Received: 16 January 2019 | Revised: 26 December 2019 | Accepted: 27 March 2020
DOI: 10.1002/asjc.2361

REGULAR PAPER

WILEY

Robust resilient H_∞ performance for finite-time boundedness of neutral-type neural networks with time-varying delays

Saravanan Shanmugam[1] | **M Syed Ali**[2] | **Keum-Shik Hong**[1] | **Quanxin Zhu**[3]

[1]School of Mechanical Engineering, Pusan National University, Busan, Korea

[2]Department of Mathematics, Thiruvalluvar University, Vellore, Tamilnadu, India

[3]Hunan Normal University, Changsha, Hunan, P. R. China

Correspondence
M Syed Ali, Department of Mathematics, Thiruvalluvar University, Vellore 632-115, Tamilnadu, India.
Email: syedgru@gmail.com

Funding information
Pusan National University and the National Research Foundation (NRF), Grant/Award Number: NRF-2017R1A2A1A17069430

Abstract

This paper discusses the resilient H_∞ performance for finite-time boundedness of neutral-type neural networks with time-varying delays. The presented theoretical analysis allows establishing the finite-time bounded of the real response of a delayed neural networks. In addition, we propose finite-time stability conditions with time-varying delay. By choosing an appropriate Lyapunov-Krasovskii functional, and employing an auxiliary function-based integral inequality and Wirtinger's based integral inequality, the sufficient criteria are derived in terms of linear matrix inequalities. The purpose is to design the system is not only finite-time bounded with a specified decay rate but also satisfies an H_∞ performance requirement. Theoretical results are tested through a numerical example.

KEYWORDS

Finite-time boundedness, H_∞ performance, Neural networks, Resilient

1 | INTRODUCTION

Neural networks constitute an effective information-processing paradigm that can conveniently address many practice problems, such as real-time pattern recognition, fault tolerance via redundant information coding, signal processing, image processing, and adaptive learning [1-7]. Moreover, to achieve design specifications in many practical applications of neural networks, the full information on the states of neural networks is often required, which is sometimes a difficult task. The more practical scenario is that we have to face the unmeasurable states of neural networks or only partially available information from the network outputs in practice. For instance, considering the external disturbance, the robust conditions have been presented in many existing works [8-12] under which the desired estimators can be designed.

The stability of a time-delay system is always a hot topic for researchers, see [13,14]. Even many researchers were more concerned about the stabilizability problems, for example, see [15-19]. As a result, to obtain stability criteria of time-delayed systems by using the Lyapunov theorem, the main efforts were concentrated on the following two directions: One is to find an appropriate positive definite functional with a negative definite-time derivative along the trajectory of the system. The other is to reduce the upper bound of the time derivative of Lyapunov-Krasovskii functional (LKF) as much as possible by developing various inequality techniques, such as Wirtinger's based integral inequality [20], free-matrix-based integral inequality [21], relaxed integral inequalities [22], the generalized free-weighting-matrix approach [23], Bessel-Legendre inequality [24], reciprocally convex approach [25], extended reciprocally convex matrix inequalities [26,27], and so on. Dynamical systems with time-delays and uncertain parameters have been of considerable interest over the past decades. But, uncertain systems and resilient control of uncertain systems with time-varying delays are rare [28-32]. Time-delays are always an important source of system instability and poor performance [33,34].

As a special class of time-delay neural networks, the neutral-type time-delayed systems have received attention in recent years. This time-delayed system contains time-delays both in its state and in the derivatives of the states. Moreover, neutral-type time-delayed systems are frequently encountered in many dynamics, such as automatic control, distributed network system containing lossless transmission line, heat exchangers, and population ecology. Various analysis approaches have been utilized to find stability criteria and control design conditions for H_∞ control of neutral-type systems and neural networks with time-delays [35-37]. Also, neural information in the biochemistry reactivity may result in a neutral-type process; that is, the involved differential expression includes not only the derivative term of the present state but also one of the past state. So, it is natural and reasonable to pay close attention to neutral-type neural networks with delays [38,39].

Compared with the widely known asymptotic stability, finite-time stability and finite-time boundedness are different concepts concerning the boundedness of the states during a fixed time interval, which almost relies on the transient response. Since many practical applications require that the state does not exceed a certain bound in a fixed time interval, e.g., to avoid saturation or excitation, we focus on the finite-time boundedness analysis in practical consideration. In recent years, many results were reported on finite-time boundedness problems: The relevant concepts of finite-time boundedness [40], finite-time stabilization [41], and finite-time H_∞ performance have been revisited in [42-45]. However, according to the authors knowledge, resilient H_∞ performance for finite-time boundedness of uncertain neural networks have not been investigated yet.

Motivated by the discussions above, in this paper, we design an appropriate resilient state feedback controller such that the closed-loop control system is finite-time bounded and satisfies the given performance index constraints. The main contributions of this paper are summarized as follows: i) A new criterion on the finite-time boundedness is established for time-delayed neutral-type neural networks by using the LMI-approach. In the approach, the key is to find a suitable Lyapunov function satisfying the derivative condition of the finite-time boundedness, which is more complicated than that of asymptotic stability. ii) Another contribution of this paper is that we present delay-dependent results on both the finite-time boundedness and finite-time H_∞ performance design. Based on the results obtained, a state feedback controller is designed such that the corresponding system is finite-time bounded. Finally, an example is provided to illustrate the efficiency of the proposed method, and conclusions are drawn.

Notation: The notation used in this paper are as follows. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, the superscript "T''" denotes the transpose, and the notation $P > 0$ (≥ 0) means P is a real symmetric positive definite (semi-definite) matrix, max(P) and min(P) denote the maximum and minimum eigenvalues of matrix P, respectively. I is an identity matrix with appropriate dimension. diag$\{a_i\}$ denotes the diagonal matrix with the diagonal elements a_i, ($i = 1, 2, \dots, n$). The asterisk $*$ in a matrix is used to denote the term induced by symmetry.

2 | PROBLEM FORMULATION AND PRELIMINARIES

Consider the following neutral-type neural networks with time-varying delays as follows:

$$\begin{aligned}\dot{x}(t) - \hat{E}\dot{x}(t-\rho(t)) &= -\hat{A}x(t) + \hat{W}_0 f(x(t)) \\ &\quad + \hat{W}_1 f(x(t-\tau(t))) + B_1 u(t) + \hat{D}_1 w(t), \\ z(t) &= \hat{C}x(t) + B_2 u(t) + \hat{D}_2 w(t), \\ x(t) &= \phi(t), t \in [-\tau_M, 0],\end{aligned} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^l$ is the control input, $w(t) \in \mathbb{R}^m$ is the disturbance input which belongs to $L_2[0, \infty)$, $z(t) \in \mathbb{R}^q$ is the controlled output. $f(x(t))$ is the neuron activation function, $\phi(t)$ is a continuous vector-valued initial function. $\hat{A} = A + \Delta A(t)$, $\hat{W}_0 = W_0 + \Delta W_0(t)$, $\hat{W}_1 = W_1 + \Delta W_1(t)$, $\hat{D}_1 = D_1 + \Delta D_1(t)$, $\hat{E} = E + \Delta E(t)$, $\hat{C} = C + \Delta C(t)$, $\hat{D}_2 = D_2 + \Delta D_2(t)$ in which A is a positive diagonal matrix, $W_0, W_1, B_1, D_1, E, C, B_2$, and D_2 are the weight connection matrices with appropriate dimensions, and $\Delta A(t), \Delta W_0(t), \Delta W_1(t), \Delta D_1(t), \Delta E(t), \Delta E(t)$, and $\Delta D_2(t)$ are uncertain real-valued matrices. The variables $\tau(t)$ and $\rho(t)$ represent the time-varying delay and the neutral delays, respectively, and satisfying $0 \leq \tau(t) \leq \bar{\tau}$, $\dot{\tau}(t) \leq \tau_D$, and $0 \leq \rho(t) \leq \bar{\rho}$, $\dot{\rho}(t) \leq \rho_D$, where $\bar{\tau}, \bar{\rho}, \tau_D$ and ρ_D are positive constants, and $\tau_M = \max\{\bar{\tau}, \bar{\rho}\}$.

