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Vibration Control of a Timoshenko Cantilever Beam with Varying Length

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Abstract: This paper addresses the vibration control of a Cartesian palletizer consisting of a trolley and a robotic arm, wherein the robotic arm is modeled as a thick cantilever beam of varying length. The Timoshenko beam theory, which describes the behavior of thick beams, is used to model the robotic arm's dynamics. A mathematical model describing the trolley's motion and the robotic arm's vibration is established based on the extended Hamilton principle. According to this dynamic model, a boundary control law is proposed to suppress the undesired transverse vibration of the robotic arm. The uniform stability of the closed-loop system is proven via the Lyapunov method. The simulation results show that the proposed control law can simultaneously control the trolley's position and the robotic arm's vibration.

Keywords: Axially moving system, boundary control, Lyapunov method, Timoshenko beam, vibration control.

1. INTRODUCTION

A pick-and-place robot is an automated material handling robot used to pick up objects from one location and place them in another. A Cartesian palletizer is a typical example of a pick-and-place robot (i.e., Fig. 1). The scheme of the placing process of a Cartesian palletizer is demonstrated in Fig. 2. As shown in this figure, the Cartesian palletizer consists of a trolley and a robotic arm. Due to the trolley's motion, undesired transverse vibration occurs along the robotic arm during the placing process. This vibration is one of the negative factors that limit the system's performance. This study, therefore, sets out to address the vibration control of a Cartesian palletizer consisting of a trolley and a flexible robotic arm. It is noted that the robotic arm of a Cartesian palletizer can be treated as a thick and short cantilever beam with varying length. Therefore, the Timoshenko beam theory, which can accurately predict the dynamical behaviors of thick beams, should be used to consider the dynamics of the robotic arm.

Fig. 1. Cartesian palletizer (www.yushinamerica.com/product-release-pa-compact-palletizing-robot).

From an engineering perspective, the robotic arm of a Cartesian palletizer can be modeled by a flexible cantilever beam with a translating base. Various beam theories, including Euler-Bernoulli, Rayleigh, and Timoshenko, have been used to model flexible beams [1–11]. The Euler-Bernoulli theory is the fundamental beam theory that considers the bending effect of a beam, whereas the Rayleigh theory further includes the influence of rotational inertia of the beam's cross-section on the dynamic model of the beam [3–5]. Based on the Rayleigh theory, Timoshenko proposed an improved theory by considering the beam's shear effect, known as the Timoshenko theory

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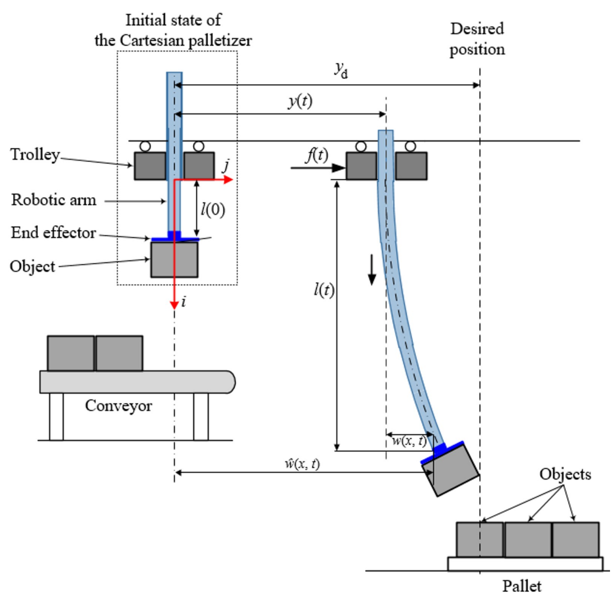

Fig. 2. Scheme of the process of placing objects onto the pallet by using a Cartesian palletizer.

[6,7]. Among these theories, the Euler-Bernoulli theory is the most common theory for modeling flexible beams. A key advantage of using this theory is that it leads to a mathematically simple dynamic model. Therefore, the Euler-Bernoulli theory is very convenient for the dynamic analysis and control design of flexible beams. Based on this beam theory, dynamic models of beams with translating bases were analyzed in [8–10]. Shah *et al.* [11] designed a control law for a refueling machine transporting fuel rods operating in the water, wherein the fuel rod was modeled by an Euler-Bernoulli beam. Contrary to the Euler-Bernoulli theory, beam models based on the Timoshenko beam are mathematically complicated. However, a study of beam theories has revealed that the Timoshenko theory predicts the beam's dynamics more accurately than the other two theories [5]. Additionally, the Timoshenko theory is more appropriate for modeling thick and short beams, wherein the shear deflection becomes more critical.

Since the robotic arm's length varies during the picking and placing process, it can be considered an axially moving beam with a time-varying length. Various dynamic models of axially moving systems, characterized by partial differential equations (PDEs), have been established using the extended Hamilton principle [12–18]. A summary of the dynamics of axially moving systems has been provided in the work of Pham and Hong [16]. A considerable amount of literature has been published on axially moving systems with time-varying length. In [17,18], a dynamic model was derived for a crane system consisting of a trolley and a hosting cable, wherein the cable was modeled as translating strings with time-varying length. In [19,20], Xing *et al.* established mathematical models for variable-length strings moving in the three-dimensional space. Few investigations have been conducted on axially moving beams with time-varying lengths [21,22].

In terms of the control of flexible systems, one of the most well-known techniques for addressing this problem is boundary control [23–38]. Numerous studies regarding the boundary control of Euler-Bernoulli beams have been published [1,2,35–38]. These studies are very diverse. In [36,37], sliding mode boundary controllers were designed for flexible Euler-Bernoulli beams in the presence of disturbances. Shah and Hong [1] proposed a control law for suppressing the vibration of a beam with a translating base operating in water. Several works on the boundary control of Timoshenko beams have been carried out [39–44]. One study by He *et al.* [41] modeled a flexible robotic manipulator based on the Timoshenko beam theory and addressed the boundary control of this system. Wu *et al.* [43] designed a control law for a manipulating system consisting of two flexible Timoshenko beams. Most of the studies presented thus far concentrate on the control of flexible beams with constant length. Works on the control of axially moving beams with time-varying length are very limited. Zhu *et al.* [21] published a study on this subject. Their work developed the dynamic model of a beam with time-varying length based on the Euler-Bernoulli beam theory. A pointwise controller was proposed to suppress the transverse vibration of the beam.

