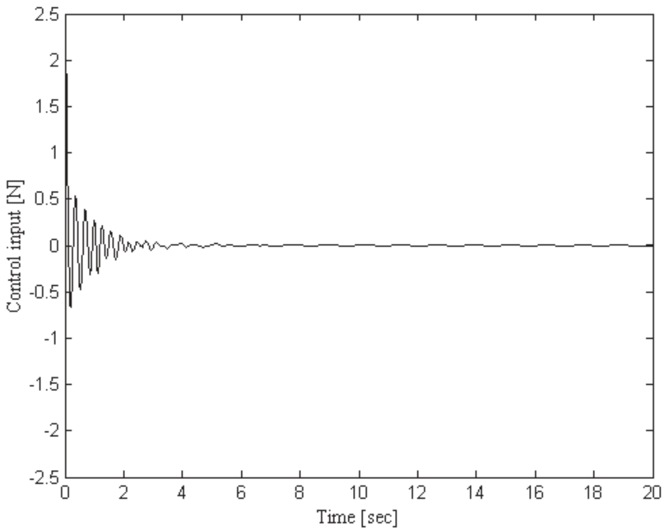

Figure 4. The control input used to obtain Figure 3.

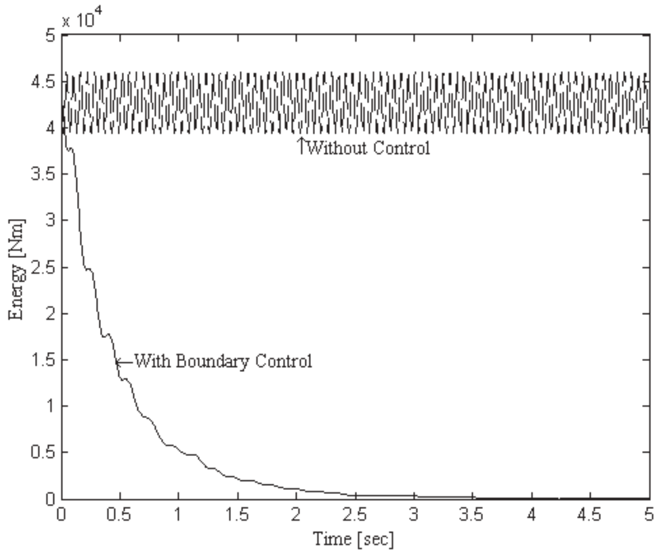

Figure 5. The exponential decay of the total energy (26) with the control input in Figure 4.

Acknowledgement. This work was supported by the Ministry of Science and Technology of Korea under a program of the National Research Laboratory, grant number NRL M1–0302–00–0039–03–J00–00-032–10.

REFERENCES

Bapat, V. A. and Srinivasan, P., 1967, "Nonlinear transverse oscillations in traveling strings by the method of harmonic balance," *Journal of Applied Mechanics* **34**, 775–777.

Benaroya, H., 1998, *Mechanical Vibration: Analysis, Uncertainties, and Control*, Prentice Hall, Englewood Cliffs, NJ.

Carrier, G. F., 1945, "On the nonlinear vibration problem of the elastic string," *Quarterly of Applied Mathematics* **3**, 157–165.

Chen, D., 1995, "Adaptive control of hot-dip galvanizing," *Automatica* **31(5)**, 715-733.

Chen, G. and Zhou, J., 1993, *Vibration and Damping in Distributed Systems*, Vols. 1 and 2, CRC Press, Boca Raton, FL.

Choi, J. Y., Hong, K. S., and Huh, C. D., 2002, "Vibration control of an axially moving strip by a nonlinear boundary control," in *Proceedings of the 15th IFAC 2002 World Congress*, Barcelona, Spain, July 21–26, T-Tu-E19-1.

Fard, M. P. and Sagatun, S. I., 2001, "Exponential stabilization of a transversely vibrating beam via boundary control," *Journal of Sound and Vibration* **240(4)**, 613–622.

Fung, R. F., Wu, J. W., and Wu, S. L., 1999a, "Exponential stability of an axially moving string by linear boundary feedback," *Automatica* **35(1)**, 177–181.

Fung, R. F., Wu, J. W., and Wu, S. L., 1999b, "Stabilization of an axially moving string by non-linear boundary feedback," *ASME Journal of Dynamic Systems, Measurement, and Control* **121(1)**, 117–120.

Fung, R. F., Chou, J. H., and Kuo, Y. L., 2002a, "Optimal boundary control of an axially moving material system," *ASME Journal of Dynamic Systems, Measurement, and Control* **124(1)**, 55–61.

