# ROBUST MRAC OF A NONAUTONOMOUS PARABOLIC SYSTEM WITH SPATIOTEMPORALLY VARYING COEFFICIENTS AND BOUNDED DISTURBANCE

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## ABSTRACT

In this paper, a robust model reference adaptive control of a parabolic system with unknown spatiotemporally varying coefficients and disturbance is investigated. In the adaptive control of time-varying plants, the derivative of a Lyapunov function candidate, which allows the derivation of adaptation laws, is not negative semi-definite in general. Under the assumption that the disturbance is uniformly bounded, the proposed robust adaptive scheme guarantees the boundedness of all signals in the closed loop system and the asymptotic convergence of the state error near to zero. With an additional persistence of excitation condition, the parameter estimation errors are shown to converge near to zero as well. Simulation results are provided.


***KeyWords:*** Robust model reference adaptive control, parabolic partial differential equation, uniform ultimate boundedness, stability, persistence of excitation.

## I. INTRODUCTION

In this paper a robust model reference adaptive control (MRAC) of a linear parabolic system with unknown spatiotemporally varying coefficients and bounded disturbance is investigated. As in the adaptive control of finite dimensional systems a robust MRAC, under the assumption that the structure of the plant is known and only parameters in the system equation are unknown, is derived. Distributed sensing and actuation are also assumed. Compared to the adaptive control/identification of finite dimensional systems, that of infinite dimensional systems is not well developed and has been recently studied [1-7,10-14,19-20,21,25,30,32-34].

The mathematical models of physical plants that control engineers adopt for the purpose of designing control systems normally contain some uncertainty. This is due to imperfect knowledge on the system parameters and/or disturbances. Parameter time-variations may be due to unmodeled dynamics, for instance, neglected frictions, neglected high order dynamics, etc., and may also arise from linear approximations along different motions over a wide range of operating conditions. Studies on the control of distributed parameter systems (DPSs) with uncertainty include various $H^{\infty}$ algorithms developed in the frequency domain [9,18,24,26], Lyapunov-based robust controller design methods [6], feedback control using Galerkin projections [23,31], and adaptive control/identification [16,29,35].

The objective of a MRAC scheme is to determine a feedback control law which forces the state of a plant to track asymptotically the state of a given reference model. At the same time, the unknown parameters in the plant model are estimated and used to update the control law. Typically, the whole adaptive system is represented as two error systems describing the evolution of the state error and the parameter estimation errors. In the infinite dimensional systems the state error and the parameter estimation errors take the forms of a partial differential equation and ordinary differential equations, respectively. The resulting closed loop system consisting of the plant, the reference model, and the estimator will be nonlinear. This is true even if the underlying plant and reference model, and the estimator are linear. The nonlinearity arises due to the coupling of error dynamics. Consequently, an adaptive scheme requires a careful stability analysis to ensure that all signals, both input and output, remain in some sense, bounded. It is also desirable, al-

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though not necessarily essential, that some sort of parameter convergence is achieved.

To conclude various stabilities in the sense of Lyapunov, the associated stability theorems require that the time derivative of a Lyapunov function candidate, $\dot{V}$, should be at least ≤ 0, i.e., negative semidefinite. Therefore, if $\dot{V}$ allows positive values near an equilibrium point, no stability can be asserted. In the MRAC of time-varying plants, the derivative of a Lyapunov function, which is introduced to derive adaptation laws, is not negative semi-definite in general.

The present paper makes the following contributions: To the authors' best knowledge this paper is the first treatment of an infinite dimensional system with unknown spatiotemporally varying parameters and additive disturbance in the frame of robust MRAC. The unknown time-varying parameters are not required to be slow, which can be allowed to vary arbitrarily fast, and the disturbance is allowed to vary in both time and spatial domains. The well posedness of the closed loop system is established. Using an appropriate Lyapunov function candidate, the tracking error convergence near to zero is established. With the additional assumption of persistence of excitation the convergence of parameter estimation errors near to zero is established as well.

The paper has the following structure. In Section 2, the standard MRAC of a linear parabolic system with spatiotemporally varying coefficients is reviewed. In Section 3, a robust MRAC algorithm in the presence of bounded disturbance is proposed. The derived control law guarantees its robustness with respect to the inaccessible disturbance and yields the desired equations of motion, thereby ensuring the adaptability of the controller. In Section 4, with the persistence of excitation condition, the adjustable parameters in the adaptive controller are shown to admit convergence to their nominal values when an appropriate reference signal is used. In Section 5, computer simulations are provided. Conclusions are given in Section 6.

## II. PROBLEM FORMULATION

In this section, the standard MRAC algorithm for a linear, 1-dimensional, parabolic partial differential equation (PDE) with spatiotemporally varying coefficients and bounded disturbance is formulated. As in the adaptive control of finite dimensional systems, under the assumption that the structure of the plant is known and only parameters in the system equation are unknown, the MRAC is investigated.

Consider the following 1-dimensional linear parabolic equation.

$$\dot{\xi}(x,t) = \big(a(x,t)\xi_x(x,t)\big)_x + b(x,t)\xi(x,t) + u(x,t) + d(x,t), \quad (1)$$

where $x \in [0,\ 1]$, $t > 0$, $a(x,t)$ and $b(x,t)$ are unknown spatiotemporally varying coefficients that are not necessarily slow-varying, $\xi(x,t)$ is the distributed state of the plant (in heat transfer, for example, it represents the temperature at position $x$ at time $t$ on a rod), $\dot{\xi} = \partial\xi/\partial t$, $\big(a(x,t)\xi_x(x,t)\big)_x = \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}\xi(x,t)\right)$, $u(x,\ t)$ is the control input function, and $d(x,t)$ is the inaccessible external disturbance. Note that $d(x,t)$ is an unknown bounded spatiotemporally varying function. Boundary conditions are given as

$$\xi(0,t) = 0, \xi(1,t) = h_1(t).$$

Initial condition is given as

$$\xi(x,0) = \xi_0(x).$$

The output $y$ of (1) in general is given by $y(x,\ t) = G\xi(x,\ t)$, where $G : C([0,\ 1] \times [0, \infty)) \to C(\Omega \subset [0,\ 1] \times [0, \infty))$ is a linear bounded time invariant operator with the form depending on the characteristics of the sensor. The following assumptions are made.

**Assumptions.** (i) The structure of (1) (plant) is *a priori* known. (ii) Boundary conditions are *a priori* known, and $h_1(\cdot) \in C^{\infty}[0,\infty)$. (iii) Distributed sensing and actuation are available, and the observation operator $G$ is a priori known ($G = I$ may be assumed, where $I$ denotes the identity operator from $C([0,\ 1]\times[0,\infty))$ onto itself. (iv) Coefficients $a(x,\ t)$ and $b(x,\ t)$ are unknown and uniformly bounded with uniformly bounded derivatives. However, $a(x,t) > 0$ is assumed, due to the parabolicity condition of the plant. (v) Disturbance $d(x,t)$ is unknown but uniformly bounded.

The parameters $a(x,\ t)$ and $b(x,\ t)$ may be piecewise values due to the presence and differing material properties of the bonding layer and patches of distributed actuators and sensors. It is then forced to differentiate discontinuous functions when considering the strong form of plant equation (1). To avoid the difficulty as well as lower smoothness requirements for approximating elements, the system (1) in weak form is considered [3,5].

To convert (1) into weak form, both sides of (1) are multiplied by a sufficiently smooth test function $\varphi$ and are integrated by parts. Assuming that $\varphi$ satisfies the boundary conditions $\varphi(x) = 0$ at $x = 0,1$, the weak form of (1) is

$$\int_0^1 \dot{\xi}(x,t)\varphi(x)dx = \int_0^1 \Big[\big(a(x,t)\xi_x(x,t)\big)_x + b(x,t)\xi(x,t) + u(x,t) + d(x,t)\Big]\varphi(x)dx. \quad (2)$$

By using the boundary conditions, the integration of the first term in the bracket in (2) yields:

$$\int_0^1 \left(a(x,t)\xi_x(x,t)\right)_x \varphi(x)dx = -\int_0^1 a(x,t)\xi_x(x,t)\varphi_x(x)dx.$$

Thus, the following weak form of the plant is derived:

$$\begin{aligned}\int_0^1 \dot{\xi}(x,t)\varphi(x)dx &= -\int_0^1 a(x,t)\xi_x(x,t)\varphi_x(x)dx \\ &+\int_0^1 b(x,t)\xi(x,t)\varphi(x)dx + \int_0^1 u(x,t)\varphi(x)dx \\ &+\int_0^1 d(x,t)\varphi(x)dx.\end{aligned} \tag{3}$$

To pose the MRAC problem, adequate function spaces are now introduced. $L_2(0, 1)$, $H^k(0, 1)$, and $H_L^1(0,1)$ are the Hilbert spaces defined as

$$\begin{aligned}L_2(0,1) &= \left\{ \eta : [0,1] \to R \ \middle|\ \int_0^1 \eta^2 dx < \infty \right\}, \\ H^k(0,1) &= \left\{ \eta \in L_2(0,1) \,\middle|\, \frac{\partial \eta}{\partial x}, \frac{\partial^2 \eta}{\partial x^2}, \cdots, \frac{\partial^{(k)}\eta}{\partial x^{(k)}} \in L_2(0,1) \right\}, \\ H_L^1(0,1) &= \left\{ \eta \in L_2(0,1) \,\middle|\, \frac{\partial \eta}{\partial x} \in L_2(0,1), \text{ and } \eta(x) = 0 \text{ at } x = 0 \right\},\end{aligned} \tag{4}$$

where the subscript $L$ in $H$ denotes $\eta(x) = 0$ at $x = 0$. The inner product in $L_2(0,1)$ and $H_L^1(0,1)$ are defined, respectively, as

$$\langle \phi(x), \psi(x) \rangle = \int_0^1 \phi(x)\psi(x)dx,$$

$$\langle \phi(x), \psi(x) \rangle_{H_L^1} = \int_0^1 \frac{\partial \phi(x)}{\partial x}\frac{\partial \psi(x)}{\partial x}dx,$$

and the corresponding induced norms are denoted by $\|\cdot\|$ and $\|\cdot\|_H$, respectively.