This paper addresses the control problem of a Cartesian palletizer, wherein the robotic arm is treated as a Timoshenko cantilever beam with time-varying length. The dynamic model describing the trolley's motion and the beam's transverse vibration and rotational angular displacement of the cross section is established by using the extended Hamilton principle. Based on this model, a boundary control law is proposed to move the trolley to the desired position and suppress the beam's vibration. The uniform stability of the closed-loop control is verified via the Lyapunov method. Finally, simulation results are provided to demonstrate the effectiveness of the designed control law.

The main contributions of this paper are the following: i) A novel dynamic model of a variable-length Timoshenko beam attached to a translating base is developed for the first time. ii) The developed control input simultaneously can position the base at a desired location and suppress the beam's vibration. iii) The uniform stability of the closed-loop system is proved, and simulation results are provided.

The remainder of this paper is organized as follows: The dynamic model of the system is derived in Section 2. In Section 3, the stability of the closed-loop system under the proposed boundary control law is analyzed. The simulation results are shown in Section 4, and the conclusions are drawn in Section 5.

2. DYNAMIC MODEL

Fig. 2 depicts the placing process of a Cartesian palletizer consisting of a trolley and a robotic arm. The trolley carrying the robotic arm moves along the j-axis, whereas the robotic arm is extended in the i-axis direction. The length of the robotic arm $l(t)$ is a prespecified time function. In this study, the robotic arm is treated as a cantilever beam with varying length. This paper assumes that i) the robotic arm is a clamp-free Timoshenko beam and ii) the end effector with the object is modeled as an end mass m with rotary inertia J. The Timoshenko beam theory considers the effects of both shear deformation and rotational inertia. Therefore, the deflection of the beam is characterized by the transverse vibration $w(x,\ t)$ and the rotational angular displacement of the cross section $\theta(x,\ t)$ (see Fig. 3). Let $\hat{w}(x,\ t) = y(t) + w(x,\ t)$, where $y(t)$ is the trolley position. It is noted that $\hat{w}_x(x,\ t) = w_x(x,\ t)$ and $\hat{w}(0,\ t) = y(t)$. In this paper, the extension/shortening of the robotic arm and the end-effector is considered as an axially moving beam, see [45]. A beam of variable length shows the gyroscopic property, wherein all the beam's elements are subjected to an axial velocity field (i.e., $\dot{l}$). Therefore, the material derivative, $D(\cdot)/Dt = (\cdot)_t + \dot{l}(\cdot)_x$, is used to describe the time rate of change of the beam element's displacement. According to the Timoshenko beam theory, the kinetic and potential energies of the system are given as follows:

$$K = \frac{1}{2}\rho A \int_0^l (\hat{w}_t + \dot{l}\hat{w}_x)^2 dx + \frac{1}{2}\rho I \int_0^l (\theta_t + \dot{l}\theta_x)^2 dx + \frac{1}{2}M\hat{w}_t(0,t)^2 + \frac{1}{2}m\left[\frac{D\hat{w}}{Dt}\bigg|_{x=l}\right]^2 + \frac{1}{2}J\left[\frac{D\theta}{Dt}\bigg|_{x=l}\right]^2, \tag{1}$$

$$U = \frac{1}{2}\int_0^l \hat{w}_x^2 P(x,t)dx + \frac{1}{2}EI\int_0^l {\theta_x}^2 dx + \frac{1}{2}\kappa GA \int_0^{l(t)} (\theta - \hat{w}_x)^2 dx, \tag{2}$$

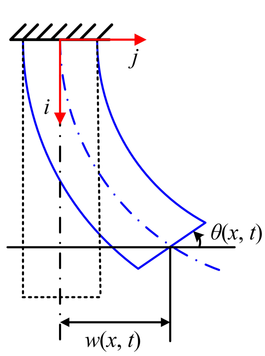

Fig. 3. Definition of the transverse vibration $w(x,t)$ and the rotational angular displacement of the cross section $\theta(x,t)$ of a Timoshenko beam.

where M and ρ denote the trolley mass and the mass density of the beam, respectively; A and I indicate the cross-sectional area and the area moment of initial of the beam, respectively; and E, G, and κ are the Young modulus, the shear modulus, and the shear coefficient, respectively. It is noted that the axial force of the beam is a spatiotemporal function defined as follows [22]:

$$P(x,t) = (m + \rho A(l - x))\,(g - \ddot{l}), \tag{3}$$

where g is the gravitational acceleration. In the Cartesian palletizer, the motion of the trolley is generated by the control force $f(t)$. Therefore, the work done is given by

$$\delta W = f\delta\hat{w}(0,t). \tag{4}$$

In this paper, $(\cdot)_x$ and $(\cdot)_t$ are used to describe the partial derivatives of spatiotemporal functions (i.e., $w(x,\ t)$ and $\theta(x,\ t)$) with respect to x and t, respectively, whereas $\dot{y}$ and $\dot{l}$ are the time derivatives of temporal functions $y(t)$ and $l(t)$.

Remark 1: It is assumed that the beam's length $l(t)$ is a prespecified time function. Therefore, it is unnecessary to consider its variation when deriving the dynamic model using the Hamilton principle.