Fung, R. F., Wu, J. W., and Lu, P. Y., 2002b, "Adaptive boundary control of an axially moving string system," *Journal of Vibration and Acoustics* **124(3)**, 435–440.

Ge, S. S., Lee, T. H., Gong, J. Q., and Wang, Z. P., 2000, "Model-free controller design for a single-link flexible smart material robot," *International Journal of Control* **73(6)**, 531–544.

Ge, S. S., Lee, T. H., Zhu, G., and Hong, F., 2001, "Variable structure control of a distributed-parameter flexible beam," *Journal of Robotic Systems* **18(1)**, 17–27.

Hong, K. S., Kim, C. W., and Hong, K. T., 2004, "Boundary control of an axially moving belt system in thin-metal production line," *International Journal of Control*, *Automation, and Systems* **2(1)**, 55–67

Laousy, H., Xu, C. Z., and Sallet, G., 1996, "Boundary feedback stabilization of a rotating body-beam system," *IEEE Transactions on Automatic Control* **41(2)**, 241–244.

Lee, S. Y. and Mote, C. D., 1996, "Vibration control of an axially moving string by boundary control," *ASME Journal of Dynamic Systems, Measurement, and Control* **118(1)**, 66–74.

Lee, S. Y. and Mote, C. D., 1999, "Wave characteristics and vibration control of translating beams by optimal boundary damping," *Journal of Vibration and Acoustics* **121(1)**, 18–25.

Li, Y. and Rahn, C. D., 2000, "Adaptive vibration isolation for axially moving beams," *IEEE/ASME Transactions on Mechatronics* **5(4)**, 419–428.

Li, Y., Aron, D., and Rahn, C. D., 2002, "Adaptive vibration isolation for axially moving strings: theory and experiment," *Automatica* **38(3)**, 379–389.

Matsuno, F., Ohno, T., and Orlov, Y., 2002, "Proportional derivative and strain (PDS) boundary feedback control of a flexible space structure with a closed-loop chain mechanism," *Automatica* **38(7)**, 1201–1211.

McIver, D. B., 1973, "Hamilton's principle for systems of changing mass," *Journal of Engineering Mathematics* **7(3)**, 249–261.

Morgul, O., 1992, "Dynamics boundary control of a Euler-Bernoulli beam," *IEEE Transactions on Automatic Control* **37(5)**, 639–642.

Mote, C. D., 1965, "A study of band saw vibration," *Journal of the Franklin Institute* **279**, 430–444.

Oostveen, J. C. and Curtain, R. F., 2000, "Robustly stabilization controllers for dissipative infinite-dimensional systems with collocated actuators and sensor," *Automatica* **36(3)**, 337–348.

Oshima, K., Takigami, T., and Hayakawa, Y., 1997, "Robust vibration control of a cantilever beam using self-sensing actuator," *JSME International Journal Series C* **40(4)**, 681–687.

Pellicano, F. and Zirilli, F., 1998, "Boundary layers and non-linear vibrations in an axially moving beam," *International Journal of Non-Linear Mechanics* **33(4)**, 691–694.

Qu, Z., 2002, "An iterative learning algorithm for boundary control of a stretched moving string," *Automatica* **38(5)**, 821–827.

Rao, S. S., 1990, *Mechanical Vibrations*, Addison Wesley, Reading, MA.

Shahruz, S. M., 1998, "Boundary control of the axially moving Kirchhoff string," *Automatica* **34(10)**, 1273–1277.

Shahruz, S. M., 2000, "Boundary control of a non-linear axially moving string," *International Journal of Robust Nonlinear Control* **10(1)**, 17–25.

Wickert, J. A., 1992, "Non-linear vibration of a traveling tensioned beam," I*nternational Journal of Non-Linear Mechanics* **27(3)**, 503–517.

Wickert, J. A. and Mote, C. D., 1988, "On the energetics of axially moving continua," *Journal of the Acoustic Society of America* **85(3)**, 1365–1368.

Wickert, J. A. and Mote, C. D., 1990, "Classical vibration analysis of axially moving continua," *Journal of Applied Mechanics* **57(3)**, 738–744.

Yang, K. J. and Hong, K. S., 2002, "Robust boundary control of an axially moving steel strip," in *Proceedings of the 15th IFAC 2002 World Congress*, Barcelona, Spain, July 21–26, T-Tu-E19-2.

Zhu, G. and Ge, S. S., 1998, "A quasi-tracking approach for finite-time control of a mass-beam system," *Automatica* **34(7)**, 881–888.