Let $a$, $b \in Q$, where $Q$ is a compact subset of $L_2(0,1)$ and is a real Hilbert space (henceforth the parameter space) with inner product $\langle \cdot,\cdot \rangle_Q$ and corresponding norm $\|\cdot\|_Q$. Note that the inner product $\langle \cdot,\cdot \rangle_Q$ is appropriately chosen according to the bonding layer and patches of distributed actuators and sensors, see [3].

With these definitions, (3) can be rewritten as

$$\langle \dot{\xi}, \varphi \rangle = -\langle a\xi_x, \varphi_x \rangle + \langle b\xi, \varphi \rangle + \langle u, \varphi \rangle + \langle d, \varphi \rangle. \tag{5}$$

The MRAC problem for plant (5), in the presence of unknown parameters $a$, $b$ and unknown disturbance $d$, is now to find a control input $u$ in feedback form, which forces the state $\xi$ to track a reference signal $\xi_m$. The reference signal $\xi_m$ is generated through a reference model with the same boundary conditions defined by

$$\langle \dot{\xi}_m, \varphi \rangle = -\langle a_m\xi_{mx}, \varphi_x \rangle + \langle b_m\xi_m, \varphi \rangle + \langle r, \varphi \rangle, \tag{6}$$

$$\xi_m(0,t) = 0, \xi_m(1,t) = h_1(t) \text{ and } \xi_m(x,0) = \xi_{m0}(x),$$

where $a_m, b_m \in R$ are the reference model parameters which are chosen so that the response $\xi_m$ can have the desired characteristics, the subscript $m$ indicates variables and parameters related to the reference model, and $r(x, t)$ is the reference input which is analytic on $[0, 1] \times [0, \infty]$. It is assumed that $a_m > 0$ and $b_m < a_m\pi^2$. It is known that if $r(\cdot, \cdot)$ is analytic in $[0, 1] \times [0, \infty]$, the solution of (6) is analytic in $[0, 1] \times \{0 < t < T < \infty\}$ [8, p. 212].

Now adopting the procedure in [12], consider a control law of the form

$$\begin{aligned}\langle u, \varphi \rangle &= -\langle a_m\xi_x, \varphi_x \rangle + \langle \hat{a}\xi_x, \varphi_x \rangle + \langle b_m\xi, \varphi \rangle \\ &\quad - \langle \hat{b}\xi, \varphi \rangle + \langle r, \varphi \rangle,\end{aligned} \tag{7a}$$

which is a weak representation of

$$u = a_m\xi_{xx} - (\hat{a}\xi_x)_x + b_m\xi - \hat{b}\xi + r, \tag{7b}$$

where $\hat{a}$ and $\hat{b}$ are adaptive estimates to be specified in the sequel. Substituting (7a) into (5) yields the following closed loop plant equation

$$\begin{aligned}\langle \dot{\xi}, \varphi \rangle &= -\langle a_m\xi_x, \varphi_x \rangle + \langle \tilde{a}\xi_x, \varphi_x \rangle + \langle b_m\xi, \varphi \rangle \\ &\quad - \langle \tilde{b}\xi, \varphi \rangle + \langle r, \varphi \rangle + \langle d, \varphi \rangle,\end{aligned} \tag{8}$$

where $\tilde{a} = \hat{a} - a$ and $\tilde{b} = \hat{b} - b$ are parameter estimation errors. Note that if $\tilde{a} = \tilde{b} = 0$ and $d = 0$, then (8) is exactly the same as (6).

Introducing the state error $e = \xi - \xi_m$, the following state error equation is obtained.

$$\begin{aligned}\langle \dot{e}, \varphi \rangle &= -\langle a_m e_x, \varphi_x \rangle + \langle \tilde{a}e_x, \varphi_x \rangle + \langle b_m e, \varphi \rangle \\ &\quad - \langle \tilde{b}e, \varphi \rangle + \langle \tilde{a}\xi_{mx}, \varphi_x \rangle - \langle \tilde{b}\xi_m, \varphi \rangle + \langle d, \varphi \rangle,\end{aligned} \tag{9}$$

$$e(0,t) = e(1,t) = 0, e(x,0) = \xi_0(x) - \xi_{m0}(x).$$

Now, consider a functional $V : L_2(0,1) \times Q^2 \to R^+$ such that

$$V(e, \tilde{a}, \tilde{b}) = \frac{1}{2}\langle e, e \rangle + \frac{1}{2\gamma_a}\langle \tilde{a}, \tilde{a} \rangle_Q + \frac{1}{2\gamma_b}\langle \tilde{b}, \tilde{b} \rangle_Q, \tag{10}$$

where $\gamma_a$ and $\gamma_b$ are positive constants, which will become the adaptation gains later. Differentiating (10) with respect to $t$ along (9) yields:

$$\dot{V} = \langle \dot{e}, e \rangle + \frac{1}{\gamma_a}\langle \tilde{a}, \dot{\tilde{a}} \rangle_Q + \frac{1}{\gamma_b}\langle \tilde{b}, \dot{\tilde{b}} \rangle_Q$$

$$= -\langle a_m e_x, e_x \rangle + \langle b_m e, e \rangle + \langle d, e \rangle + \frac{1}{\gamma_a}\langle \tilde{a}, \dot{\tilde{a}} \rangle_Q + \langle \tilde{a}e_x, e_x \rangle$$

$$+\langle \tilde{a}\xi_{mx}, e_x\rangle + \frac{1}{\gamma_b}\left\langle \tilde{b}, \dot{\tilde{b}}\right\rangle_Q - \left\langle \tilde{b}e, e\right\rangle - \left\langle \tilde{b}\xi_m, e\right\rangle. \tag{11}$$

Let the differential equations of the adaptive estimators in (7) be given as

$$\dot{\hat{a}} = -\gamma_a\left( e_x e_x + \xi_{mx} e_x \right), \tag{12a}$$

$$\dot{\hat{b}} = \gamma_b\left( ee + \xi_m e \right). \tag{12b}$$

From (12a,b) the differential equations of the adaptive estimate errors can be given as

$$\left\langle \dot{\tilde{a}}, p_a \right\rangle_Q = -\gamma_a\left( \langle p_a e_x, e_x\rangle + \langle p_a \xi_{mx}, e_x\rangle \right) - \langle \dot{a}, p_a\rangle_Q, \tag{13a}$$

$$\left\langle \dot{\tilde{b}}, p_b \right\rangle_Q = \gamma_b\left( \langle p_b e, e\rangle + \langle p_b \xi_m, e\rangle \right) - \left\langle \dot{b}, p_b\right\rangle_Q, \tag{13b}$$

where $p_a, p_b \in Q$. By Poincare's inequality [17, p.67], it is concluded that

$$\pi^2 \langle e, e\rangle \le \langle e_x, e_x\rangle \tag{14}$$

for all $t \ge 0$. Then, substituting (13a,b) and (14) into (11) yields:

$$\dot{V} \le -\left( a_m\pi^2 - b_m \right)\|e\|^2 - \frac{1}{\gamma_a}\langle \tilde{a}, \dot{a}\rangle_Q - \frac{1}{\gamma_b}\left\langle \tilde{b}, \dot{b}\right\rangle_Q + \langle d, e\rangle. \tag{15}$$

Now, from (15) the following observations are made.

**Case (1)** Assume that the system coefficients $a$ and $b$ are constant, i.e., $\dot{a} = \dot{b} = 0$, and the disturbance $d = 0$. Then, (15) becomes $\dot{V} \le 0$, i.e., negative semidefinite, which implies that (10) is a Lyapunov function. Therefore, the stability of an equilibrium point $(e, \tilde{a}, \tilde{b}) = (0, 0, 0)$ is guaranteed. Furthermore, the convergence of the state error $e$ to zero is also guaranteed by the uniqueness and semigroup properties of the solution [11].

**Case (2)** Assume that $d = 0$, but $a$ and $b$ are time-varying. Then, (15) becomes

$$\dot{V} \le -\left( a_m\pi^2 - b_m \right)\|e\|^2 - \frac{1}{\gamma_a}\langle \tilde{a}, \dot{a}\rangle_Q - \frac{1}{\gamma_b}\left\langle \tilde{b}, \dot{b}\right\rangle_Q. \tag{15a}$$

In this case, $\dot{V}$ may take positive values because of the last two terms. Hence, no stability conclusion can be drawn from the Lyapunov function candidate (10).