Using (1), (2), and (4) for the extended Hamilton principle yields

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} (\delta K - \delta U + \delta W)dt \\ &= \int_{t_1}^{t_2} \delta \int_0^l L(x,t)dxdt \\ &\quad + \int_{t_1}^{t_2} (M\hat{w}_t(0,t)\delta\hat{w}_t(0,t) + f\delta\hat{w}(0,t))dt \\ &\quad + \frac{1}{2}\int_{t_1}^{t_2} \delta\left(m\left[\frac{D\hat{w}}{Dt}\bigg|_{x=l}\right]^2 + J\left[\frac{D\theta}{Dt}\bigg|_{x=l}\right]^2\right)dt, \end{aligned} \tag{5}$$

where

$$\begin{aligned} L(x,t) &= \frac{1}{2}\rho A(\hat{w}_t + \dot{l}\hat{w}_x)^2 + \frac{1}{2}\rho I(\theta_t + \dot{l}\theta_x)^2 \\ &\quad - \frac{1}{2}\hat{w}_x^2 P(x,t) - \frac{1}{2}EI{\theta_x}^2 - \frac{1}{2}\kappa GA(\theta - \hat{w}_x)^2. \end{aligned} \tag{6}$$

Since the domain of integration for the spatial variable x in the first term of (5) changes with time, the Leibnitz integration rule must be used to express this term. Accordingly, (5) is expressed as follows:

$$\begin{aligned} 0 &= \int_{t_1}^{t_2}\int_0^l [-\rho A(\hat{w}_{tt} + \ddot{l}\hat{w}_x + 2\dot{l}\hat{w}_{xt} + \dot{l}^2\hat{w}_{xx}) + (\hat{w}_x P)_x \\ &\quad - \kappa GA(\theta_x - w_{xx})]\delta w dxdt \\ &\quad + \int_{t_1}^{t_2}\int_0^l [-\kappa GA(\theta - w_x) \end{aligned}$$

$$
\begin{aligned}
&-\rho I(\theta_{tt}+\ddot{l}\theta_x+2\dot{l}\theta_{xt}+\dot{l}^2\theta_{xx})+EI\theta_{xx}]\delta\theta dxdt\\
&+\int_{t_1}^{t_2}[f-M\hat{w}_{tt}-\rho A\dot{l}(\hat{w}_t+\dot{l}\hat{w}_x)-\kappa GA(\theta-\hat{w}_x)\\
&+\hat{w}_xP(x,t)]|_{x=0}\delta\hat{w}(0,t)dt\\
&-\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial\theta_x}-\frac{\dot{l}\partial L}{\partial\theta_t}\right)\delta\theta\bigg|_{x=0}dt\\
&+\int_{t_1}^{t_2}\left(m\frac{D^2\hat{w}}{Dt^2}-\frac{\partial L}{\partial\hat{w}_x}-\frac{\dot{l}\partial L}{\partial\hat{w}_t}\right)\delta\hat{w}\bigg|_{x=l}dt\\
&+\int_{t_1}^{t_2}\left(J\frac{D^2\theta}{Dt^2}-\frac{\partial L}{\partial\theta_x}+\dot{l}\frac{\partial L}{\partial\theta_t}\right)\delta\theta\bigg|_{x=l}dt. \qquad (7)
\end{aligned}
$$

The PDEs governing the transverse vibration $w(x,\ t)$ and the rotational angular displacement of the cross section $\theta(x,t)$ are derived based on the first two terms in (7). That is,

$$
\begin{aligned}
&\rho A(\hat{w}_{tt}+\ddot{l}\hat{w}_x+2\dot{l}\hat{w}_{xt}+\dot{l}^2\hat{w}_{xx})-(\hat{w}_xP)_x\\
&\quad+\kappa GA\left(\theta_x-\hat{w}_{xx}\right)=0, \qquad (8)\\
&\rho I(\theta_{tt}+\ddot{l}\theta_x+2\dot{l}\theta_{xt}+\dot{l}^2\theta_{xx})-EI\theta_{xx}\\
&\quad+\kappa GA\left(\theta-\hat{w}_x\right)=0. \qquad (9)
\end{aligned}
$$

The boundary conditions can be obtained by considering the rest of (7), namely,

$$
\begin{aligned}
&M\hat{w}_{tt}(0,t)+\rho A\dot{l}(\hat{w}_t(0,t)+\dot{l}\hat{w}_x(0,t))-\hat{w}_x(0,t)P(0,t)\\
&\quad+\kappa GA\left(\theta(0,t)-\hat{w}_x(0,t)\right)=f(t), \qquad (10)\\
&\theta(0,t)=w(0,t)=0, \qquad (11)\\
&\left[m\frac{D^2\hat{w}}{Dt^2}+\hat{w}_xP+\kappa GA\left(\hat{w}_x-\theta\right)\right]\bigg|_{x=l}=0,\\
&\left[J\frac{D^2\theta}{Dt^2}+EI\theta_x\right]\bigg|_{x=l}=0. \qquad (12)
\end{aligned}
$$

It is noted that (10) is the ordinary differential equation (ODE) describing the dynamics of the trolley. Substituting $\hat{w}(x,\ t)=y(t)+w(x,\ t)$ into (8)-(12), the dynamic model and boundary conditions of the considered system are rewritten as follows:

$$
\begin{aligned}
&\rho A(\ddot{y}+w_{tt}+\ddot{l}w_x+2\dot{l}w_{xt}+\dot{l}^2w_{xx})-w_{xx}P\\
&\quad-w_xP_x+\kappa GA\left(\theta_x-w_{xx}\right)=0, \qquad (13)\\
&w(0,t)=0,\\
&\left[m\left(\ddot{y}+\frac{D^2w}{Dt^2}\right)+w_xP+\kappa GA\left(\hat{w}_x-\theta\right)\right]\bigg|_{x=l}=0, \qquad (14)\\
&\rho I(\theta_{tt}+\ddot{l}\theta_x+2\dot{l}\theta_{xt}+\dot{l}^2\theta_{xx})-EI\theta_{xx}\\
&\quad+\kappa GA\left(\theta-w_x\right)=0, \qquad (15)\\
&\theta(0,t)=0,\\
&\left[J\frac{D^2\theta}{Dt^2}+EI\theta_x\right]\bigg|_{x=l}=0, \qquad (16)\\
&M\ddot{y}+\rho A\dot{l}\left(\dot{y}+\dot{l}w_x(0,t)\right)-P(0,t)w_x(0,t)\\
&\quad-\kappa GAw_x(0,t)=f(t). \qquad (17)
\end{aligned}
$$