The uncertain matrices satisfy

$$\begin{aligned}&[\Delta A(t) \quad \Delta W_0(t) \quad \Delta W_1(t) \quad \Delta E(t) \quad \Delta D_1(t)] \\ &= F_1\eta(t)[G_1 \quad G_2 \quad G_3 \quad G_4 \quad G_5], \\ &\left[\Delta C(t) \;\; \Delta D_2(t)\right] = F_2\eta(t)\left[G_1 \;\; G_4\right]\end{aligned} \tag{2}$$

where $F_1, F_2, G_1, G_2, G_3, G_4$, and G_5 are known real matrices with suitable dimension and $\eta(t)$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$\eta^T(t)\eta(t) \leq I. \tag{3}$$

In this paper, we design the state feedback controller as the following form: $u(t) = (K + \Delta K(t))x(t)$, where $K \in \mathcal{R}^n$ is the gain matrix to be designed and $\Delta K(t)$ is the time-varying controller gain which satisfies:

$$\Delta K(t) = H_1\delta(t)H_2, \qquad \delta^T(t)\delta(t) \leq I. \tag{4}$$

The system 1 with the controller 4 can be converted to the following form

$$\begin{aligned} \dot{x}(t) - \hat{E}\dot{x}(t-\rho(t)) &= -\bar{A}x(t) + \hat{W}_0 f(x(t)) \\ &\quad + \hat{W}_1 f(x(t-\tau(t))) + B_1u(t) + \hat{D}_1w(t), \\ z(t) &= \bar{C}x(t) + B_2u(t) + \hat{D}_2w(t), \\ x(t) &= \phi(t), t \in [-\tau_M, 0], \end{aligned} \tag{5}$$

where $\bar{A} = (A + B_1K) + (\Delta A(t) + B_1\Delta K(t))$ and $\bar{C} = (C + B_2K) + (\Delta C(t) + B_2\Delta K(t))$.

Assumption 1. The activation functions satisfy the following condition, for any $a = 1, 2, \ldots, n$ there exist constants F_a^-, F_a^+ such that

$$F_a^- \leq \frac{f_a(x_1) - f_a(x_2)}{x_1 - x_2} \leq F_a^+ \quad \forall \quad x_1, x_2 \in \mathbb{R}, \quad x_1 \neq x_2.$$

For presentation convenience, we denote $M_t = diag\{F_1^-F_1^+, F_2^-F_2^+, \ldots F_n^-F_n^+\}$,
$M_u = diag\{\frac{F_1^-+F_1^+}{2}, \frac{F_2^-+F_2^+}{2}, \ldots, \frac{F_n^-+F_n^+}{2}\}$.

Definition 1 [46]. (finite-time boundedness). Given a positive matrix $R > 0$, positive constants $c_1 > 0$, $c_2 > 0$, $(c_2 > c_1)$, and T the neural networks (1) with $u(t) = 0$ is said to be finite-time bounded with respect to (c_1, c_2, T, R, d), if the following inequalities hold:

$$\begin{aligned} &\limsup_{-\tau_M \leq t_0 \leq 0} \{x^T(t_0)Rx(t_0), \dot{x}^T(t_0)R\dot{x}(t_0)\} \leq c_1 \\ &\Rightarrow x^T(t)Rx(t) < c_2, t \in [0, T]. \end{aligned}$$

Definition 2 [43]. Given a positive matrix $R > 0$, positive constants $T > 0, c_1 > 0$ and $c_2 > 0$ with $c_2 > c_1$, the neural networks (1) is said to be finite-time bounded with respect to (c_1, c_2, T, R, d) with a prescribed level of noise attenuation $\gamma > 0$, and under a zero initial condition, it holds that

$$\int_0^T z^T(s)z(s)ds \leq \gamma^2 \int_0^T w^T(t)w(t)dt.$$

Definition 3 [47]. The neural networks (1) is said to be finite-time stabilizable with respect to (c_1, c_2, T, R, d), if there exists a controller, $u(t) = (K + \Delta K(t))x(t), t \in [0, T]$, such that the corresponding closed-loop neural networks is finite-time bounded with respect to (c_1, c_2, T, R, α).

Lemma 1 [20]. *For any constant matrix $M > 0$, the following inequality holds for all continuously differentiable function φ on $[a, b] \to \mathcal{R}^{n\times n}$:*

$$\begin{aligned} &(b-a)\int_a^b \varphi^T(s)M\varphi(s)ds \\ &\quad \geq \left(\int_a^b \varphi(s)ds\right)^T M \left(\int_a^b \varphi(s)ds\right) + 3\Omega^T M\Omega, \end{aligned}$$

where $\Omega = \int_a^b \varphi(s)ds - \frac{2}{b-a}\int_a^b\int_a^s \varphi(\theta)d\theta ds.$

Lemma 2 [48]. *Let $\mathbb{M} > 0$ be any constant matrix, and for given scalars a and b with $a < b$, the following relation is well defined for any differentiable function η in $[a, b] \to \mathbb{R}^n$:*

$$-\frac{b^2-a^2}{2}\int_{-a}^{-b}\int_{t+\theta}^{t} \dot{\eta}^T(s)\mathbb{M}\dot{\eta}(s)dsd\theta \leq -\Omega_1^T\mathbb{M}\Omega_1 - 2\Omega_2^T\mathbb{M}\Omega_2,$$

where $\Omega_1 = (b-a)\eta(t) - \int_{t-a}^{t-b} \eta(s)ds,$

$$\Omega_2 = -\frac{(b-a)}{2}\eta(t) - \int_{t-a}^{t-b}\eta(s)ds + \frac{3}{b-a}\int_{-a}^{-b}\int_{t+\theta}^{t}\eta(s)dsd\theta.$$

Lemma 3 [49]. *For a positive definite matrix $M > 0$, and a differentiable function $\{x(u)|u \in [a, b]\}$, the following inequality holds:*

$$\int_a^b \dot{x}^T(s)Mx(s)ds \geq \frac{1}{b-a}\pi_1^TM\pi_1 + \frac{3}{b-a}\pi_2^TM\pi_2 + \frac{5}{b-a}\pi_3^TM\pi_3,$$

where $\pi_1 = x(b) - x(a), \quad \pi_2 = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds,$

$$\pi_3 = x(b) - x(a) + \frac{6}{b-a}\int_a^b x(s)ds - \frac{12}{(b-a)^2}\int_a^b\int_\theta^b x(s)dsd\theta.$$

Lemma 4 [50]. *Given matrices J, E and $\Theta = \Theta^T$, then $\Theta + EF(t)G + G^TF^T(t)E^T < 0$ holds for any $F(t)$ satisfying $F^T(t)F(t) \leq I$, if there exists a scalar $\epsilon > 0$ such that $\Theta + \epsilon^{-1}EE^T + \epsilon G^TG < 0$.*

3 | MAIN RESULTS

3.1 | Finite-time boundedness

In this section, we first provide the finite-time boundedness condition for the following system:

$$\begin{aligned} \dot{x}(t) - \hat{E}\dot{x}(t-\rho(t)) &= -\bar{A}x(t) + \hat{W}_0 f(x(t)) \\ &\quad + \hat{W}_1 f(x(t-\tau(t))) + \hat{D}_1w(t), \\ x(t) &= \phi(t), t \in [-\tau_M, 0], \end{aligned} \tag{6}$$