(a) But, further assume that $\tilde{a}$ and $\tilde{b}$ are bounded and $\dot{a}, \dot{b} \in L_1$. Then, the integration of both sides of (15a) gives

$$\int_0^\infty \left( a_m\pi^2 - b_m \right)\|e\|^2\, dt \le V(0) - V(\infty)$$
$$-\int_0^\infty \left( \frac{1}{\gamma_a}\langle \tilde{a}, \dot{a}\rangle_Q + \frac{1}{\gamma_b}\left\langle \tilde{b}, \dot{b}\right\rangle_Q \right) dt$$
$$\le V(0) - V(\infty) + \int_0^\infty \left( \frac{1}{\gamma_a}\|\tilde{a}\|_Q \|\dot{a}\|_Q + \frac{1}{\gamma_b}\left\|\tilde{b}\right\|_Q \left\|\dot{b}\right\|_Q \right) dt < \infty.$$

Therefore, the tracking error $e(t)$ converges to zero as $t \to \infty$.

(b) Now, if only the boundedness of $\tilde{a}$, $\dot{a}$, $\tilde{b}$, and $\dot{b}$ is assumed, then no asymptotic convergence to zero can be asserted. However, if $\langle \tilde{a}, \dot{a}\rangle_Q / \gamma_a + \left\langle \tilde{b}, \dot{b}\right\rangle_Q / \gamma_b$ is sufficiently small, then it is guaranteed that $e$ is uniformly ultimately bounded within an arbitrarily small neighborhood of zero [15].

**Remark 1.** In order to have the boundedness of $\langle \tilde{a}, \dot{a}\rangle_Q / \gamma_a + \left\langle \tilde{b}, \dot{b}\right\rangle_Q / \gamma_b$, the boundedness of individual $\tilde{a}$, $\tilde{b}$, $\dot{a}$, and $\dot{b}$ is necessarily required. The boundedness of $\dot{a}$ and $\dot{b}$ depends on the plant, which can be assumed. However, the boundedness of $\tilde{a}$ and $\tilde{b}$ can be proved through more rigorous theoretical analysis. In the case that $a$ and $b$ are slowly varying scalars and the disturbance $d = 0$, the boundedness of $\tilde{a}$ and $\tilde{b}$ was shown through averaging analysis [14]. Now, more general problem would be one in which $a$, $b$, $\dot{a}$ and $\dot{b}$ vary in unknown fashion but bounded (no slow varying assumption) and the disturbance $d \ne 0$. The aim of this paper is for a nonautonomous parabolic system with unknown spatiotemporally varying, but bounded, coefficients and disturbance to show not only the boundedness of all signals in the closed loop system including $\hat{a}$, $\hat{b}$, and an estimator of $d$ but also the convergence of the state error and parameter estimation errors near to zero.

## III. ROBUST MRAC: STABILITY

As discussed in Section 2, the adaptation laws (12a,b) with the boundedness of $\dot{a}(t)$ and $\dot{b}(t)$ were not able to assure the boundedness of $e$, $\tilde{a}$ and $\tilde{b}$ due to disturbance and unknown time-varying behavior. In this section, the control and adaptation laws are modified so that the boundedness of all signals in the closed loop system and the convergence of the state error $e$ near to zero can be assured. Assume that $\|d(x,t)\|$ is uniformly bounded by $\mu_d$, i.e., $\mu_d \ge \|d(x,t)\|$, where $\mu_d$ is an unknown positive constant. The main idea is to consider the worst case of the uncertainties in the form of possible bounds. Based upon the worst case, the following control algorithms are proposed.

*Control Law.*

$$\begin{aligned}\langle u,\varphi\rangle &= -\langle a_m\xi_x,\varphi_x\rangle+\langle \hat{a}\xi_x,\varphi_x\rangle+\langle b_m\xi,\varphi\rangle \\ &\quad -\langle \hat{b}\xi,\varphi\rangle+\langle r,\varphi\rangle+\langle f,\varphi\rangle,\end{aligned} \tag{16a}$$

which is a weak representation of

$$u = a_m\xi_{xx}-(\hat{a}\xi_x)_x+b_m\xi-\hat{b}\xi+r+f, \tag{16b}$$

where the additional term $f(x, t)$ is regarded as a new input signal to be determined based on robust control strategy. Let the additional input $f(x, t)$ be given by

$$f(x,t) = -\frac{\hat{\mu}_d^2}{\hat{\mu}_d\,\|e(x,t)\|+\varepsilon_d}e(x,t), \tag{17}$$

where $\varepsilon_d > 0$ and $\hat{\mu}_d$ is the estimate of $\mu_d$.

*Adaptation Laws.*

$$\dot{\hat{a}} = -\delta_a\hat{a}-\gamma_a\left(e_xe_x+\xi_{mx}e_x-g_a\right), \tag{18a}$$

$$\dot{\hat{b}} = -\delta_b\hat{b}+\gamma_b\left(ee+\xi_me+g_b\right),, \tag{18b}$$

$$\dot{\hat{\mu}}_d = -\delta_d\hat{\mu}_d+\gamma_d\|e\|, \tag{18c}$$

where

$$\delta_a>0,\ g_a = -\frac{\mu_a^2}{\|\hat{a}\|_Q\,\mu_a+\varepsilon_a}\hat{a},\ \varepsilon_a>0,\ \mu_a\ge\|f_a\|_Q,$$

$$f_a \overset{\Delta}{=} -\frac{\delta_a}{\gamma_a}\left(a+\frac{\dot{a}}{\delta_a}\right),$$

$$\delta_b>0,\ g_b = -\frac{\mu_b^2}{\|\hat{b}\|_Q\,\mu_b+\varepsilon_b}\hat{b},\ \varepsilon_b>0,\ \mu_b\ge\|f_b\|_Q,$$

$$f_b \overset{\Delta}{=} -\frac{\delta_b}{\gamma_b}\left(b+\frac{\dot{b}}{\delta_b}\right),$$

$\delta_d > 0$ and $\gamma_d > 0$.

Note that the adaptation laws (18a-c) are implementable. The terms $-\delta_a\hat{a}$, $-\delta_b\hat{b}$, and $-\delta_d\hat{\mu}_d$ in (18a-c) are purposely inserted to enhance the convergence of $\hat{a}$, $\hat{b}$, and $\hat{\mu}_d$ respectively; $g_a$ and $g_b$ are introduced to cope with the variations of $a$ and $b$, respectively. Since $a$, $\dot{a}$, $b$ and $\dot{b}$ are assumed to be bounded, $\mu_a$ and $\mu_b$ can be selected at reasonable values by making $\gamma_a$ and $\gamma_b$ sufficiently large. It is also noted that the control magnitudes $\mu_a$, $\mu_b$, and $\hat{\mu}_d$ are to compensate the maximum possible bounds of $f_a$, $f_b$, and $d$, respectively, for both positive and negative cases.

Substituting (16b) into (5) yields the following closed loop plant equation:

$$\begin{aligned}\langle \dot{\xi},\varphi\rangle &= -\langle a_m\xi_x,\varphi_x\rangle+\langle \tilde{a}\xi_x,\varphi_x\rangle+\langle b_m\xi,\varphi\rangle \\ &\quad -\langle \tilde{b}\xi,\varphi\rangle+\langle r,\varphi\rangle+\langle f,\varphi\rangle+\langle d,\varphi\rangle.\end{aligned} \tag{19}$$

Then, the following state error equation is derived:

$$\begin{aligned}\langle \dot{e},\varphi\rangle &= -\langle a_me_x,\varphi_x\rangle+\langle \tilde{a}e_x,\varphi_x\rangle+\langle b_me,\varphi\rangle \\ &\quad -\langle \tilde{b}e,\varphi\rangle+\langle \tilde{a}\xi_{mx},\varphi_x\rangle-\langle \tilde{b}\xi_m,\varphi\rangle+\langle f,\varphi\rangle+\langle d,\varphi\rangle,\end{aligned}$$

$$e(0,t)=e(1,t)=0,\ e(x,0)=\xi_0(x)-\xi_{m0}(x). \tag{20}$$

Now, consider a functional $V_0 : L_2(0,1)\times Q^2\times R \to R^+$ such that

$$\begin{aligned}V_0(e,\tilde{a},\tilde{b},\tilde{\mu}_d) &= \frac{1}{2}\langle e,e\rangle+\frac{1}{2\gamma_a}\langle \tilde{a},\tilde{a}\rangle_Q+\frac{1}{2\gamma_b}\langle \tilde{b},\tilde{b}\rangle_Q \\ &\quad +\frac{1}{2\gamma_d}\tilde{\mu}_d{}^2,\end{aligned} \tag{21}$$

where $\tilde{\mu}_d = \hat{\mu}_d-\mu_d$. Differentiating (21) with respect to $t$ along (20) yields:

$$\begin{aligned}\dot{V}_0 &= -\langle a_me_x,e_x\rangle+\langle b_me,e\rangle+\langle f,e\rangle+\langle d,e\rangle+\frac{1}{\gamma_d}\tilde{\mu}_d\dot{\hat{\mu}}_d \\ &\quad +\frac{1}{\gamma_a}\langle \tilde{a},\dot{\tilde{a}}\rangle_Q+\langle \tilde{a}e_x,e_x\rangle+\langle \tilde{a}\xi_{mx},e_x\rangle \\ &\quad +\frac{1}{\gamma_b}\langle \tilde{b},\dot{\tilde{b}}\rangle_Q-\langle \tilde{b}e,e\rangle-\langle \tilde{b}\xi_m,e\rangle.\end{aligned} \tag{22}$$