3. BOUNDARY CONTROL

It is noted that the deflection of the considered Timoshenko beam is characterized by the transverse vibration $w(x,t)$ and the rotational angular displacement of the cross section $\theta(x,t)$. In practice, the measurement of $\theta(x,t)$ is a significant challenge. Therefore, this paper aims to design a control scheme that can guarantee a stable system without the feedback signal of $\theta(x,t)$. The control design aims to achieve the following control objectives: i) Moving the trolley and the robotic arm to the desired position y_d and ii) suppressing the transverse vibration of the robotic arm's tip. To achieve these objectives, we propose the following boundary control law.

$$
\begin{aligned}
f=&\dot{l}\rho A(\dot{y}+\dot{l}w_x(0,t))+(k_1-1)P(0,t)w_x(0,t)\\
&+(k_1-1)\kappa GAw_x(0,t)-k_2(y-y_{\mathrm{d}})-k_3\dot{y}, \qquad (18)
\end{aligned}
$$

where k_1, k_2, and k_3 are positive control gains. The stability of the system under control law (18) is verified based on the Lyapunov method. The design procedure of the control law is depicted in Fig. 4.

The following lemma and remarks are utilized for stability analysis.

Lemma 1 [46]**:** Let $\varphi(x,t)\in$ R be a function defined on $x\in[0,l]$ and $t\in[0,\infty)$ that satisfies the condition $\varphi(0,t)=0$, $\forall t\in[0,\ \infty)$, the following inequalities hold.

$$
\varphi^2(x,t)\le l\int_0^l\varphi_x^2(x,t)dx,\ \forall x\in[0,l]. \qquad (19)
$$

Remark 2: For an axially moving beam with varying length, the mechanical energy of the beam consists of the energy in the longitudinal motion. However, since the proposed control law aims to eliminate the transverse vibration, the longitudinal motion can be omitted from the stability analysis of the closed-loop system.

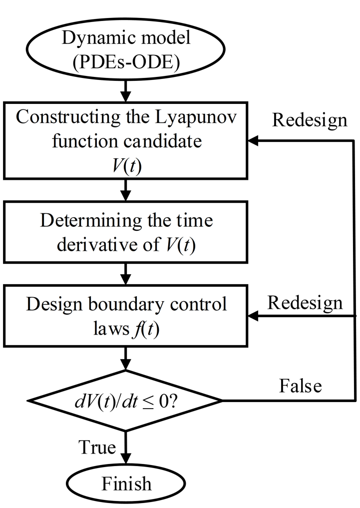

Fig. 4. Design procedure of the boundary control.

Remark 3: In the process of extending the length of the beam, the beam's axial velocity $\dot{l}$ is a non-negative function (i.e., $\dot{l} \geq 0$). Furthermore, the control strategy proposed in this paper treats the beam's length as a second-order polynomial in time (i.e., $l(t) = at^2 + bt + c$, where the jerk is zero). For instance, for given $l(0) = l_0$, t_a, $l(t_a)$, and $\dot{l}(t_a) = 0$, we can solve the unknown two parameters from the following two equations: $l(t_a) = a{t_a}^2 + bt_a + l_0$, $\dot{l}(t_a) = 2at_a + b = 0$. It follows that the axial jerk of the beam is ignored (i.e., $\dddot{l} = 0$). Any remaining destabilizing effect (due to the jerk) can then be suppressed by choosing the control gain large enough, see [17,18,21].

Theorem 1: Consider the system given by (13)-(17). Under the boundary control law in (18), the closed-loop system is uniformly stable in the sense of Lyapunov, and $(y - y_d)$ and $\dot{y}$ go to zero.

Proof: Let us consider the following Lyapunov function candidate.

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{20}$$

where

$$V_1(t) = \frac{1}{2}\rho A k_1 \int_0^l (\dot{y} + w_t + \dot{l} w_x)^2 dx + \frac{1}{2} k_1 \int_0^l {w_x}^2 P dx, \tag{21}$$

$$\begin{aligned} V_2(t) = &\frac{1}{2} k_1 \rho I \int_0^l (\theta_t + \dot{l}\theta_x)^2 dx + \frac{1}{2} k_1 EI \int_0^l {\theta_x}^2 dx \\ &+ \frac{1}{2} k_1 \kappa GA \int_0^l (\theta - w_x)^2 dx, \end{aligned} \tag{22}$$

$$\begin{aligned} V_3(t) = &\frac{1}{2} M \dot{y}^2 + \frac{1}{2} k_1 m (\dot{y} + [Dw/Dt]|_{x=l})^2 \\ &+ \frac{1}{2} k_1 J ([D\theta/Dt]|_{x=l})^2 + k_2 (y - y_d)^2. \end{aligned} \tag{23}$$

Note that V_1 represents the mechanical energy about the transverse vibration of the axially moving beam, V_2 is the mechanical energy concerning the rotational angular displacement of the cross section of the beam, and V_3 is the mechanical energy of the trolley and payload.