Theorem 1. *For given positive scalars T, c_1, c_2, d, $\bar{\tau}$, $\bar{\rho}$, $\bar{\tau}_D$, $\bar{\rho}_D$ and α, the system 6 is finite-time bounded if there exist symmetric positive definite matrices $P > 0$, $Q_i > 0$ $(i = 1, 2, 3, 4)$, $S_j > 0$ $(j = 1, 2, 3)$, the appropriate dimensional matrices $\mathcal{N}_k > 0$ $(k = 1, 2, 3, 4)$, and positive diagonal matrices $\mathcal{S}_t > 0$ and $\mathcal{S}_u > 0$ such that the following LMIs holds:*

$$\Omega = [\Omega_{ij}]_{10\times 10} < 0, \tag{7}$$

$$e^{\alpha T}\left(\Lambda c_1 + d\lambda_{10}\right) < \lambda_1 c_2, \tag{8}$$

where the elements of $\Omega = [\Omega_{ij}]_{10\times 10}$ *are the following* $\Omega_{11} = -P\bar{A} - \bar{A}^T P^T + Q_1 + Q_2 + \bar{\tau}S_1 - \alpha P_i - \frac{1}{\bar{\tau}}S_2 - \frac{3}{\bar{\tau}}S_2 + \frac{5}{\bar{\tau}}S_2 - \bar{\tau}^2 S_3 - \frac{\bar{\tau}^2}{4}S_3 - M_t\mathcal{S}_t - \mathcal{N}_1\bar{A} - \bar{A}^T\mathcal{N}_1^T$, $\Omega_{12} = -\bar{A}^T\mathcal{N}_2^T$, $\Omega_{13} = \frac{1}{\bar{\tau}}S_2 + \frac{3}{\bar{\tau}}S_2 + \frac{5}{\bar{\tau}}S_2 - \bar{A}^T\mathcal{N}_3^T$, $\Omega_{14} = -\mathcal{N}_1 - \bar{A}^T\mathcal{N}_4^T$, $\Omega_{15} = \frac{6}{\bar{\tau}^2}S_2 - \frac{30}{\bar{\tau}^2}S_2 + \bar{\tau}S_3 + \frac{\bar{\tau}}{2}S_3$, $\Omega_{16} = \frac{60}{\bar{\tau}^3}S_2 - \frac{3}{2}S_3$, $\Omega_{17} = P\hat{W}_0 + M_u\mathcal{S}_t + \mathcal{N}_1\hat{W}_0$, $\Omega_{18} = P\hat{W}_1 + \mathcal{N}_1\hat{W}_1$, $\Omega_{19} = P\hat{E} + \mathcal{N}_1\hat{E}$, $\Omega_{110} = P\hat{D}_1 + \mathcal{N}_1\hat{D}_1$, $\Omega_{22} = -(1 - \bar{\tau}_D)Q_1 - M_t\mathcal{S}_u$, $\Omega_{23} = 0$, $\Omega_{24} = -\mathcal{N}_2$, $\Omega_{25} = 0$, $\Omega_{26} = 0$, $\Omega_{27} = \mathcal{N}_2\hat{W}_0$, $\Omega_{28} = M_u\mathcal{S}_u + \mathcal{N}_2\hat{W}_1$, $\Omega_{29} = \mathcal{N}_2\hat{E}$, $\Omega_{210} = \mathcal{N}_2\hat{D}_1$, $\Omega_{33} = -Q_2 - \frac{1}{\bar{\tau}}S_2 - \frac{3}{\bar{\tau}}S_2$, $\Omega_{34} = -\mathcal{N}_3$, $\Omega_{35} = \frac{6}{\bar{\tau}^2}S_2 + \frac{30}{\bar{\tau}^2}S_2$, $\Omega_{36} = -\frac{60}{\bar{\tau}^3}S_2$, $\Omega_{37} = \mathcal{N}_3\hat{W}_0$, $\Omega_{38} = \mathcal{N}_3\hat{W}_1$, $\Omega_{39} = \mathcal{N}_3\hat{E}$, $\Omega_{310} = \mathcal{N}_3\hat{D}_1$, $\Omega_{44} = Q_3 + \bar{\tau}S_2 - \mathcal{N}_4$, $\Omega_{45} = 0$, $\Omega_{46} = 0$, $\Omega_{47} = \mathcal{N}_4\hat{W}_0$, $\Omega_{48} = \mathcal{N}_4\hat{W}_1$, $\Omega_{49} = \mathcal{N}_4\hat{E}$, $\Omega_{410} = \mathcal{N}_4\hat{D}_1$, $\Omega_{55} = -\frac{1}{\bar{\tau}}S_1 - \frac{3}{\bar{\tau}}S_1 - \frac{12}{\bar{\tau}^3}S_2 - \frac{180}{\bar{\tau}^3}S_2 - S_3 - S_3$, $\Omega_{56} = \frac{6}{\bar{\tau}^2}S_1\frac{360}{\bar{\tau}^4}S_2 + \frac{3}{\bar{\tau}}S_3$, $\Omega_{57} = 0$, $\Omega_{58} = 0$, $\Omega_{59} = 0$, $\Omega_{510} = 0$, $\Omega_{66} = -\frac{12}{\bar{\tau}^3}S_1 - \frac{720}{\bar{\tau}^5}S_2 - \frac{9}{\bar{\tau}^2}S_3$, $\Omega_{67} = 0$, $\Omega_{68} = 0$, $\Omega_{69} = 0$, $\Omega_{610} = 0$, $\Omega_{77} = Q_4 - \mathcal{S}_t$, $\Omega_{78} = 0$, $\Omega_{79} = 0$, $\Omega_{710} = 0$, $\Omega_{88} = -(1 - \tau_D)Q_4 - \mathcal{S}_u$, $\Omega_{89} = 0$, $\Omega_{810} = 0$, $\Omega_{99} = -(1 - \rho_D)Q_3$, $\Omega_{910} = 0$, $\Omega_{1010} = -\alpha X$.

$$\begin{aligned} &\lambda_1 R < P < \lambda_2 R, \quad Q_1 < \lambda_3 R, \quad Q_2 < \lambda_4 R, \\ &Q_3 < \lambda_5 R, \quad Q_4 < \lambda_6 R, \quad S_1 < \lambda_7 R, \\ &S_2 < \lambda_8 R, \quad S_3 < \lambda_9 R, \quad X < \lambda_{10} R, \end{aligned} \tag{9}$$

$\lambda_1 = \lambda_{\min}(\bar{P})$, $\lambda_2 = \lambda_{\max}(\bar{P})$, $\lambda_3 = \lambda_{\max}(\bar{Q}_1)$, $\lambda_4 = \lambda_{\max}(\bar{Q}_2)$, $\lambda_5 = \lambda_{\max}(\bar{Q}_3)$, $\lambda_6 = \lambda_{\max}(\bar{Q}_4)$, $\lambda_7 = \lambda_{\max}(\bar{S}_1)$, $\lambda_8 = \lambda_{\max}(\bar{S}_2)$, $\lambda_9 = \lambda_{\max}(\bar{S}_3)$, $\lambda_{10} = \lambda_{\max}(\bar{X})$.