From (18a,b) and $\dot{\tilde{a}} = \dot{\hat{a}}-\dot{a}$ and $\dot{\tilde{b}} = \dot{\hat{b}}-\dot{b}$, the differential equations of the adaptive estimates errors can be given as

$$\begin{aligned}\langle \dot{\tilde{a}},p_a\rangle_Q &= -\langle \delta_a\hat{a},p_a\rangle_Q-\gamma_a\left(\langle p_ae_x,e_x\rangle+\langle p_a\xi_{mx},e_x\rangle\right. \\ &\quad \left.-\langle g_a,p_a\rangle_Q\right)-\langle \dot{a},p_a\rangle_Q,\end{aligned}$$

$$\begin{aligned}\langle \dot{\tilde{b}},p_b\rangle_Q &= -\langle \delta_b\hat{b},p_b\rangle_Q+\gamma_b\left(\langle p_be,e\rangle+\langle p_b\xi_m,e\rangle\right. \\ &\quad \left.+\langle g_b,p_b\rangle_Q\right)-\langle \dot{b},p_b\rangle_Q,\end{aligned}$$

where $p_a, p_b \in Q$. Therefore, (22) yields:

$$\dot{V}_0 = -\langle a_me_x,e_x\rangle+\langle b_me,e\rangle+\langle f,e\rangle+\langle d,e\rangle+\frac{1}{\gamma_d}\tilde{\mu}_d\dot{\hat{\mu}}_d$$

$$-\frac{\delta_a}{\gamma_a}\langle \hat{a},\tilde{a}\rangle_Q+\langle g_a,\tilde{a}\rangle_Q-\frac{1}{\gamma_a}\langle \dot{a},\tilde{a}\rangle_Q-\frac{\delta_b}{\gamma_b}\langle \hat{b},\tilde{b}\rangle_Q \\ +\langle g_b,\tilde{b}\rangle_Q-\frac{1}{\gamma_b}\langle \dot{b},\tilde{b}\rangle_Q. \tag{23}$$

The right hand side terms of (23) satisfy the following inequalities, respectively:

$$\langle f,e\rangle+\langle d,e\rangle+\frac{1}{\gamma_d}\tilde{\mu}_d\dot{\hat{\mu}}_d\le-\frac{\delta_d}{\gamma_d}\tilde{\mu}_d^{\,2}+\varepsilon_d+\frac{\delta_d}{2\gamma_d}\mu_d,$$

$$-\frac{\delta_a}{\gamma_a}\langle \hat{a},\tilde{a}\rangle_Q+\langle g_a,\tilde{a}\rangle_Q-\frac{1}{\gamma_a}\langle \dot{a},\tilde{a}\rangle_Q$$

$$\le-\frac{\delta_a}{\gamma_a}\|\tilde{a}\|_Q^2+\varepsilon_a+\mu_a\|a\|_Q+\frac{\delta_a}{\gamma_a}\|a\|_Q^2+\frac{1}{\gamma_a}\langle \dot{a},a\rangle_Q,$$

$$-\frac{\delta_b}{\gamma_b}\langle \hat{b},\tilde{b}\rangle_Q+\langle g_b,\tilde{b}\rangle_Q-\frac{1}{\gamma_b}\langle \dot{b},\tilde{b}\rangle_Q$$

$$\le-\frac{\delta_b}{\gamma_b}\|\tilde{b}\|_Q^2+\varepsilon_b+\mu_b\|b\|_Q+\frac{\delta_b}{\gamma_b}\|b\|_Q^2+\frac{1}{\gamma_b}\langle \dot{b},b\rangle_Q. \tag{24a, b, c}$$

Therefore, the derivative of the Lyapunov function candidate is bounded as follows:

$$\dot{V}_0\le-\left(a_m\pi^2-b_m\right)\|e\|^2-\frac{\delta_d}{\gamma_d}\tilde{\mu}_d^{\,2}-\frac{\delta_a}{\gamma_a}\|\tilde{a}\|_Q^2 \\ -\frac{\delta_b}{\gamma_b}\|\tilde{b}\|_Q^2+\upsilon(t), \tag{25}$$

where

$$\upsilon(t)=\varepsilon_d+\frac{\delta_d}{2\gamma_d}\mu_d+\varepsilon_a+\mu_a\|a\|_Q+\frac{\delta_a}{\gamma_a}\|a\|_Q^2+\frac{1}{\gamma_a}\langle \dot{a},a\rangle_Q \\ +\varepsilon_b+\mu_b\|b\|_Q+\frac{\delta_b}{\gamma_b}\|b\|_Q^2+\frac{1}{\gamma_b}\langle \dot{b},b\rangle_Q.$$

Note that $\upsilon(t)$ is bounded because of the assumption that $a$, $b$, $\dot{a}$, $\dot{b}$, and $\mu_d$ are bounded.

**Remark 2.** The existence and uniqueness of the solutions for coupled nonautonomous dynamical systems (20) and (18a-c) are addressed in Appendix A. Since $\upsilon(t)$ is bounded, the solutions $e$, $\tilde{a}$, $\tilde{b}$, and $\tilde{\mu}_d$ are uniformly ultimately bounded [15].

**Remark 3.** The equations of $\|f_a\|_Q$ and $\|f_b\|_Q$ can be rewritten as

$$\|f_a\|_Q=\left\|-\frac{\delta_a}{\gamma_a}\left(a+\frac{\dot{a}}{\delta_a}\right)\right\|_Q=\frac{1}{\gamma_a}\|-\delta_a a+\dot{a}\|_Q$$

and

$$\|f_b\|_Q=\left\|-\frac{\delta_b}{\gamma_b}\left(b+\frac{\dot{b}}{\delta_b}\right)\right\|_Q=\frac{1}{\gamma_b}\|-\delta_b b+\dot{b}\|_Q.$$

From $\mu_a\ge\|f_a\|_Q$ and $\mu_b\ge\|f_b\|_Q$, $\mu_a$ and $\mu_b$ can be chosen at reasonable values according to $\|f_a\|_Q$ and $\|f_b\|_Q$, respectively. Thus, $\upsilon(t)$ can be pushed in an arbitrarily small boundedness region by making sufficiently small $\varepsilon_a$, $\varepsilon_b$, $\varepsilon_d$, $\delta_a$, $\delta_b$, $\delta_d$ and sufficiently large $\gamma_a$, $\gamma_b$, $\gamma_d$.

All the above developments are now summarized as follows:

**Theorem 1.** Consider the coupled nonautonomous dynamical system (20) and (18a-c). Then, all signals in the system are uniformly ultimately bounded. Furthermore, the uniform ultimate boundedness region of the state error $e$ can be made arbitrarily small near to zero by a suitable choice of $\varepsilon_a$, $\varepsilon_b$, $\varepsilon_d$, $\delta_a$, $\delta_b$, $\delta_d$, $\gamma_a$, $\gamma_b$, and $\gamma_d$.

## IV. PARAMETER ERROR CONVERGENCE

Theorem 1 implies that the basic control objective is now achieved, i.e., all the signals in the closed loop are bounded and the trajectory following is achieved. In addition to the state error convergence near to zero, it is also desirable to have an adaptive control scheme to provide parameter estimation error convergence near to zero as well, i.e., the parameters $\hat{a}$ and $\hat{b}$ converge near to the true parameters $a$ and $b$ as quickly as possible. If the parameter error convergence is established, the robustness of the entire adaptive algorithm can be improved. To assure this, the following additional persistency of excitation condition on the reference model is required.

Let $H\overset{\Delta}{=}H_L^1(0,1)$ be a Hilbert space that is densely and continuously embedded in $L_2(0,1)$ [27, p.54-56], and $H^*$ be the continuous dual space of $H$. From (6), let $A_m:(D(A_m)\equiv H^2(0,1)\cap H)\to H^*$ be the reference model dynamic operator such that

$$\langle A_m\xi_m,\varphi\rangle\overset{\Delta}{=}-\langle a_m\xi_{mx},\varphi_x\rangle+\langle b_m\xi_m,\varphi\rangle,\ \xi_m\in H.$$

And, let $A_e(q):(D(A_e(q))\equiv H^2(0,1)\cap H)\to H^*$ be a differential operator such that

$$\langle A_e(q)\xi,\varphi\rangle\overset{\Delta}{=}-\langle q_1\xi_x,\varphi_x\rangle+\langle q_2\xi,\varphi\rangle,\ \xi\in H,$$

where $q=(q_1,q_2,q_3)\in W$, $W\overset{\Delta}{=}Q^2\times R$. Using the operators $A_m$ and $A_e(q)$, (6) and (20) can be rewritten, respectively, as

$$\langle \dot{\xi}_m, \varphi \rangle = \langle A_m \xi_m, \varphi \rangle + \langle r, \varphi \rangle, \quad (26a)$$

$$\langle \dot{e}, \varphi \rangle = \langle A_m e, \varphi \rangle - \langle A_e(\theta)\{e + \xi_m\}, \varphi \rangle + \langle f + d, \varphi \rangle \quad (26b)$$

where $\theta = (\tilde{a}, \tilde{b}, \tilde{\mu}_d) \in W$. Note that the following is obtained:

$$|\langle f + d, \varphi \rangle| \le \|f + d\| \|\varphi\| \le (\hat{\mu}_{d,\max} + \mu_d)\|\varphi\|,$$

where $\|f(x,t)\| \le |\hat{\mu}_d(t)| \le \hat{\mu}_{d,\max}$ for $t \ge 0$ with a positive constant $\hat{\mu}_{d,\max}$.