The time rates of V_1 and V_2 can be determined based on the Reynolds transport theorem for a translating medium with variable length [17]. Using the equations of motion in (13) and (15) yields

$$\begin{aligned} \dot{V}_1 = &- k_1 \dddot{l} \frac{1}{2} \int_0^l {w_x}^2 (l - x) dx \\ &+ k_1 \int_0^l \{ [(\dot{y} + w_t + \dot{l} w_x) w_x P]_x \\ &- (\dot{y} + w_t + \dot{l} w_x) \kappa GA (\theta - w_x)_x \} dx \\ = &k_1 [(\dot{y} + w_t + \dot{l} w_x) w_x P] |_{x=0}^{x=l} \\ &- k_1 \int_0^l (\dot{y} + w_t + \dot{l} w_x) \kappa GA (\theta - w_x)_x dx, \end{aligned} \tag{24}$$

$$\begin{aligned} \dot{V}_2 = &k_1 \int_0^l \{ EI [\theta_x (\theta_t + \dot{l}\theta_x)]_x \\ &- \kappa GA (\theta - w_x)(w_{xt} + \dot{l} w_{xx}) \} dx \\ = &k_1 EI [\theta_x (\theta_t + \dot{l}\theta_x)] |_{x=0}^{x=l} \\ &- k_1 \int_0^l \kappa GA (\theta - w_x)(w_{xt} + \dot{l} w_{xx}) dx. \end{aligned} \tag{25}$$

The time derivative of V_3 using the equation of motion of the trolley in (17) is given by

$$\begin{aligned} \dot{V}_3 = &M\dot{y}\ddot{y} + k_1 m [(\ddot{y} + D^2 w/Dt^2)(\dot{y} + w_t + \dot{l} w_x)]|_{x=l} \\ &+ k_1 J [(D^2\theta/Dt^2)(\theta_t + \dot{l}\theta_x)]|_{x=l} + k_2 \dot{y}(y - y_d) \\ = &k_1 m [(\ddot{y} + D^2 w/Dt^2)(\dot{y} + w_t + \dot{l} w_x)]|_{x=l} \\ &+ k_1 J [(D^2\theta/Dt^2)(\theta_t + \dot{l}\theta_x)]|_{x=l} \\ &+ \dot{y} [f - \rho A \dot{l} (\dot{y} + \dot{l} w_x(0,t)) \\ &+ (P(0,t) + \kappa GA) w_x(0,t) \\ &+ k_2 (y - y_d)]. \end{aligned} \tag{26}$$

According to (24)-(26), the time rate of the Lyapunov function candidate is expressed as follows:

$$\begin{aligned} \dot{V} = &\dot{V}_1 + \dot{V}_2 + \dot{V}_3 \\ = &k_1 [(\dot{y} + w_t + \dot{l} w_x) w_x P] |_{x=0}^{x=l} \\ &- k_1 \int_0^l (\dot{y} + w_t + \dot{l} w_x) \kappa GA (\theta - w_x)_x dx \\ &+ k_1 EI [\theta_x (\theta_t + \dot{l}\theta_x)] |_{x=0}^{x=l} \\ &- k_1 \int_0^l \kappa GA (\theta - w_x)(w_{xt} + \dot{l} w_{xx}) dx \\ &+ k_1 m [(\ddot{y} + D^2 w/Dt^2)(\dot{y} + w_t + \dot{l} w_x)]|_{x=l} \\ &+ k_1 J [(D^2\theta/Dt^2)(\theta_t + \dot{l}\theta_x)]|_{x=l} \\ &+ \dot{y} [f - \dot{l}\rho A (\dot{y} + \dot{l} w_x(0,t)) \\ &+ (P(0,t) + \kappa GA) w_x(0,t) + k_2 (y - y_d)] \\ = &k_1 [(\dot{y} + w_t + \dot{l} w_x) w_x P] |_{x=0}^{x=l} \\ &+ k_1 EI [\theta_x (\theta_t + \dot{l}\theta_x)] |_{x=0}^{x=l} \\ &- k_1 \int_0^l [(\dot{y} + w_t + \dot{l} w_x) \kappa GA (\theta - w_x)]_x dx \\ &+ k_1 m [(\ddot{y} + D^2 w/Dt^2)(\dot{y} + w_t + \dot{l} w_x)]|_{x=l} \\ &+ k_1 J [(D^2\theta/Dt^2)(\theta_t + \dot{l}\theta_x)]|_{x=l} \\ &\times \dot{y} [f - \dot{l}\rho A (\dot{y} + \dot{l} w_x(0,t)) \\ &+ (P(0,t) + \kappa GA) w_x(0,t) + k_2 (y - y_d)] \\ = &k_1 m [(\ddot{y} + D^2 w/Dt^2)(\dot{y} + w_t + \dot{l} w_x)]|_{x=l} \\ &+ k_1 (\dot{y} + w_t + \dot{l} w_x)(P w_x - \kappa GA (\theta - w_x))|_{x=0}^{x=l} \\ &+ k_1 J [(D^2\theta/Dt^2)(\theta_t + \dot{l}\theta_x)]|_{x=l} \\ &+ k_1 EI \theta_x (\theta_t + \dot{l}\theta_x)|_{x=0}^{x=l} \\ &+ \dot{y} [f - \dot{l}\rho A (\dot{y} + \dot{l} w_x(0,t)) \\ &+ (P(0,t) + \kappa GA) w_x(0,t) + k_2 (y - y_d)]. \end{aligned} \tag{27}$$

By using the boundary conditions in (14) and (16), (27) can be rewritten as follows:

$$\begin{aligned}\dot{V} =& -k_1\dot{l}\left(P(0,t)+\kappa GA\right)w_x(0,t)^2 - k_1EI\dot{l}\theta_x(0,t)^2\\ &-k_1\dot{y}\left(P(0,t)+\kappa GA\right)w_x(0,t)+\dot{y}\left[f-\dot{l}\rho A\left(\dot{y}\right.\right.\\ &\left.+\dot{l}w_x(0,t)\right)+\left(P(0,t)+\kappa GA\right)w_x(0,t)\\ &\left.+k_2(y-y_{\mathrm{d}})\right].\end{aligned} \tag{28}$$

Substituting the proposed control law in (18) into (28) yields

$$\begin{aligned}\dot{V} =& -k_1\dot{l}\left(P(0,t)+\kappa GA\right)w_x(0,t)^2\\ &-k_1EI\dot{l}\theta_x(0,t)^2-k_3\dot{y}^2.\end{aligned} \tag{29}$$