Proof. Choose the following LKF for the system 6 as

$$V(t) = \sum_{i=1}^{5} V_i(t), \tag{10}$$

where

$$\begin{aligned} V_1(t) &= x^T(t)Px(t), \\ V_2(t) &= \int_{t-\tau(t)}^{t} x^T(s)Q_1x(s)ds + \int_{t-\bar{\tau}}^{t} x^T(s)Q_2x(s)ds, \\ V_3(t) &= \int_{t-\rho(t)}^{T} \dot{x}^T(s)Q_3\dot{x}(s)ds \\ &\quad + \int_{t-\tau(t)}^{t} f^T(x(s))Q_4 f(x(s))ds, \\ V_4(t) &= \int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} x^T(s)S_1x(s)dsd\theta \\ &\quad + \int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)S_2\dot{x}(s)dsd\theta, \\ V_5(t) &= \frac{\bar{\tau}^2}{2}\int_{-\bar{\tau}}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)S_3\dot{x}(s)dsd\theta. \end{aligned}$$

□

Calculating the time derivatives of the above LKF along the trajectory of the system 6, we have

$$\dot{V}_1 = 2x^T(t)P\dot{x}(t), \tag{11}$$

$$\begin{aligned} \dot{V}_2 = &x^T(t)\left(Q_1 + Q_2\right)x(t) - (1 - \tau_D) \\ &\times x^T(t - \tau(t))Q_1x(t - \tau(t)) \\ &- x^T(t - \bar{\tau})Q_2x(t - \bar{\tau}), \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{V}_3 = &\dot{x}^T(t)Q_3\dot{x}(t) - (1 - \rho_D)x^T(t - \rho(t)) \\ &\times Q_3x(t - \rho(t)) + f^T(x(t))Q_4 f(x(t)) \\ &- f^T(x(t - \tau(t)))Q_4 f(x(t - \tau(t))), \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{V}_4 = &\bar{\tau}x^T(t)S_1x(t) - \int_{t-\bar{\tau}}^{t} x^T(s)S_1x(s)ds \\ &+ \bar{\tau}\dot{x}^T(t)S_2\dot{x}(t) - \int_{t-\bar{\tau}}^{t} \dot{x}^T(s)S_2\dot{x}(s)ds, \end{aligned} \tag{14}$$

$$\begin{aligned} \dot{V}_5 = &\left(\frac{\bar{\tau}^2}{2}\right)^2\dot{x}^T(t)S_3\dot{x}(t) \\ &-\frac{\bar{\tau}^2}{2}\int_{\bar{\tau}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)S_3\dot{x}(s)dsd\theta. \end{aligned} \tag{15}$$

By applying Lemma 3, we can get,

$$\begin{aligned} &-\int_{t-\bar{\tau}}^{t} \dot{x}^T(s)S_2\dot{x}(s)ds \\ &\leq -\frac{1}{\bar{\tau}}\pi_1^T S_2\pi_1 - \frac{3}{\bar{\tau}}\pi_2^T S_2\pi_2 - \frac{5}{\bar{\tau}}\pi_3^T S_2\pi_3. \end{aligned} \tag{16}$$

where $\pi_1 = x(t) - x(t - \bar{\tau})$, $\pi_2 = x(t) + x(t - \bar{\tau}) - \frac{2}{\bar{\tau}}\int_{t-\bar{\tau}}^{t} x(s)ds$, $\pi_3 = x(t) - x(t - \bar{\tau}) + \frac{6}{\bar{\tau}}\int_{t-\bar{\tau}}^{t} x(s)ds - \frac{12}{\bar{\tau}^2}\int_{\bar{\tau}}^{0}\int_{t+\theta}^{t} x(s)dsd\theta$. By using Lemma 1, we can obtain,

$$\begin{aligned} &-\int_{t-\bar{\tau}}^{t} x^T(s)Q_4x(s)ds \\ &\leq \frac{-1}{\bar{\tau}}\left(\int_{t-\bar{\tau}}^{t} x(s)ds\right)^T S_1\left(\int_{t-\bar{\tau}}^{t} x(s)ds\right) - \frac{3}{\bar{\tau}}\pi_4^T S_1\pi_4, \end{aligned} \tag{17}$$

where $\pi_4 = \int_{t-\bar{\tau}}^{t} x(s)ds - \frac{2}{\bar{\tau}}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} x(s)dsd\theta$. And

$$\begin{aligned} &-\frac{\bar{\tau}^2}{2}\int_{\bar{\tau}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)S_3\dot{x}(s)dsd\theta \\ &\leq -\begin{bmatrix} \pi_5 \\ \pi_6 \end{bmatrix}^T \begin{bmatrix} S_3 & 0 \\ 0 & 2S_3 \end{bmatrix} \begin{bmatrix} \pi_5 \\ \pi_6 \end{bmatrix}, \end{aligned} \tag{18}$$

where $\pi_5 = \bar{\tau}x(t) - \int_{t-\bar{\tau}}^{t} x(s)ds$, $\pi_6 = -\frac{\bar{\tau}}{2}x(t) - \int_{t-\bar{\tau}}^{t} x(s)ds + \frac{3}{\bar{\tau}}\int_{-\bar{\tau}}^{0}\int_{t+\theta}^{t} x(s)dsd\theta$. Based on Assumption 1, we obtain

$$[f_a(x_a(t)) - M_a^- x_a(t)][f_a(x_a(t)) - M_a^- x_a(t)] \leq 0,$$

$$\begin{aligned} &[f_a(x_a(t - \tau(t))) - M_a^- x_a(t - \tau(t))][f_a(x_a(t - \tau(t))) \\ &-M_a^- x_a(t - \tau(t))] \leq 0, \end{aligned}$$

where $a = 1, 2, \dots, n$, The above equations can be compactly written as follows.

$$\begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} M_t & -M_u \\ * & I \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} x(t - \tau(t)) \\ f(x(t - \tau(t))) \end{bmatrix}^T \begin{bmatrix} M_t & -M_u \\ * & I \end{bmatrix} \begin{bmatrix} x(t - \tau(t)) \\ f(x(t - \tau(t))) \end{bmatrix} \leq 0.$$

Then for any positive matrices $\mathcal{S}_t = diag\{s_1, s_2, \dots, s_n\}$ and $\mathcal{S}_u = diag\{\hat{s}_1, \hat{s}_2, \dots, \hat{s}_n\}$, the following inequalities hold:

$$\begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix}^T \begin{bmatrix} M_t\mathcal{S}_t & -M_u\mathcal{S}_t \\ * & \mathcal{S}_t \end{bmatrix} \begin{bmatrix} x(t) \\ f(x(t)) \end{bmatrix} \leq 0, \tag{19}$$

$$\begin{bmatrix} x(t-\tau(t)) \\ f(x(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} M_t S_u & -M_u S_u \\ * & S_u \end{bmatrix} \times \begin{bmatrix} x(t-\tau(t)) \\ f(x(t-\tau(t))) \end{bmatrix} \leq 0. \tag{20}$$

On the other hand, for any matrices $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$ with appropriate dimensions, it follows that

$$\begin{aligned} 0 =& 2\left[x^T(t)\mathcal{N}_1 + x^T(t-\tau(t))\mathcal{N}_2 \right. \\ &\left. + x^T(t-\bar{\tau}(t))\mathcal{N}_3 + \dot{x}^T(t)\mathcal{N}_4\right] \\ &\times \left[-\bar{A}x(t) + \hat{W}_0 f(x(t)) + \hat{W}_1 f(x(t-\tau(t))) \right. \\ &\left. + \hat{D}_1 w(t) + \hat{\mathrm{E}}\dot{x}(t-\rho(t)) - \dot{x}(t)\right]. \end{aligned} \tag{21}$$

Therefore, for given $\alpha > 0$ and from 11-21, one can obtain that

$$\dot{V}(t) - \alpha V(x(t)) - \alpha w^T(t) X w(t) \leq \vartheta^T(t)\Omega\vartheta(t), \tag{22}$$

where $\vartheta^T(t) = [x^T(t)x^T(t-\tau(t))x^T(t-\bar{\tau})\dot{x}^T(t)(\int_{t-\bar{\tau}}^t x(s)ds^T(\int_{-\bar{\tau}}^0\int_{t+\theta}^t x(s)dsd\theta)^T f^T(x(t))f(x(t-\tau(t)))\dot{x}^T(t-\rho(t))w^T(t)]$ Then we can write that

$$\dot{V}(x(t)) \leq \alpha V(x(t)) + \alpha w^T(t) X w(t), \tag{23}$$

Multiplying both sides in 23 by $e^{-\alpha t}$ we get

$$\frac{d}{dt}\left(e^{-\alpha t}V(t)\right) \leq \alpha w^T(t) X w(t). \tag{24}$$

Then integrating inequality 24 0 to t, where $t \in [0, T]$, we get

$$e^{-\alpha t}V(t) \leq V(0) + \alpha \int_0^t e^{\alpha s} w^T(s) X w(s) ds,$$

$$V(t) < e^{\alpha t}\left(V(0) + \alpha \int_0^t e^{\alpha s} w^T(s) X w(s) ds\right), \tag{25}$$

$$V(t) < e^{\alpha T}\left(V(0) + \lambda_6 d\right). \tag{26}$$

Define $\bar{P} = R^{-1/2}PR^{-1/2}$, $\bar{Q}_1 = R^{-1/2}Q_1R^{-1/2}$, $\bar{Q}_2 = R^{-1/2}Q_2R^{-1/2}$, $\bar{Q}_3 = R^{-1/2}Q_3R^{-1/2}$, $\bar{Q}_4 = R^{-1/2}Q_4R^{-1/2}$, $\bar{S}_1 = R^{-1/2}S_1R^{-1/2}$, $\bar{S}_2 = R^{-1/2}S_2R^{-1/2}$, $\bar{S}_3 = R^{-1/2}S_3R^{-1/2}$.