The following definition is then adopted.

**Definition.** The reference model (6), or the triple $\{A_m, r, \xi_{m0}\}$, is persistently exciting if there exist positive constants $\tau_0$, $\delta_0$, $\varepsilon_0$, and $c_0$, such that for each $q \in W$ with $|q|_W = 1$ and $t \ge 0$ sufficiently large, there exists $\bar{t} \in [t, t + \tau_0]$ for which

$$\left\| \int_{\bar{t}}^{\bar{t}+\delta_0} A_e(q)\{e(\tau) + \xi_m(\tau)\} d\tau \right\|_{H^*} \ge \varepsilon_0 + (\hat{\mu}_{d,\max} + \mu_d)\frac{\delta_0}{c_0}. \quad (27)$$

**Theorem 2.** If $r \in L_\infty(0,\infty;H)$ and $\xi_{m0} \in H$, and if the reference model (6) is persistently exciting, then the uniform ultimate boundedness region of the parameter estimation error vector $\theta = (\tilde{a}, \tilde{b}, \tilde{\mu}_d)$ can be made arbitrarily small near to zero by a suitable choice of $\varepsilon_a$, $\varepsilon_b$, $\varepsilon_d$, $\delta_a$, $\delta_b$, $\delta_d$, $\gamma_a$, $\gamma_b$, and $\gamma_d$.

**Proof.** In this proof the following notation is used: $|\cdot|_2 = \|\cdot\|_{L_2}$, $\|\cdot\| = \|\cdot\|_H$, and $\|\cdot\|_* = \|\cdot\|_{H^*}$. For the operators defined above, there exist $\alpha_0, \alpha_1 > 0$ and $K_0 > 0$ such that for $\psi_1, \psi_2 \in H$ and $q \in W$

$$|\langle A_m \psi_1, \psi_2 \rangle| \le \alpha_0 \|\psi_1\| \|\psi_2\|, \quad (28)$$

$$|\langle A_e(q) \psi_1, \psi_2 \rangle| \le \alpha_1 |q|_W \|\psi_1\| \|\psi_2\|, \quad (29)$$

$$\|\psi_1\|_* \le K_0 |\psi_1|_2. \quad (30)$$

From (18a-c), the parameter estimation error is rewritten as

$$\langle \dot{\theta}, q \rangle_W = \langle A_e(\gamma q)\{e + \xi_m\}, e \rangle + \langle -\delta\hat{\theta} + \gamma_0 |e|_2 + \gamma g - \dot{\theta}^*, q \rangle_W, \quad (31)$$

where $\gamma q = (\gamma_a q_1, \gamma_b q_2, \gamma_d q_3)$, $\delta\hat{\theta} = (\delta_a \hat{a}, \delta_b \hat{b}, \delta_d \hat{\mu}_d)$, $\gamma_0 = (0, 0, \gamma_d)$, $\gamma g = (\gamma_a g_a, \gamma_b g_b, 0)$, and $\dot{\theta}^* = (\dot{a}, \dot{b}, \dot{\mu}_d)$.

Now assume that $r \in L_\infty(0,\infty;H)$ and $\xi_{m0} \in H$. Then, $\xi_m \in L_\infty(0,\infty;H)$, see Theorem 2.2 of [5]. Assume that $|\theta(t)/\gamma|_W$ is uniformly bounded by $\rho$ where $\theta/\gamma = (\tilde{a}/\gamma_a, \tilde{b}/\gamma_b, \tilde{\mu}_d/\gamma_d)$. Then, from (29) and (31) it follows that for $q = \frac{1}{\rho}\left(\frac{\theta(t)}{\gamma}\right)$

$$\begin{aligned}
|\theta(t_2) - \theta(t_1)|_W &= \sup_{|q|_W \le 1} |\langle \theta(t_2) - \theta(t_1), q \rangle_W| \\
&= \sup_{|q|_W \le 1} \left| \left\langle \int_{t_1}^{t_2} \dot{\theta}(t) dt, q \right\rangle_W \right| \\
&\le \int_{t_1}^{t_2} \sup_{|q|_W \le 1} |\langle A_e(\gamma q)\{e(t) + \xi_m(t)\}, e(t) \rangle| dt \\
&\quad + \int_{t_1}^{t_2} \sup_{|q|_W \le 1} |\langle -\delta\hat{\theta} + \gamma_0 |e|_2 + \gamma g - \dot{\theta}^*, q \rangle_W| dt \\
&\le \alpha_1 \int_{t_1}^{t_2} \|e(t)\|^2 dt + \alpha_1 \|\xi_m(t)\|_{L_\infty(0,\infty;H)} \int_{t_1}^{t_2} \|e(t)\| dt \\
&\quad + \frac{1}{\rho} \int_{t_1}^{t_2} |\langle -\delta\hat{\theta} + \gamma_0 |e|_2 + \gamma g - \dot{\theta}^*, \theta/\gamma \rangle_W| dt \\
&\le \alpha_1 \int_{t_1}^{t_2} \|e(t)\|^2 dt + \alpha_1 \|\xi_m(t)\|_{L_\infty(0,\infty;H)} (t_2 - t_1)^{1/2} \\
&\quad \cdot \left\{ \int_{t_1}^{t_2} \|e(t)\|^2 dt \right\}^{1/2} + \frac{1}{\rho} \int_{t_1}^{t_2} |\langle -\delta\hat{\theta} + \gamma_0 |e|_2 \\
&\quad + \gamma g - \dot{\theta}^*, \theta/\gamma \rangle_W| dt.
\end{aligned} \quad (32)$$

For $t_2 > t_1$, (26b), (28), (29), and (30) imply that

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} A_e(\theta(t))\{e(t) + \xi_m(t)\} dt \right\|_* \le \|e(t_2)\|_* + \|e(t_1)\|_* \\
&\quad + \int_{t_1}^{t_2} \|A_m e(t)\|_* dt + \int_{t_1}^{t_2} \|f(t) + d(t)\|_* dt \\
&\le K_0 |e(t_2)|_2 + K_0 |e(t_1)|_2 + \alpha_0 (t_2 - t_1)^{1/2} \left\{ \int_{t_1}^{t_2} \|e(t)\|^2 dt \right\}^{1/2} \\
&\quad + \int_{t_1}^{t_2} \|f(t) + d(t)\|_* dt \le K_0 |e(t_2)|_2 + K_0 |e(t_1)|_2 \\
&\quad + \alpha_0 (t_2 - t_1)^{1/2} \left\{ \int_{t_1}^{t_2} \|e(t)\|^2 dt \right\}^{1/2} + (|\hat{\mu}_d| + \mu_d)(t_2 - t_1).
\end{aligned} \quad (33)$$

Once again assume that $\lim_{t\to\infty} |\theta(t)|_W > 0$, and let $\{t_k\}_{k=1}^{\infty}$ be an increasing sequence of positive numbers for which $\lim_{k\to\infty} t_k = \infty$ and

$$|\theta(t_k)|_W \ge c_0, \quad k = 1, 2, \cdots, \quad (34)$$

for some $c_0 > 0$. Assume further that the reference (6) is persistently exciting, and for each $k = 1, 2, \cdots$, let $\bar{t}_k \in [t_k, t_k + \tau_0]$ be such that

$$\left\| \int_{\bar{t}_k}^{\bar{t}_k+\delta_0} A_e\left(\frac{\theta(t_k)}{|\theta(t_k)|_W}\right) \xi(t) dt \right\|_* \ge \varepsilon_0 + (\hat{\mu}_{d,\max} + \mu_d)\frac{\delta_0}{c_0}. \quad (35)$$

Then, using (28), (29), (32) and (33) we obtain the estimate

$$
\begin{aligned}
&0 < c_0\varepsilon_0 + \left(\hat{\mu}_{d,\max} + \mu_d\right)\delta_0 = c_0\left(\varepsilon_0 + \left(\hat{\mu}_{d,\max} + \mu_d\right)\frac{\delta_0}{c_0}\right) \\
&\le |\theta(t_k)|_W \left\| \int_{\bar{t}_k}^{\bar{t}_k+\delta_0} A_e\left(\frac{\theta(t_k)}{|\theta(t_k)|_W}\right)\xi(t)dt \right\|_* \\
&= \left\| \int_{\bar{t}_k}^{\bar{t}_k+\delta_0} A_e\big(\theta(t_k)\big)\xi(t)dt \right\|_* \le \left\| \int_{\bar{t}_k}^{\bar{t}_k+\delta_0} A_e\big(\theta(t)\big)\xi(t)dt \right\|_* \\
&\quad + \left\| \int_{\bar{t}_k}^{\bar{t}_k+\delta_0} A_e\big(\theta(t_k)-\theta(t)\big)\{ e(t)+\xi_m(t)\}dt \right\|_* \\
&\le K_0 |e(\bar{t}_k+\delta_0)|_2 + K_0|e(\bar{t}_k)|_2 + \alpha_0\sqrt{\delta_0}\left\{\int_{\bar{t}_k}^{\bar{t}_k+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} \\
&\quad + \left(\hat{\mu}_{d,\max} + \mu_d\right)\delta_0 + \alpha_1 |\theta(\bar{t}_k+\tau_0+\delta_0)-\theta(\bar{t}_k)|_W \\
&\quad \cdot \int_{\bar{t}_k}^{\bar{t}_k+\delta_0}\left\{ \|e(t)\| + \|\xi_m(t)\| \right\}dt \le K_0 |e(\bar{t}_k+\delta_0)|_2 + K_0|e(\bar{t}_k)|_2 \\
&\quad + \alpha_0\sqrt{\delta_0}\left\{\int_{\bar{t}_k}^{\bar{t}_k+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} + \left(\hat{\mu}_{d,\max} + \mu_d\right)\delta_0 \\
&\quad + \alpha_1 \times \begin{bmatrix} \alpha_1\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\|e(t)\|^2 dt + \alpha_1\|\xi_m(t)\|_{L_\infty(0,\infty;H)} \\ \cdot\ \sqrt{\tau_0+\delta_0}\left\{\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} \\ + \dfrac{1}{\rho}\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\left|\left\langle -\delta\hat{\theta} + \gamma_0|e|_2 + \gamma g - \dot{\theta}^*, \theta/\gamma\right\rangle_W\right| dt \end{bmatrix} \\
&\quad \times\left[\sqrt{\delta_0}\left\{\int_{\bar{t}_k}^{\bar{t}_k+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} + \delta_0\|\xi_m(t)\|_{L_\infty(0,\infty;H)}\right].
\end{aligned} \tag{36}
$$