If the prespecified time function $l(t)$ is generated such that the beam's axial acceleration $\ddot{l}$ is smaller than the gravitational acceleration (i.e., $\ddot{l} < g$), it follows $P(0,t) > 0$ from (3). Therefore, $\dot{V}$ is negative semi-definite and the closed-loop system is uniformly stable. Moreover, by integrating both sides of (29) and using the Barbalat's lemma, we can see that $w_x(0,t)$, $\theta_x(0,t)$, and $\dot{y} \to 0$ as $t \to \infty$. Also, from (21), (22), and Lemma 1, we have

$$\frac{k_1}{2l}w^2 \le \frac{1}{2}k_1\int_0^l {w_x}^2 P dx \le V < \infty, \tag{30}$$

$$\frac{k_1EI}{2l}\theta^2 \le \frac{1}{2}k_1EI\int_0^l {\theta_x}^2 dx \le V < \infty. \tag{31}$$

Inequalities (30) and (31) imply the boundedness of the transverse vibration and the rotational angular displacement of the cross section. Specifically, $w(x,t)$ and $\theta(x,t)$ are bounded by $2lV(0)/k_1$ and $2lV(0)/k_1EI$, where $V(0)$ depends on initial conditions.

Remark 4: The significant control gains can lead to a rapid reduction of $V(t)$, see (29). It follows that the vibration energy decreases quickly. Additionally, the large control gains can suppress the destabilizing effect due to the jerk. Nevertheless, choosing large control gains makes the initial value of $V(t)$, see (20), large and, therefore, the control input becomes large too. Subsequently, a large control input requires an expansive actuator. The control gains need to be tuned so that the value of $V(t)$ is minimized in a short time as possible.

4. SIMULATION RESULTS

In this section, a simulation is performed to verify the effectiveness of the proposed control law. A Cartesian palletizer with the following parameters is considered: $\rho = 2700$ kg/m^3, $M = 100$ kg, $m = 20$ kg, $J = 0.05$ kg·m^2, $A = 1.9\times10^{-3}$ m^2, $I = 2.8\times10^{-6}$ m^4, $E = 69$ GPa, $G = 24$ GPa, and $\kappa = 0.8$. The trolley moves from the initial position to the desired position at $y_d = 5$ m, whereas the beam's length is extended from 1 m to 3 m in 1 second. The initial conditions are given as follows: $w(x,0) = 0$ and $\theta(x,0) = 0$. The simulation is performed by using MATLAB. The equations of motion of the considered system are solved based on the finite difference method.

Figs. 5-8 show the system's performance in two cases: i) Without vibration control and (ii) with vibration control. A PD control law is applied to guarantee that the trolley moves to the desired position for the case without vibration control. The proposed control law (18) is implemented in the system in the case of vibration control. Fig. 5 reveals that the trolley can reach the desired position by the PD controller and the proposed controller. Fig. 6 shows the responses of the transverse vibration at the beam's tip $w(l,t)$. As shown in this figure, the control law in (18) can significantly suppress the transverse vibration of the beam. It is observed that the transverse vibration at the beam's tip is eliminated almost completely when the trolley reaches $y_d = 5$ m (i.e., at t $\approx$ 2 seconds). Additionally, the rotational angular displacement of the cross section at the beam's tip $\theta(l,t)$ is substantially suppressed (see Fig. 7). These results reveal that the control law in (18) can guarantee the precise movement of an object (i.e., the beam's tip) to the desired position (see Fig. 8).

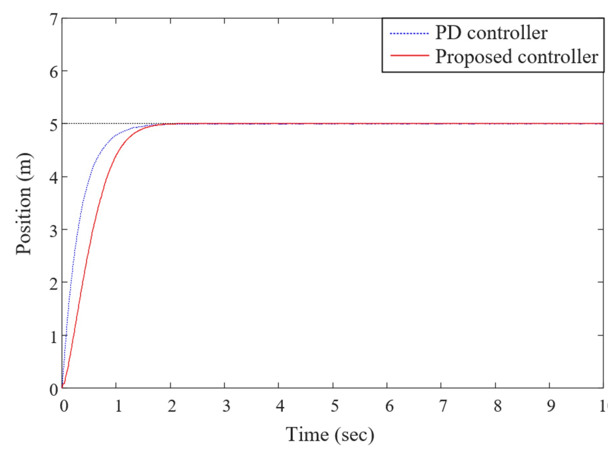

Fig. 5. Trolley's position $y(t)$.

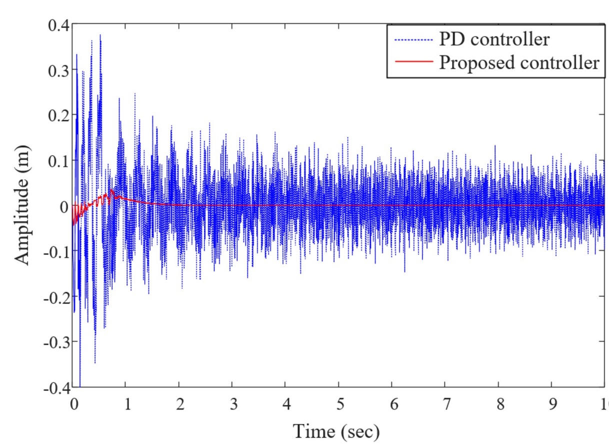

Fig. 6. Transverse vibration at the beam's tip $w(l,t)$.

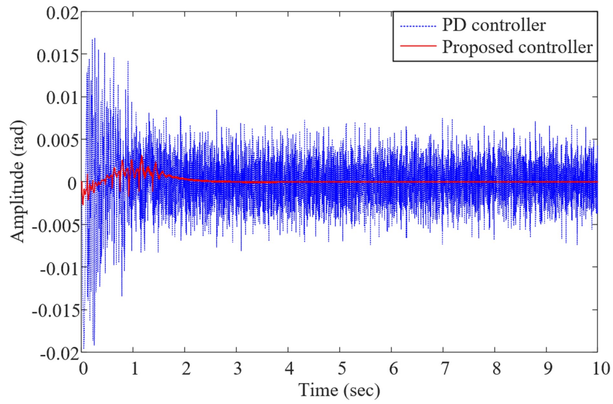

Fig. 7. Rotational angular displacement of the cross section at the beam's tip $\theta(l,t)$.