On the other hand,

$$\begin{aligned} V(x_0, 0) =& \lambda_{max}(\bar{P})x^T(0)Rx(0) \\ &+ \lambda_{max}(\bar{Q}_1)\int_{-\tau(0)}^0 x^T(s)Rx(s)ds \\ &+ \lambda_{max}(\bar{Q}_2)\int_{-\bar{\tau}}^0 x^T(s)Rx(s)ds \\ &+ \lambda_{max}(\bar{Q}_3)\int_{-\rho(0)}^0 \dot{x}^T(s)R\dot{x}(s)ds \\ &+ \lambda_{max}(\bar{Q}_4)\max|M_t^-, M_u^+|^2 \\ &\int_{-\tau(0)}^0 x^T(s)Rx(s)ds \\ &+ \lambda_{max}(\bar{S}_1)\int_{-\bar{\tau}}^0\int_\theta^0 x^T(s)Rx(s)dsd\theta \\ &+ \lambda_{max}(\bar{S}_2)\int_{-\bar{\tau}}^0\int_\theta^0 \dot{x}^T(s)R\dot{x}(s)dsd\theta \\ &+ \lambda_{max}(\bar{S}_3)\int_{-\bar{\tau}}^0\int_\beta^0\int_\theta^0 \dot{x}^T(s)R\dot{x}(s)dsd\theta \\ \leq& \left\{\lambda_{max}(\bar{P}) + \bar{\tau}\lambda_{max}(\bar{Q}_1) + \bar{\tau}\lambda_{max}(\bar{Q}_2) \right. \\ &+ \bar{\rho}\lambda_{max}(\bar{Q}_3) + \bar{\tau}\max|M_t^-, M_u^+|^2\lambda_{max}(\bar{Q}_4) \\ &\left. + \frac{\bar{\tau}^2}{2}\lambda_{max}(\bar{S}_1) + \frac{\bar{\tau}^2}{2}\lambda_{max}(\bar{S}_2) + \frac{\bar{\tau}^3}{6}\lambda_{max}(\bar{S}_3)\right\} \\ &\times \sup_{-\tau_M \leq s \leq 0}\{x^T(s)Rx(s), \dot{x}^T(s)R\dot{x}(s)\}, \end{aligned}$$

$$V(x(t)) \leq e^{\alpha T}\left(\Lambda c_1 + d\lambda_{10}\right). \tag{27}$$

Noting that

$$V(x(t)) \geq \lambda_{min}(\bar{P})x^T(t)Rx(t) = \lambda_1 x^T(t)Rx(t).$$

From 8, we have

$$x^T(t)Rx(t) < c_2. \tag{28}$$

By Definition 1, the system 6 is finite-time boundedness. This completes the proof.

3.2 | Finite-time H_∞ performance

In this section, we consider the following neutral-type neural networks with disturbance:

$$\begin{aligned} \dot{x}(t) - \hat{\mathrm{E}}\dot{x}(t-\rho(t)) =& -\bar{A}x(t) + \hat{W}_0 f(x(t)) \\ &+ \hat{W}_1 f(x(t-\tau(t))) + \hat{D}_1 w(t), \\ z(t) =& \bar{C}x(t) + \hat{D}_2 w(t), \\ x(t) =& \phi(t), t \in [-\tau_M, 0], \end{aligned} \tag{29}$$

Theorem 2. *For given positive scalars T, c_1, c_2, d, $\bar{\tau}$, $\bar{\rho}$, $\bar{\tau}_D$, $\bar{\rho}_D$ and α with a prescribed level of noise attenuation $\gamma > 0$. The system 29 is finite-time bounded if there exist symmetric positive definite matrices $P > 0$, $Q_i > 0$ $(i = 1, 2, 3, 4)$, $S_j > 0$ $(j = 1, 2, 3)$, appropriate dimensional* *matrices* $\mathcal{N}_k > 0$ $(k = 1, 2, 3, 4)$*, and positive diagonal matrices* $\mathcal{S}_t > 0$ *and* $\mathcal{S}_u > 0$ *such that the following LMIs holds:*

$$\bar{\Omega} = \begin{bmatrix} \bar{\Omega}_1 & \overline{C} \\ * & -I \end{bmatrix} < 0, \tag{30}$$

$$e^{\alpha T}(\Lambda c_1 + d\lambda_{10}) < \lambda_1 c_2, \tag{31}$$

where the elements of $\bar{\Omega}_1 = [\bar{\Omega}_{1ij}]_{10\times 10}$ are the following