Note that (36) is rewritten as

$$
\begin{aligned}
&0 < c_0\varepsilon_0 \\
&\le K_0 |e(\bar{t}_k+\delta_0)|_2 + K_0|e(\bar{t}_k)|_2 + \alpha_0\sqrt{\delta_0}\left\{\int_{\bar{t}_k}^{\bar{t}_k+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} \\
&\quad + \alpha_1 \times \begin{bmatrix} \alpha_1\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\|e(t)\|^2 dt + \alpha_1\|\xi_m(t)\|_{L_\infty(0,\infty;H)}\sqrt{\tau_0+\delta_0} \\ \cdot\ \left\{\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} \\ + \dfrac{1}{\rho}\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\left|\left\langle -\delta\hat{\theta} + \gamma_0|e|_2 + \gamma g - \dot{\theta}^*, \theta/\gamma\right\rangle_W\right| dt \end{bmatrix} \\
&\quad \times\left[\sqrt{\delta_0}\left\{\int_{\bar{t}_k}^{\bar{t}_k+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} + \delta_0\|\xi_m(t)\|_{L_\infty(0,\infty;H)}\right].
\end{aligned} \tag{37}
$$

From the adaptive laws (18a-c) and (24b,c) we can have

$$\left\langle -\delta\hat{\theta} + \gamma g - \dot{\theta}^*, \theta/\gamma\right\rangle_W \le \upsilon'(t),$$

where

$$
\begin{aligned}
\upsilon'(t) &= \varepsilon_a + \mu_a\|a\|_Q + \frac{\delta_a}{\gamma_a}\|a\|_Q^2 + \frac{1}{\gamma_a}\langle \dot{a}, a\rangle_Q + \varepsilon_b + \mu_b\|b\|_Q \\
&\quad + \frac{\delta_b}{\gamma_b}\|b\|_Q^2 + \frac{1}{\gamma_b}\langle \dot{b}, b\rangle_Q
\end{aligned}
$$

and $\upsilon'(t)$ can be made arbitrary small near to zero by making sufficiently small $\varepsilon_a$, $\varepsilon_b$, $\delta_a$, $\delta_b$, and sufficiently large $\gamma_a$, $\gamma_b$. Now, from Appendix B, for any $M > 0$, $\lim_{t\to\infty}\int_t^{t+M}\|e(s)\|^2 ds \approx 0$. Therefore, letting $k \to \infty$ in (37) and sufficiently small $\varepsilon_d$, $\varepsilon_a$, $\varepsilon_b$, $\delta_a$, $\delta_b$, $\delta_d$, and sufficiently large $\gamma_a$, $\gamma_b$, $\gamma_d$, Theorem 1 and Appendix B imply that

$$
\begin{aligned}
&0 < c_0\varepsilon_0 \le K_0 \lim_{k\to\infty}|e(\bar{t}_k+\delta_0)|_2 + K_0\lim_{k\to\infty}|e(\bar{t}_k)|_2 \\
&\quad + \alpha_0\sqrt{\delta_0}\left\{\lim_{k\to\infty}\int_{\bar{t}_k}^{\bar{t}_k+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} \\
&\quad + \alpha_1 \times \begin{bmatrix} \lim_{k\to\infty}\alpha_1\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\|e(t)\|^2 dt \\ +\alpha_1\|\xi_m(t)\|_{L_\infty(0,\infty;H)}\sqrt{\tau_0+\delta_0} \\ \cdot\left\{\lim_{k\to\infty}\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} \\ + \dfrac{1}{\rho}\lim_{k\to\infty}\int_{\bar{t}_k}^{\bar{t}_k+\tau_0+\delta_0}\left|\left\langle \delta\hat{\theta} - \gamma_0|e|_2 \right.\right. \\ \left.\left. -\gamma g + \dot{\theta}^*, \theta/\gamma\right\rangle_W\right| dt \end{bmatrix} \\
&\quad \times\left[\sqrt{\delta_0}\left\{\lim_{k\to\infty}\int_{\bar{t}_k}^{\bar{t}_k+\delta_0}\|e(t)\|^2 dt\right\}^{1/2} + \delta_0\|\xi_m(t)\|_{L_\infty(0,\infty;H)}\right] \approx 0,
\end{aligned}
$$

which is a contradiction, and the theorem is proved. ■

## V. SIMULATIONS

To illustrate the application of the algorithm developed in the previous sections, a heat transfer equation with a time-varying coefficient and a spatiotemporally varying disturbance is chosen. Let the heat transfer equation be given with known homogeneous boundary conditions as

$$
\begin{aligned}
&\dot{\xi}(x,t) = a(t)\xi_{xx}(x,t) + u(x,t) + d(x,t),\ x \in [0,\ 1],\ t > 0, \\
&\xi(0,t) = \xi(1,t) = 0,\ t > 0, \text{ and} \\
&\xi(x,0) = \xi_0(x) = 0.2\sin(2\pi x),
\end{aligned} \tag{38}
$$

where $a(t)$ is the time-varying conductivity and $d(x, t)$ is the spatiotemporally varying latent heat of transformation which is treated as disturbance. $a(t)$ and $d(x, t)$ are unknown, but for simulation purpose $a(t) = 3 - 2.5\sin(3t)$ and $d(x, t) = 0.3 + 0.5\sin(3\pi x)\sin(5t)$ are assumed. The reference model is chosen as

$$\begin{aligned} &\dot{\xi}_m(x,t) = 0.5\xi_{mxx}(x,t) + 5,\ x \in [0,1],\ t > 0, \\ &\xi_m(0,t) = \xi_m(1,t) = 0,\ t > 0, \text{ and} \\ &\xi_m(x,0) = -\sin(\pi x). \end{aligned} \tag{39}$$

The control gains in (17), (18a), and (18c) are chosen as $\varepsilon_d = 0.1$, $\delta_a = 0.01$, $\gamma_a = 50$, $\mu_a = 0.2$, $\varepsilon_a = 0.01$, $\delta_d = 0.1$, and $\gamma_d = 30$. Figure 1 shows the convergence of the state error $e(x,t)$ near to zero. Figure 2 shows the convergence of the estimated parameter $\hat{a}(t)$ near to the plant parameter $a(t)$. Figure 3 shows the convergence of the estimated parameter $\hat{\mu}_d(t)$ near to the bounded value of the disturbance norm $\|d(x,t)\|$.

## VI. CONCLUSIONS

A robust MRAC algorithm for a linear parabolic system with unknown spatiotemporally varying coefficients and disturbance was developed. The coefficients were assumed to be uniformly bounded with uniformly bounded derivatives, but they were allowed to vary arbitrarily fast. The unknown disturbance was also assumed to be uniformly bounded. Under the unknown plant parameters and external disturbances, the robust MRAC law developed in this paper assured the closed loop system to track a desired signal which comes from a reference model. Because the derivative of a Lyapunov function candidate was not negative semidefinite, only uniform ultimate boundedness would have been concluded. However, further analysis in this paper has shown that the state error, which remains in the derivative of the Lyapunov function candidate, converges near to zero. Also, with the additional persistence of excitation condition, the algorithm guaranteed the convergence of the adjustable controller parameters near to their nominal values. The application of the robust MRAC scheme proposed can be extended to flexible robots/structures including MEMS which are described by linear hyperbolic systems.

However, any feedback controller for such a DPS must be a finite-dimensional (and discrete-time) system to be implemented with on-line digital computers with a finite number of actuators and sensors. Therefore, the research issues like how to synthesize finite-dimensional controllers that can be implemented by a finite number of actuators and sensors and on-line computers and how to assess the stability and performance of these controllers with the actual DPSs are left for future work.

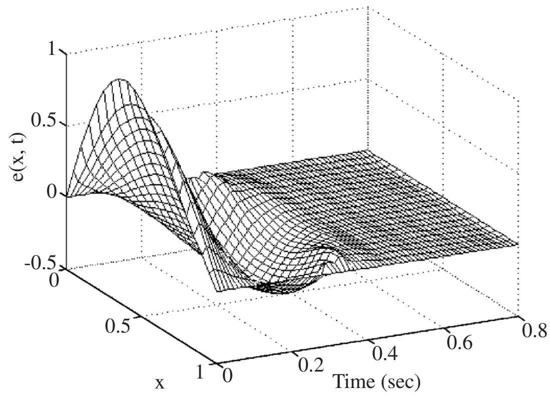


Fig. 1. Convergence of state error $e(x,t)$ near to zero.