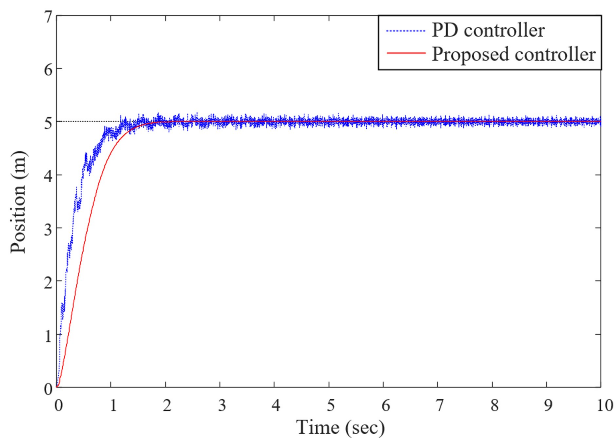

Fig. 8. Position of the beam' tip $\hat{w}(l,t) = y(t) + w(l,t)$.

5. CONCLUSION

This paper investigated the vibration control of a Cartesian palletizer consisting of a trolley and a robotic arm, wherein the robotic arm is treated as a Timoshenko cantilever beam with time-varying length. The governing equations describing the beam's vibration and the trolley's motion were developed based on the extended Hamilton principle. Subsequently, a boundary control law was proposed to suppress the transverse vibration of the beam. The stability of the closed-loop system was analyzed via the Lyapunov method. The simulation results showed that the proposed control law could simultaneously control the trolley's position and the beam's vibration.

REFERENCES

[1] U. H. Shah and K.-S. Hong, "Active vibration control of a flexible rod moving in water: Application to nuclear refueling machines," *Automatica*, vol. 93, pp. 231-243, 2018.

[2] K.-S. Hong and U. H. Shah, *Dynamics and Control of Industrial Cranes*, Springer, Singapore, 2019.

[3] O. A. Bauchau and J. I. Craig, *Structural Analysis*, Springer, Dordrecht, 2009.

[4] J. R. Chang, W. J. Lin, C. J. Huang, and S. T. Choi, "Vibration and stability of an axially moving Rayleigh beam," *Applied Mathematical Modelling*, vol. 34, no. 6, pp. 1482-1497, 2010.

[5] A. Labuschagne, N. J. van Rensburg, and A. J. van der Merwe, "Comparison of linear beam theories," *Mathematical and Computer Modelling*, vol. 49, no. 1-2, pp. 20-30, 2009.

[6] S. P. Timoshenko, "On the correction factor for shear of the differential equation for transverse vibrations of bars of uniform cross-section," *Philosophical Magazine*, vol. 41, pp. 744-746, 1921.

[7] S. P. Timoshenko, "On the transverse vibrations of bars of uniform cross-section," *Philosophical Magazine*, vol. 43, pp. 125-131, 1922.

[8] H. H. Yoo and S. H. Shin, "Vibration analysis of rotating cantilever beams," *Journal of Sound and Vibration*, vol. 212, no. 5, pp. 807-828, 1998.

[9] T. R. Kane, R. R. Ryan, and A. K. Banerjee, "Dynamics of a cantilever beam attached to a moving base," *Journal of Guidance Control and Dynamics*, vol. 10, no. 2, pp. 139-151, 1987.

[10] Y. Xie, P. Liu, and G. P. Cai, "Frequency identification of flexible hub-beam system using control data," *International Journal of Acoustics and Vibration*, vol. 21, no. 3, pp. 257-265, 2016.

[11] U. H. Shah, K.-S. Hong, and S.-H. Choi, "Open-loop vibration control of an underwater system: Application to refueling machine," *IEEE/ASME Trans. on Mechatronics*, vol. 22, no. 4, pp.1622-1632, 2017.

[12] J. A. Wickert, "Nonlinear vibration of a traveling tensioned beam," *International Journal of Non-Linear Mechanics*, vol. 27, no. 3, pp. 503-517, 1992.

[13] W. D. Zhu and C. D. Mote, "Free and forced response of an axially moving string transporting a damped linear-oscillator," *Journal of Sound and Vibration*, vol. 177, no. 5, pp. 591-610, 1994.

[14] S. Mahalingam, "Transverse vibrations of power transmission chains," *British Journal of Applied Physics*, vol. 8, no. 4, p. 145, 1957.

[15] A. L. Thurman and C. D. Mote, "Free, periodic, nonlinear oscillation of an axially moving strip," *Journal of Applied Mechanics-Trans. of the ASME*, vol. 36, no. 1, pp. 83-91, 1969.

[16] P.-T. Pham and K.-S. Hong, "Dynamic models of axially moving systems: A review," *Nonlinear Dynamics*, vol. 100, no. 1, pp. 315-349, 2020.

[17] C.-S. Kim and K.-S. Hong, "Boundary control of container cranes from the perspective of controlling an axially moving string system," *International Journal of Control, Automation, and Systems*, vol. 7, no. 3, pp. 437-445, 2009.

[18] Q. H. Ngo, K.-S. Hong, and I. H. Jung, "Adaptive control of an axially moving system," *Journal of Mechanical Science and Technology*, vol. 23, no. 11, pp. 3071-3078, 2009.

[19] X. Xing, J. Liu, and Z. Liu, "Dynamic modeling and vibration control of a three-dimensional flexible string with variable length and spatiotemporally varying parameters subject to input constraints," *Nonlinear Dynamics*, vol. 95, no. 2, pp. 1395-1413, 2019.

[20] X. Xing, H. Yang, J. Liu, and S. Wang, "Vibration control of nonlinear three-dimensional length-varying string with input quantization," *Journal of Vibration and Control*, vol. 26, no. 19-20, pp. 1835-1847, 2020.

[21] W. D. Zhu, J. Ni, and J. Huang, "Active control of translating media with arbitrarily varying length," *Journal of Vibration and Acoustics-Trans. of the ASME*, vol. 123, no. 3, pp. 347-358, 2001.