$$\begin{aligned}
&\overline{C} = \left[\, \bar{C}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \hat{D}_2 \,\right],\\
&\bar{\Omega}_{11} = -P\bar{A} - \bar{A}^T P^T + Q_1 + Q_2 + \bar{\tau} S_1 - \alpha P_i - \tfrac{1}{\bar{\tau}} S_2\\
&-\tfrac{3}{\bar{\tau}} S_2 + \tfrac{5}{\bar{\tau}} S_2 - \bar{\tau}^2 S_3 - \tfrac{\bar{\tau}^2}{4} S_3 - M_t \mathcal{S}_t - \mathcal{N}_1 \bar{A} - \bar{A}^T \mathcal{N}_1^T,\\
&\bar{\Omega}_{12} = -\bar{A}^T \mathcal{N}_2^T, \quad \bar{\Omega}_{13} = \tfrac{1}{\bar{\tau}} S_2 + \tfrac{3}{\bar{\tau}} S_2 + \tfrac{5}{\bar{\tau}} S_2 - \bar{A}^T \mathcal{N}_3^T,\\
&\bar{\Omega}_{14} = -\mathcal{N}_1 - \bar{A}^T \mathcal{N}_4^T, \quad \bar{\Omega}_{15} = \tfrac{6}{\bar{\tau}^2} S_2 - \tfrac{30}{\bar{\tau}^2} S_2 + \bar{\tau} S_3 + \tfrac{\bar{\tau}}{2} S_3,\\
&\bar{\Omega}_{16} = \tfrac{60}{\bar{\tau}^3} S_2 - \tfrac{3}{2} S_3, \quad \bar{\Omega}_{17} = P\hat{W}_0 + M_u \mathcal{S}_t + \mathcal{N}_1 \hat{W}_0,\\
&\bar{\Omega}_{18} = P\hat{W}_1 + \mathcal{N}_1 \hat{W}_1, \quad \bar{\Omega}_{19} = P\hat{\mathrm{E}} + \mathcal{N}_1 \hat{\mathrm{E}},\\
&\bar{\Omega}_{110} = P\hat{D}_1 + \mathcal{N}_1 \hat{D}_1, \quad \bar{\Omega}_{22} = -(1 - \bar{\tau}_D) Q_1 - M_t \mathcal{S}_u,\\
&\bar{\Omega}_{23} = 0, \quad \bar{\Omega}_{24} = -\mathcal{N}_2, \quad \bar{\Omega}_{25} = 0, \quad \bar{\Omega}_{26} = 0,\\
&\bar{\Omega}_{27} = \mathcal{N}_2 \hat{W}_0, \bar{\Omega}_{28} = M_u \mathcal{S}_u + \mathcal{N}_2 \hat{W}_1, \quad \bar{\Omega}_{29} = \mathcal{N}_2 \hat{\mathrm{E}},\\
&\bar{\Omega}_{210} = \mathcal{N}_2 \hat{D}_1, \quad \bar{\Omega}_{33} = -Q_2 - \tfrac{1}{\bar{\tau}} S_2 - \tfrac{3}{\bar{\tau}} S_2, \quad \bar{\Omega}_{34} = -\mathcal{N}_3,\\
&\bar{\Omega}_{35} = \tfrac{6}{\bar{\tau}^2} S_2 + \tfrac{30}{\bar{\tau}^2} S_2, \quad \bar{\Omega}_{36} = -\tfrac{60}{\bar{\tau}^3} S_2, \quad \bar{\Omega}_{37} = \mathcal{N}_3 \hat{W}_0,\\
&\bar{\Omega}_{38} = \mathcal{N}_3 \hat{W}_1, \quad \bar{\Omega}_{39} = \mathcal{N}_3 \hat{\mathrm{E}}, \quad \bar{\Omega}_{310} = \mathcal{N}_3 \hat{D}_1,\\
&\bar{\Omega}_{44} = Q_3 + \bar{\tau} S_2 - \mathcal{N}_4, \quad \bar{\Omega}_{45} = 0, \quad \bar{\Omega}_{46} = 0, \quad \bar{\Omega}_{47} = \mathcal{N}_4 \hat{W}_0,\\
&\bar{\Omega}_{48} = \mathcal{N}_4 \hat{W}_1, \quad \bar{\Omega}_{49} = \mathcal{N}_4 \hat{\mathrm{E}}, \quad \bar{\Omega}_{410} = \mathcal{N}_4 \hat{D}_1,\\
&\bar{\Omega}_{55} = -\tfrac{1}{\bar{\tau}} S_1 - \tfrac{3}{\bar{\tau}} S_1 - \tfrac{12}{\bar{\tau}^3} S_2 - \tfrac{180}{\bar{\tau}^3} S_2 - S_3 - S_3,\\
&\bar{\Omega}_{56} = \tfrac{6}{\bar{\tau}^2} S_1 + \tfrac{360}{\bar{\tau}^4} S_2 + \tfrac{3}{\bar{\tau}} S_3, \quad \bar{\Omega}_{57} = 0, \quad \bar{\Omega}_{58} = 0, \quad \bar{\Omega}_{59} = 0,\\
&\bar{\Omega}_{510} = 0, \quad \bar{\Omega}_{66} = -\tfrac{12}{\bar{\tau}^3} S_1 - \tfrac{720}{\bar{\tau}^5} S_2 - \tfrac{9}{\bar{\tau}^2} S_3, \quad \bar{\Omega}_{67} = 0,\\
&\bar{\Omega}_{68} = 0, \quad \bar{\Omega}_{69} = 0, \quad \bar{\Omega}_{610} = 0, \quad \bar{\Omega}_{77} = Q_4 - \mathcal{S}_t, \quad \bar{\Omega}_{78} = 0,\\
&\bar{\Omega}_{79} = 0, \quad \bar{\Omega}_{710} = 0, \quad \bar{\Omega}_{88} = -(1 - \tau_D) Q_4 - \mathcal{S}_u, \quad \bar{\Omega}_{89} = 0,\\
&\bar{\Omega}_{810} = 0, \quad \bar{\Omega}_{99} = -(1 - \rho_D) Q_3, \quad \bar{\Omega}_{910} = 0, \quad \bar{\Omega}_{1010} = -\gamma^2 I.
\end{aligned}$$

Proof. By the following similar derivatives in Theorem 1, we have,

$$\begin{aligned}
&\dot{V}(t, e(t)) - \alpha V(t) + z^T(t) z(t)\\
&\quad - \gamma^2 w^T(t) w(t) < \vartheta^T(t) \bar{\Omega} \vartheta(t) < 0.
\end{aligned} \tag{32}$$

□

Define

$$J = \gamma^2 w^T(t) w(t) - z^T(t) z(t). \tag{33}$$

Multiplying 33 by $e^{-\alpha t}$, we have,

$$\frac{d}{dt}\{e^{-\alpha t} V(t)\} < e^{-\alpha t} J(t). \tag{34}$$

Integrating this inequality on $[0, T]$ yields

$$0 \le e^{-\alpha T} V(t) < \int_0^T e^{-\alpha t} J(t) dt. \tag{35}$$

We have

$$\begin{aligned}
&e^{-\alpha T} \int_0^T z^T(t) z(t) dt < \int_0^T e^{-\alpha t} z^T(t) z(t) dt\\
&< \gamma^2 \int_0^T e^{-\alpha t} w^T(t) w(t) dt < \gamma^2 \int_0^T w^T(t) w(t) dt.
\end{aligned} \tag{36}$$

By Definition 2, the system 29 is finite-time bounded with respect to (c_1, c_2, T, R, d) and with a prescribed level of noise attenuation $\gamma > 0$. This completes the proof.

3.3 | Resilient finite-time H_∞ performance

In this section we consider the uncertain neural networks with time-varying delays:

$$\begin{aligned}
\dot{x}(t) - \hat{\mathrm{E}}\dot{x}(t - \rho(t)) &= -\bar{A} x(t) + \hat{W}_0 f(x(t))\\
&\quad + \hat{W}_1 f(x(t - \tau(t))) + B_1 u(t) + \hat{D}_1 w(t),\\
z(t) &= \bar{C} x(t) + B_2 u(t) + \hat{D}_2 w(t),\\
x(t) &= \phi(t), t \in [-\tau_M, 0],
\end{aligned} \tag{37}$$

Theorem 3. *For given positive scalars T, c_1, c_2, d, $\bar{\tau}$, $\bar{\rho}$, $\bar{\tau}_D$, $\bar{\rho}_D$ and α with a prescribed level of noise attenuation $\gamma > 0$. The neural networks under state feedback controller 37 is finite-time bounded if there exist symmetric positive definite matrices $P > 0$, $Q_i > 0$ $(i = 1, 2, 3, 4)$, $S_j > 0$ $(j = 1, 2, 3)$, appropriate dimensional matrices $\mathcal{N}_k > 0$ $(k = 1, 2, 3, 4)$, and positive diagonal matrices $\mathcal{S}_t > 0$, $\mathcal{S}_u > 0$ such that the following LMIs holds:*

$$\left[\begin{array}{cccccccccccc}
\bar{\Sigma} & \overline{C}_1 & \Upsilon_1 & \varrho_1 \Upsilon_2 & \Upsilon_3 & \varrho_2 \Upsilon_4 & \varrho_3 \Upsilon_6 & \Upsilon_7 & \varrho_4 \Upsilon_8 & \Upsilon_9 & \varrho_5 \Upsilon_{10} & 0\\
* & -I & -\varrho_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & -\varrho_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -\varrho_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & -\varrho_2 I & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -\varrho_3 I & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & -\varrho_3 I & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -\varrho_4 I & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -\varrho_4 I & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & -\varrho_5 I & 0\\
* & * & * & * & * & * & * & * & * & * & * & -\varrho_5 I
\end{array}\right] < 0, \tag{38}$$

$$e^{\alpha T}(\Lambda c_1 + d\lambda_{10}) < \lambda_1 c_2, \tag{39}$$

where the elements of $[\bar{\Sigma}]_{10\times 10}$ are the following.