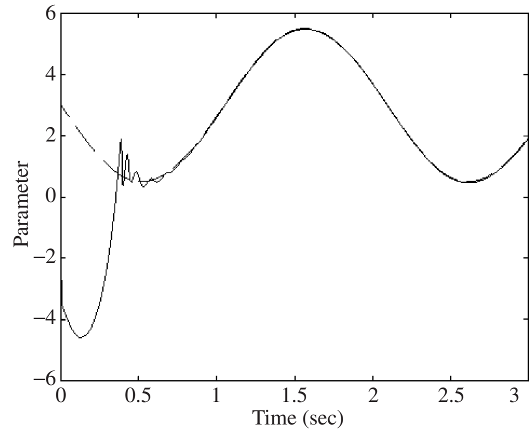


Fig. 2. Convergence of an estimated parameter $\hat{a}(t)$ (solid line) near to the true plant parameter $a(t)$ (dashed line).

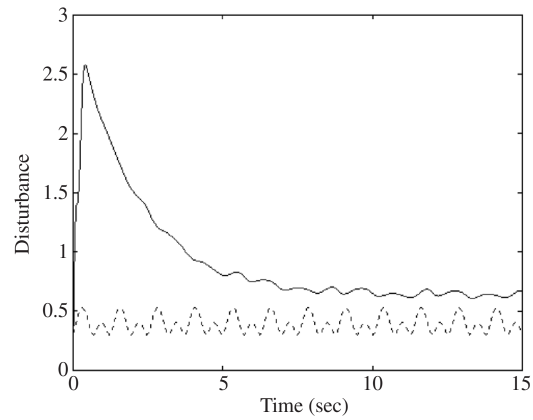


Fig. 3. Convergence of an estimated parameter $\hat{\mu}_d(t)$ (solid line) near to a bounded value of disturbance norm $\|d(x,t)\|$ (dashed line).

## APPENDIX A.

**Existence and uniqueness of the coupled system.**

Let $H_L^{1*}(0,1)$ be the algebraic dual space of $H_L^1(0,1)$, and let

$$A_1 : (D(A_1) \equiv H^2(0,1) \cap H_L^1(0,1)) \rightarrow H_L^{1*}(0,1)$$

and

$$A_2(t) : (D(A_2) \equiv H^2(0,1) \cap H_L^1(0,1)) \rightarrow H_L^{1*}(0,1)$$

be differential operators such that

$$A_1 \triangleq \frac{\partial}{\partial x}\left(a_m \frac{\partial}{\partial x}\right) + b_m \text{ and } A_2(t) \triangleq \frac{\partial}{\partial x}\left(a \frac{\partial}{\partial x}\right) + b.$$

Then, nonlinear coupled system (20) and (18a-c) can be rewritten as follows:

$$\begin{aligned}\langle \dot{e}, \varphi \rangle &= \langle A_1 e, \varphi \rangle + \langle A_2(t) e, \varphi \rangle - \langle (\hat{a} e_x)_x, \varphi \rangle - \langle \hat{b} e, \varphi \rangle \\ &\quad - \langle (\tilde{a}\xi_{mx})_x, \varphi \rangle - \langle \tilde{b}\xi_m, \varphi \rangle + \langle f, \varphi \rangle + \langle d, \varphi \rangle,\end{aligned} \tag{A1}$$

$$\begin{aligned}\langle \dot{\hat{a}}, p_a \rangle_Q &= -\langle \delta_a \hat{a}, p_a \rangle_Q \\ &\quad - \gamma_a \left( \langle e_x e_x, p_a \rangle_Q + \langle \xi_{mx} e_x, p_a \rangle_Q - \langle g_a, p_a \rangle_Q \right),\end{aligned} \tag{A2}$$

$$\begin{aligned}\langle \dot{\hat{b}}, p_b \rangle_Q &= -\langle \delta_b \hat{b}, p_b \rangle_Q \\ &\quad + \gamma_b \left( \langle ee, p_b \rangle_Q + \langle \xi_m e, p_b \rangle_Q + \langle g_b, p_b \rangle_Q \right),\end{aligned} \tag{A3}$$

$$\dot{\hat{\mu}}_d r_d = -\delta_d \hat{\mu}_d r_d + \gamma_d \|e\| r_d, \tag{A4}$$

where $r_d \in R$. From the system (A1)-(A4), the following is obtained:

$$\begin{aligned}&\langle \dot{e}, \varphi \rangle + \langle \dot{\hat{a}}, p_a \rangle_Q + \langle \dot{\hat{b}}, p_b \rangle_Q + \dot{\hat{\mu}}_d r_d \\ &= \langle A_1 e, \varphi \rangle + \langle A_2(t) e, \varphi \rangle - \langle \delta_a \hat{a}, p_a \rangle_Q - \langle \delta_b \hat{b}, p_b \rangle_Q \\ &\quad - \delta_d \hat{\mu}_d r_d - \langle (\hat{a} e_x)_x, \varphi \rangle - \langle \hat{b} e, \varphi \rangle - \langle ((\hat{a} - a)\xi_{mx})_x, \varphi \rangle \\ &\quad - \langle (\hat{b} - b)\xi_m, \varphi \rangle + \langle f, \varphi \rangle + \langle d, \varphi \rangle \\ &\quad - \gamma_a \left( \langle e_x e_x, p_a \rangle_Q + \langle \xi_{mx} e_x, p_a \rangle_Q - \langle g_a, p_a \rangle_Q \right) \\ &\quad + \gamma_b \left( \langle ee, p_b \rangle_Q + \langle \xi_m e, p_b \rangle_Q + \langle g_b, p_b \rangle_Q \right) \\ &\quad + \gamma_d \|e\| r_d.\end{aligned} \tag{A5}$$

Define a state space as $Y \triangleq L_2(0,1) \times Q^2 \times R$. The system (A5) is then rewritten as

$$\begin{aligned}&\left\langle \begin{pmatrix} \dot{e} \\ \dot{\hat{a}} \\ \dot{\hat{b}} \\ \dot{\hat{\mu}}_d \end{pmatrix}, \begin{pmatrix} \varphi \\ p_a \\ p_b \\ r_d \end{pmatrix} \right\rangle_Y \\ &= \left\langle \begin{bmatrix} A_1 + A_2(t) & 0 & 0 & 0 \\ 0 & -\delta_a & 0 & 0 \\ 0 & 0 & -\delta_b & 0 \\ 0 & 0 & 0 & -\delta_d \end{bmatrix} \begin{pmatrix} e \\ \hat{a} \\ \hat{b} \\ \hat{\mu}_d \end{pmatrix}, \begin{pmatrix} \varphi \\ p_a \\ p_b \\ r_d \end{pmatrix} \right\rangle_Y \\ &\quad + \left\langle \begin{bmatrix} -(\hat{a} e_x)_x - \hat{b} e - ((\hat{a} - a)\xi_{mx})_x - (\hat{b} - b)\xi_m + f + d \\ -\gamma_a (e_x e_x + \xi_{mx} e_x - g_a) \\ \gamma_b (ee + \xi_m e + g_b) \\ \gamma_d \|e\| \end{bmatrix}, \begin{pmatrix} \varphi \\ p_a \\ p_b \\ r_d \end{pmatrix} \right\rangle_Y,\end{aligned} \tag{A6}$$

where $(\varphi, p_a, p_b, r_d)^T \in Y$.

From (A6), the following system in weak form is then obtained:

$$\langle \dot{z}, \Theta \rangle_Y = \langle A(t) z, \Theta \rangle_Y + \langle F(t,z), \Theta \rangle_Y, \; z(0) = z_0, \tag{A7}$$

where $z = (e, \hat{a}, \hat{b}, \hat{\mu}_d)^T \in Y, \Theta = (\varphi, p_a, p_b, r_d)^T \in Y$, and

$$A(t) = \begin{bmatrix} A_0(t) & 0 & 0 & 0 \\ 0 & -\delta_a & 0 & 0 \\ 0 & 0 & -\delta_b & 0 \\ 0 & 0 & 0 & -\delta_d \end{bmatrix},$$

$$F(t,z) = \begin{bmatrix} -(\hat{a} e_x)_x - \hat{b} e - ((\hat{a} - a)\xi_{mx})_x - (\hat{b} - b)\xi_m + f + d \\ -\gamma_a (e_x e_x + \xi_{mx} e_x - g_a) \\ \gamma_b (ee + \xi_m e + g_b) \\ \gamma_d \|e\| \end{bmatrix},$$

where $A_0(t) \triangleq A_1 + A_2(t)$.