[22] W. D. Zhu and J. Ni, "Energetics and stability of translating media with an arbitrarily varying length," *Journal of Vibration and Acoustics-Trans. of the ASME*, vol. 122, no. 3, pp. 295-304, 2000.

[23] Q. C. Nguyen and K.-S. Hong, "Transverse vibration control of axially moving membranes by regulation of axial velocity," *IEEE Trans. on Control Systems Technology,* vol. 20, no. 4, pp. 1124-1131, 2012.

[24] Q. C. Nguyen, M. Piao, and K.-S. Hong, "Multivariable adaptive control of the rewinding process of a roll-to-roll system governed by hyperbolic partial differential equations," *International Journal of Control, Automation, and Systems*, vol. 16, no. 5, pp. 2177-2186, 2018.

[25] Z. Liu, J. Liu, and W. He, "Modeling and vibration control of a flexible aerial refueling hose with variable lengths and input constraint," *Automatica*, vol. 77, pp. 302-310, 2017.

[26] Q. C. Nguyen and K.-S. Hong, "Simultaneous control of longitudinal and transverse vibrations of an axially moving string with velocity tracking," *Journal of Sound and Vibration,* vol. 331, no. 13, pp. 3006-3019, 2012.

[27] G. H. Kim, P.-T. Pham, Q. H. Ngo, and Q. C. Nguyen, "Neural network-based robust anti-sway control of an industrial crane subjected to hoisting dynamics and uncertain hydrodynamic forces," *International Journal of Control, Automation, and Systems*, vol. 19, no. 5, pp. 1953-1961, 2021.

[28] K.-S. Hong, P.-T. Pham, "Control of axially moving systems: A review," *International Journal of Control, Automation, and Systems*, vol. 17, no. 12, pp. 2983-3008, 2019.

[29] Y. Zhou, B. Cui, and X. Lou, "Dynamic H∞ feedback boundary control for a class of parabolic systems with a spatially varying diffusivity," *International Journal of Control, Automation, and Systems*, vol. 19, no. 2, pp. 999-1012, 2021.

[30] G.-H. Kim, K.-S. Hong, "Adaptive sliding mode control of an offshore container crane with unknown disturbances," *IEEE/ASME Trans. on Mechatronics*, vol. 24, no. 6, pp. 2850-2861, 2019.

[31] K.-S. Hong, C. W. Kim, and K. T. Hong, "Boundary control of an axially moving belt system in a thin-metal production line," *International Journal of Control, Automation, and Systems*, vol. 2, no. 1, pp. 55-67, 2004.

[32] J. Y. Choi, K.-S. Hong, and K. J. Yang, "Exponential stabilization of an axially moving tensioned strip by passive damping and boundary control," *Journal of Vibration and Control,* vol. 10, no. 5, pp. 661-682, 2004.

[33] K. J. Yang, K.-S. Hong, and F. Matsuno, "Boundary control of an axially moving steel strip under a spatiotemporally varying tension," *JSME International Journal Series C-Mechanical Systems Machine Elements and Manufacturing*, vol. 47, no. 2, pp. 665-674, 2004.

[34] Q. C. Nguyen, T. H. Le, and K.-S. Hong, "Transverse vibration control of axially moving web systems by regulation of axial tension," *International Journal of Control, Automation, and Systems,* vol. 13, no. 3, pp. 689-696, 2015.

[35] W. He and S. S. Ge, "Vibration control of a flexible beam with output constraint," *IEEE Trans. on Industrial Electronics*, vol. 62, no. 8, pp. 5023-5030, 2015.

[36] M. A. Eshag, L. Ma, Y. Sun, and K. Zhang, "Robust boundary vibration control of uncertain flexible robot manipulator with spatiotemporally-varying disturbance and boundary disturbance," *International Journal of Control, Automation, and Systems*, vol. 19, no. 2, pp. 788-798, 2021.

[37] F. Han, Y. Jia, "Sliding mode boundary control for a planar two-link rigid-flexible manipulator with input disturbances," *International Journal of Control, Automation, and Systems*, vol.18, no. 2, pp. 351-362, 2020.

[38] P.-T. Pham, G.-H. Kim, Q.-C. Nguyen, and K.-S. Hong, "Control of a non-uniform flexible beam: Identification of first two modes," *International Journal of Control, Automation, and Systems*, vol. 19, pp. 3698-3707, 2021

[39] Z. Zhao, Z. Liu, W. He, K.-S. Hong, and H. X. Li, "Boundary adaptive fault-tolerant control for a flexible Timoshenko arm with backlash-like hysteresis," *Automatica*, vol. 130, 109690, 2021.

[40] M. Grobbelaar-Van Dalsen, "Uniform stability for the Timoshenko beam with tip load," *Journal of Mathematical Analysis and Applications,* vol. 361, no. 2, pp. 392-400, 2010.

[41] W. He, S. Zhang, and S. S. Ge, "Boundary output-feedback stabilization of a Timoshenko beam using disturbance observer," *IEEE Trans. on Industrial Electronics*, vol. 60, no. 11, pp. 5186-5194, 2012.

[42] K. D. Do, "Stochastic boundary control design for Timoshenko beams with large motions," *Journal of Sound and Vibration*, vol. 402, pp. 164-184, 2017.

[43] D. Wu, T. Endo, and F. Matsuno, "Exponential stability of two Timoshenko arms for grasping and manipulating an object," *International Journal of Control, Automation, and Systems*, vol. 19, no. 3, pp. 1328-1339, 2021.

[44] W. He, T. Meng, J. K. Liu, and H. Qin, "Boundary control of a Timoshenko beam system with input dead-zone," *International Journal of Control*, vol. 88, no. 6, pp.1257-1270, 2015.

[45] K.-S. Hong, L.-Q. Chen, P.-T. Pham, and X.-D. Yang, *Control of Axially Moving Systems*, Springer, 2021.

[46] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1959.

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