$$
\begin{aligned}
&\bar{\Sigma}_{11} = -AX + B_1L - XA^T + L^TB_1^T + \bar{Q}_1 + \bar{Q}_2 + \bar{\tau}\bar{S}_1\\
&-\alpha X - \tfrac{1}{\bar{\tau}}\bar{S}_2 - \tfrac{3}{\bar{\tau}}\bar{S}_2 + \tfrac{5}{\bar{\tau}}\bar{S}_2 - \bar{\tau}^2\bar{S}_3 - \tfrac{\bar{\tau}^2}{4}\bar{S}_3 - M_t\bar{\mathcal{S}}_t\\
&-\bar{\mathcal{N}}_1A + LB_1 - A^T\bar{\mathcal{N}}_1^T + B_1^TL, \quad \bar{\Sigma}_{12} = -A\bar{\mathcal{N}}_2^T + L^TB_1^T,\\
&\bar{\Sigma}_{13} = \tfrac{1}{\bar{\tau}}\bar{S}_2 + \tfrac{3}{\bar{\tau}}\bar{S}_2 + \tfrac{5}{\bar{\tau}}\bar{S}_2 - \bar{\mathcal{N}}_3^TA^T + L^TB_1^T,\\
&\bar{\Sigma}_{14} = -\bar{\mathcal{N}}_1 - A^T\bar{\mathcal{N}}_4^T + L^TB_1^T, \quad \bar{\Sigma}_{15} = \tfrac{6}{\bar{\tau}^2}\bar{S}_2 - \tfrac{30}{\bar{\tau}^2}\bar{S}_2\\
&+\bar{\tau}\bar{S}_3 + \tfrac{\bar{\tau}}{2}\bar{S}_3, \quad \bar{\Sigma}_{16} = \tfrac{60}{\bar{\tau}^3}\bar{S}_2 - \tfrac{3}{2}\bar{S}_3, \quad \bar{\Sigma}_{17} = W_0 + M_u\bar{\mathcal{S}}_t\\
&+\bar{\mathcal{N}}_1W_0, \quad \bar{\Sigma}_{18} = W_1 + \bar{\mathcal{N}}_1W_1, \quad \bar{\Sigma}_{19} = E + \bar{\mathcal{N}}_1E,\\
&\bar{\Sigma}_{110} = D_1 + \bar{\mathcal{N}}_1D_1, \quad \bar{\Sigma}_{22} = -(1-\bar{\tau}_D)\bar{Q}_1 - M_t\bar{\mathcal{S}}_u,\\
&\bar{\Sigma}_{23} = 0, \quad \bar{\Sigma}_{24} = -\bar{\mathcal{N}}_2, \quad \bar{\Sigma}_{25} = 0, \quad \bar{\Sigma}_{26} = 0, \quad \bar{\Sigma}_{27} = \bar{\mathcal{N}}_2W_0,\\
&\bar{\Sigma}_{28} = M_u\bar{\mathcal{S}}_u + \bar{\mathcal{N}}_2W_1, \quad \bar{\Sigma}_{29} = \bar{\mathcal{N}}_2E, \quad \bar{\Sigma}_{210} = \bar{\mathcal{N}}_2D_1,\\
&\bar{\Sigma}_{33} = -\bar{Q}_2 - \tfrac{1}{\bar{\tau}}\bar{S}_2 - \tfrac{3}{\bar{\tau}}\bar{S}_2, \quad \bar{\Sigma}_{34} = -\bar{\mathcal{N}}_3,\\
&\bar{\Sigma}_{35} = \tfrac{6}{\bar{\tau}^2}\bar{S}_2 + \tfrac{30}{\bar{\tau}^2}\bar{S}_2, \quad \bar{\Sigma}_{36} = -\tfrac{60}{\bar{\tau}^3}\bar{S}_2, \quad \bar{\Sigma}_{37} = \bar{\mathcal{N}}_3W_0,\\
&\bar{\Sigma}_{38} = \bar{\mathcal{N}}_3W_1, \quad \bar{\Sigma}_{39} = \bar{\mathcal{N}}_3E, \quad \bar{\Sigma}_{310} = \bar{\mathcal{N}}_3D_1,\\
&\bar{\Sigma}_{44} = \bar{Q}_3 + \bar{\tau}\bar{S}_2 - \bar{\mathcal{N}}_4, \quad \bar{\Sigma}_{45} = 0, \quad \bar{\Sigma}_{46} = 0, \quad \bar{\Sigma}_{47} = \bar{\mathcal{N}}_4W_0,\\
&\bar{\Sigma}_{48} = \bar{\mathcal{N}}_4W_1, \quad \bar{\Sigma}_{49} = \bar{\mathcal{N}}_4E, \quad \bar{\Sigma}_{410} = \bar{\mathcal{N}}_4D_1,\\
&\bar{\Sigma}_{55} = -\tfrac{1}{\bar{\tau}}\bar{S}_1 - \tfrac{3}{\bar{\tau}}\bar{S}_1 - \tfrac{12}{\bar{\tau}^3}\bar{S}_2 - \tfrac{180}{\bar{\tau}^3}\bar{S}_2 - \bar{S}_3 - \bar{S}_3,\\
&\bar{\Sigma}_{56} = \tfrac{6}{\bar{\tau}^2}\bar{S}_1 + \tfrac{360}{\bar{\tau}^4}\bar{S}_2 + \tfrac{3}{\bar{\tau}}\bar{S}_3, \quad \bar{\Sigma}_{57} = 0, \quad \bar{\Sigma}_{58} = 0,\\
&\bar{\Sigma}_{59} = 0, \quad \bar{\Sigma}_{510} = 0, \quad \bar{\Sigma}_{66} = -\tfrac{12}{\bar{\tau}^3}\bar{S}_1 - \tfrac{720}{\bar{\tau}^5}\bar{S}_2 - \tfrac{9}{\bar{\tau}^2}\bar{S}_3,\\
&\bar{\Sigma}_{67} = 0, \quad \bar{\Sigma}_{68} = 0, \quad \bar{\Sigma}_{69} = 0, \quad \bar{\Sigma}_{610} = 0, \quad \bar{\Sigma}_{77} = \bar{Q}_4 - \bar{\mathcal{S}}_t,\\
&\bar{\Sigma}_{78} = 0, \quad \bar{\Sigma}_{79} = 0, \quad \bar{\Sigma}_{710} = 0, \quad \bar{\Sigma}_{88} = -(1-\tau_D)\bar{Q}_4 - \bar{\mathcal{S}}_u,\\
&\bar{\Sigma}_{89} = 0, \quad \bar{\Sigma}_{810} = 0, \quad \bar{\Sigma}_{99} = -(1-\rho_D)\bar{Q}_3, \quad \bar{\Sigma}_{910} = 0,\\
&\bar{\Sigma}_{1010} = -\gamma^2 I,\\
&\overline{C}_1 = \begin{bmatrix} C + B_2K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_2 \end{bmatrix}.
\end{aligned}
$$

Let $K = P^{-1}L$, $X = P^{-1}$, $\bar{Q}_i = XQ_iX^T$ $(i = 1,2,3,4)$, $\bar{S}_j = XS_jX^T$ $(j = 1,2,3)$, $\bar{\mathcal{N}}_k = X\mathcal{N}_kX^T$ $(k = 1,2,3,4)$, $\overline{\mathcal{S}}_t = X\mathcal{S}_tX^T$, and $\overline{\mathcal{S}}_u = X\mathcal{S}_uX^T$.