The weak form (A7) is formally equivalent to the system

$$\dot{z} = A(t) z + F(t,z), \; z(0) = z_0. \tag{A8}$$

Therefore, the existence of a unique solution to the system (A7) can be established by establishing the existence of a unique strong solution to the initial value problem in $Y$ given by (A8). The domain of the operator $A(t)$ is given by

$$D(A)=\left\{(e,\hat{a},\hat{b},\hat{\mu}_d)\in Y: e\in H^2(0,1)\cap H_0^1(0,1) \text{ with } e(0)=0=e(1), \text{ and } \hat{a},\hat{b}\in Q, \hat{\mu}_d\in R\right\},$$

where the subscript 0 in $H$ denotes $\eta(x) = 0$ at both $x = 0$ and $x = 1$ in (4). Note that the boundary conditions of (20) have been incorporated in the space $H^2(0,1)\cap H_0^1(0,1)$, which is the domain of the differential operator $A_0$. $D(A)$ is dense, and $A$ is a closed operator ([28]).

For $z \in D(A)$

$$\begin{aligned}\langle z, Az\rangle_Y &= -\langle a_m e_x, e_x\rangle + \langle b_m e, e\rangle - \langle a e_x, e_x\rangle + \langle be, e\rangle \\ &\quad -\delta_a\|\hat{a}\|_Q^2 - \delta_b\|\hat{b}\|_Q^2 - \delta_d\hat{\mu}_d^2 \\ &\le -a_m\|e_x\|^2 + b_m\|e\|^2 - \beta_0(a)\|e_x\|^2 + \|b\|_Q\|e\|^2 \\ &\quad -\delta_a\|\hat{a}\|_Q^2 - \delta_b\|\hat{b}\|_Q^2 - \delta_d\hat{\mu}_d^2 \\ &\le -\left(a_m\pi^2 - b_m + \beta_0(a)\pi^2 - \|b\|_Q\right)\|e\|^2 \\ &\quad -\delta_a\|\hat{a}\|_Q^2 - \delta_b\|\hat{b}\|_Q^2 - \delta_d\hat{\mu}_d^2,\end{aligned} \tag{A9}$$

where $\beta_0(a) > 0$ such that $\langle a e_x, e_x\rangle \ge \beta_0(a)\|e_x\|^2$, $\pi^2\|e\|^2 \le \|e_x\|^2$ from (14), and $a_m\pi^2 + \beta_0(a)\pi^2 \ge b_m + \|b\|_Q$ is assumed. Hence, from (A9) the following is obtained:

$$\langle z, Az\rangle_Y \le -C_1\langle z, z\rangle_Y,$$

where

$$C_1 = \min\left\{a_m\pi^2 - b_m + \beta_0(a)\pi^2 - \|b\|_Q, \delta_a, \delta_b, \delta_d\right\} > 0.$$

By the linearity of $A$, we see that $\omega I - A$ is monotone (accretive) for every $\omega \in C_1$. Hence, $A : D(A) \subset Y \to Y$ is the infinitesimal generator of a linear process

$$\{S(t)\}_{t\ge0} = \left\{(\Phi(t,0), \hat{A}(t), \hat{B}(t), \hat{E}(t))\right\}_{t\ge0}$$

on $Y$, see [28, p. 92, Theorem 3.2]. Note that the first component $\Phi(t, 0)$ is generated by $A_0$. Note also that $\Phi(t, 0)e_0$ is the strong solution of the evolution equation $\dot{e}(t) = A_0 e(t)$ for every $e_0 \in D(A_0)$.

Now, set

$$z=\left(e,\hat{a},\hat{b},\hat{\mu}_d\right) \text{ and } z'=\left(e',\hat{a}',\hat{b}',\hat{\mu}_d'\right).$$

Then,

$$\begin{aligned}\|F(t,z)-F(t,z')\|_Y^2 &= \Big\| -(\hat{a}e_x)_x + (\hat{a}'e_x')_x - \hat{b}e + \hat{b}'e' \\ &\quad -((\hat{a}-a)\xi_{mx})_x + ((\hat{a}'-a)\xi_{mx})_x - (\hat{b}-b)\xi_m \\ &\quad +(\hat{b}'-b)\xi_m + f - f' + d - d\Big\|^2 \\ &\quad +\gamma_a^2\|e_x e_x - e_x'e_x' + \xi_{mx}e_x - \xi_{mx}e_x' - g_a - g_a'\|_Q^2 \\ &\quad +\gamma_b^2\|ee - e'e' + \xi_m e - \xi_m e' - g_b - g_b'\|_Q^2 + \gamma_d^2\|\|e\| - \|e'\|\|^2 \\ &\le \left(\beta_e^2 K_a^2\|\hat{a}'\|_Q^2 + \|\hat{b}'\|_Q^2 + \frac{2|\hat{\mu}_d'|^6\|e'\|^2 + \varepsilon_d^2|\hat{\mu}_d'|^4}{|(\hat{\mu}_d'\|e\| + \varepsilon_d)(\hat{\mu}_d'\|e'\| + \varepsilon_d)|^2}\right. \\ &\quad +\gamma_a^2 K_a^2\left(\|e_x\|^2 + \|e_x'\|^2 + \|\xi_{mx}\|^2\right) \\ &\quad \left.+\gamma_b^2\left(\|e\|^2 + \|e'\|^2 + \|\xi_m\|^2\right) + \gamma_d^2 K_d^2\right)\|e-e'\|^2 \\ &\quad +\left(\beta_a^2\left(\|e_x\|^2 + \|\xi_{mx}\|^2\right) + \frac{\gamma_a^2\left(2\mu_a^6\|\hat{a}'\|_Q^2 + \varepsilon_a^2\mu_a^4\right)}{(\|\hat{a}\|_Q\mu_a + \varepsilon_a)(\|\hat{a}'\|_Q\mu_a + \varepsilon_a)}\right) \\ &\quad \cdot\|\hat{a}-\hat{a}'\|_Q^2 \\ &\quad +\left(\|e\|^2 + \|\xi_m\|^2 + \frac{\gamma_b^2\left(2\mu_b^6\|\hat{b}'\|_Q^2 + \varepsilon_b^2\mu_b^4\right)}{\left(\|\hat{b}\|_Q\mu_b + \varepsilon_b\right)\left(\|\hat{b}'\|_Q\mu_b + \varepsilon_b\right)}\right)\|\hat{b}-\hat{b}'\|_Q^2 \\ &\quad +\left|\frac{\hat{\mu}_d\hat{\mu}_d'\|e\| + \varepsilon_d(\hat{\mu}_d + \hat{\mu}_d')}{(\hat{\mu}_d\|e\| + \varepsilon_d)(\hat{\mu}_d'\|e\| + \varepsilon_d)}\right|^2\|e\|^2|\hat{\mu}_d - \hat{\mu}_d'|^2,\end{aligned} \tag{A10}$$

where

$$f' = -\frac{\hat{\mu}_d'^2}{\hat{\mu}_d'\|e'\| + \varepsilon_d}e', \quad g_a' = -\frac{\mu_a^2}{\|\hat{a}'\|_Q\mu_a + \varepsilon_a}\hat{a}',$$

and

$$g_b' = -\frac{\mu_b^2}{\|\hat{b}'\|_Q\mu_b + \varepsilon_b}\hat{b}'.$$

Hence,

$$\|F(t,z) - F(t,z')\|_Y \le C_2\|z - z'\|_Y,$$

where $C_2$ is a positive constant. Therefore $F : Y \to Y$ is locally Lipschitz continuous in $Y$. Thus a unique solution exists. Finally, the strong solution of (20) can be written in the following variation of constant formula ([22])

$$e(t) = \Phi(t,0)e(0)$$

$$+\int_0^t \Phi(t,\tau)\Big(-(\hat{a}(\tau)e(\tau)_x)_x - \hat{b}(\tau)e(\tau) - (\tilde{a}(\tau)\xi_{mx}(\tau))_x$$
$$-\tilde{b}(\tau)\xi_m(\tau) + f(\tau) + d(\tau)\Big)d\tau,$$

where $\Phi(t, s)$ is the evolution operator associated with $A_0$ in the space $L_2(0, 1)$.

## APPENDIX B.

**Tracking error convergence**

From (14), (23), and (25), we have

$$\dot{V}_0 \le -\langle a_m e_x, e_x\rangle + \langle b_m e, e\rangle + \upsilon(t)$$
$$\le -a_m \langle e_x, e_x\rangle + \frac{b_m}{\pi^2}\langle e_x, e_x\rangle + \upsilon(t)$$
$$\le -\beta_0 \| e\|^2 + \upsilon(t),$$

where

$$a_m\pi^2 > b_m,\ \beta_0 > 0,$$

and

$$\upsilon(t) = \varepsilon_d + \frac{\delta_d}{2\gamma_d}\mu_d + \varepsilon_a + \mu_a \|a\|_Q + \frac{\delta_a}{\gamma_a}\|a\|_Q^2$$
$$+\frac{1}{\gamma_a}\langle \dot{a}, a\rangle_Q + \varepsilon_b + \mu_b\|b\|_Q + \frac{\delta_b}{\gamma_b}\|b\|_Q^2$$
$$+\frac{1}{\gamma_b}\langle \dot{b}, b\rangle_Q.$$

$\upsilon(t)$ can be made arbitrary small near to zero by making sufficiently small $\varepsilon_d$, $\varepsilon_a$, $\varepsilon_b$, $\delta_a$, $\delta_b$, $\delta_d$ and sufficiently large $\gamma_a$, $\gamma_b$, $\gamma_d$, and be assumed to be uniformly bounded by $\upsilon_0$. Thus, from Theorem 1 the following is satisfied:

$$\lim_{t\to\infty}\int_t^{t+M} \|e(s)\|^2\, ds \approx 0 \text{ for any } M > 0.$$

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