Proof. Replace $\bar{A}$ by $(A + B_1K) + (\Delta A(t) + B_1\Delta K(t))$, $\bar{C}$ by $(C + B_2K) + (\Delta C(t) + B_2\Delta K(t))$, $\hat{W}_0$ by $W_0 + \Delta W_0(t)$, $\hat{W}_1$ by $W_1 + \Delta W_1(t)$, $\hat{D}_1$ by $D_1 + \Delta D_1(t)$, $\hat{E}$ by $E + \Delta E(t)$, $\hat{D}_2$ by $D_2 + \Delta D_2(t)$ in 30, we get

$$
\hat{\Omega} = \begin{bmatrix} \hat{\Omega}_1 & \overline{C} \\ * & -I \end{bmatrix} < 0, \tag{40}
$$

□

We can rewrite equation 40 as follows.

$$
\hat{\Omega} = \bar{\Psi} + \Delta\hat{\Psi} < 0, \tag{41}
$$

where the elements of $\bar{\Psi} = [\bar{\Psi}_{ij}]_{10\times 10}$ and $\hat{\Psi} = [\hat{\Psi}_{ij}]_{10\times 10}$ are the following.

$$
\begin{aligned}
&\bar{\Psi}_{11} = -P(A + B_1K) - (A + B_1K)^TP^T + Q_1\\
&+Q_2 + \bar{\tau}S_1 - \alpha P_i - \tfrac{1}{\bar{\tau}}S_2 - \tfrac{3}{\bar{\tau}}S_2 + \tfrac{5}{\bar{\tau}}S_2 - \bar{\tau}^2S_3 - \tfrac{\bar{\tau}^2}{4}S_3\\
&-M_t\mathcal{S}_t - \mathcal{N}_1(A + B_1K) - (A + B_1K)^T\mathcal{N}_1^T,\\
&\bar{\Psi}_{12} = -(A + B_1K)^T\mathcal{N}_2^T, \quad \bar{\Psi}_{13} = \tfrac{1}{\bar{\tau}}S_2 + \tfrac{3}{\bar{\tau}}S_2 + \tfrac{5}{\bar{\tau}}S_2\\
&-(A + B_1K)^T\mathcal{N}_3^T, \quad \bar{\Psi}_{14} = -\mathcal{N}_1 - (A + B_1K)^T\mathcal{N}_4^T,\\
&\bar{\Psi}_{15} = \tfrac{6}{\bar{\tau}^2}S_2 - \tfrac{30}{\bar{\tau}^2}S_2 + \bar{\tau}S_3 + \tfrac{\bar{\tau}}{2}S_3, \quad \bar{\Psi}_{16} = \tfrac{60}{\bar{\tau}^3}S_2 - \tfrac{3}{2}S_3,\\
&\bar{\Psi}_{17} = PW_0 + M_u\mathcal{S}_t + \mathcal{N}_1W_0, \quad \bar{\Psi}_{18} = PW_1 + \mathcal{N}_1W_1,\\
&\bar{\Psi}_{19} = PE + \mathcal{N}_1E, \quad \bar{\Psi}_{110} = PD_1 + \mathcal{N}_1D_1,\\
&\bar{\Psi}_{22} = -(1-\bar{\tau}_D)Q_1 - M_t\mathcal{S}_u, \quad \bar{\Psi}_{23} = 0, \quad \bar{\Psi}_{24} = -\mathcal{N}_2,\\
&\bar{\Psi}_{25} = 0, \quad \bar{\Psi}_{26} = 0, \quad \bar{\Psi}_{27} = \mathcal{N}_2W_0, \quad \bar{\Psi}_{28} = M_u\mathcal{S}_u + \mathcal{N}_2W_1,\\
&\bar{\Psi}_{29} = \mathcal{N}_2E, \quad \bar{\Psi}_{210} = \mathcal{N}_2D_1, \quad \bar{\Psi}_{33} = -Q_2 - \tfrac{1}{\bar{\tau}}S_2 - \tfrac{3}{\bar{\tau}}S_2,\\
&\bar{\Psi}_{34} = -\mathcal{N}_3, \quad \bar{\Psi}_{35} = \tfrac{6}{\bar{\tau}^2}S_2 + \tfrac{30}{\bar{\tau}^2}S_2, \quad \bar{\Psi}_{36} = -\tfrac{60}{\bar{\tau}^3}S_2,\\
&\bar{\Psi}_{37} = \mathcal{N}_3W_0, \quad \bar{\Psi}_{38} = \mathcal{N}_3W_1, \quad \bar{\Psi}_{39} = \mathcal{N}_3E, \quad \bar{\Psi}_{310} = \mathcal{N}_3D_1,\\
&\bar{\Psi}_{44} = Q_3 + \bar{\tau}S_2 - \mathcal{N}_4, \quad \bar{\Psi}_{45} = 0, \quad \bar{\Psi}_{46} = 0, \quad \bar{\Psi}_{47} = \mathcal{N}_4W_0,\\
&\bar{\Psi}_{48} = \mathcal{N}_4W_1, \quad \bar{\Psi}_{49} = \mathcal{N}_4E, \quad \bar{\Psi}_{410} = \mathcal{N}_4D_1,\\
&\bar{\Psi}_{55} = -\tfrac{1}{\bar{\tau}}S_1 - \tfrac{3}{\bar{\tau}}S_1 - \tfrac{12}{\bar{\tau}^3}S_2 - \tfrac{180}{\bar{\tau}^3}S_2 - S_3 - S_3,\\
&\bar{\Psi}_{56} = \tfrac{6}{\bar{\tau}^2}S_1 + \tfrac{360}{\bar{\tau}^4}S_2 + \tfrac{3}{\bar{\tau}}S_3, \quad \bar{\Psi}_{57} = 0, \quad \bar{\Psi}_{58} = 0, \quad \bar{\Psi}_{59} = 0,\\
&\bar{\Psi}_{510} = 0, \quad \bar{\Psi}_{66} = -\tfrac{12}{\bar{\tau}^3}S_1 - \tfrac{720}{\bar{\tau}^5}S_2 - \tfrac{9}{\bar{\tau}^2}S_3, \quad \bar{\Psi}_{67} = 0,\\
&\bar{\Psi}_{68} = 0, \quad \bar{\Psi}_{69} = 0, \quad \bar{\Psi}_{610} = 0, \quad \bar{\Psi}_{77} = Q_4 - \mathcal{S}_t, \quad \bar{\Psi}_{78} = 0,\\
&\bar{\Psi}_{79} = 0, \quad \bar{\Psi}_{710} = 0, \quad \bar{\Psi}_{88} = -(1-\tau_D)Q_4 - \mathcal{S}_u, \quad \bar{\Psi}_{89} = 0,\\
&\bar{\Psi}_{810} = 0, \quad \bar{\Psi}_{99} = -(1-\rho_D)Q_3, \quad \bar{\Psi}_{910} = 0, \quad \bar{\Psi}_{1010} = -\gamma^2 I,\\
&\overline{C}_1 = \begin{bmatrix} C + B_2K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_2 \end{bmatrix}.\\
&\hat{\Psi}_{11} = -P(\Delta A(t) + B_1\Delta K(t)) - (\Delta A(t) + B_1\Delta K(t))^T\\
&\quad P^T + Q_1 + Q_2 + \bar{\tau}S_1 - \alpha P_i - \tfrac{1}{\bar{\tau}}S_2 - \tfrac{3}{\bar{\tau}}S_2\\
&+\tfrac{5}{\bar{\tau}}S_2 - \bar{\tau}^2S_3 - \tfrac{\bar{\tau}^2}{4}S_3 - M_t\mathcal{S}_t\\
&-\mathcal{N}_1(\Delta A(t) + B_1\Delta K(t)) - (\Delta A(t) + B_1\Delta K(t))^T\\
&\quad \mathcal{N}_1^T, \quad \hat{\Psi}_{12} = -(\Delta A(t) + B_1\Delta K(t))^T\mathcal{N}_2^T,\\
&\quad \hat{\Psi}_{13} = \tfrac{1}{\bar{\tau}}S_2 + \tfrac{3}{\bar{\tau}}S_2 + \tfrac{5}{\bar{\tau}}S_2 - (\Delta A(t) + B_1\Delta K(t))^T\mathcal{N}_3^T,\\
&\quad \hat{\Psi}_{14} = -\mathcal{N}_1 - (\Delta A(t) + B_1\Delta K(t))^T\mathcal{N}_4^T, \quad \hat{\Psi}_{15} = 0,\\
&\quad \hat{\Psi}_{16} = 0, \quad \hat{\Psi}_{17} = P\Delta W_0(t) + M_u\mathcal{S}_t + \mathcal{N}_1\Delta W_0(t),\\
&\quad \hat{\Psi}_{18} = P\Delta W_1(t) + \mathcal{N}_1\Delta W_1(t), \quad \hat{\Psi}_{19} = P\Delta E(t)\\
&\quad +\mathcal{N}_1\Delta E(t), \quad \hat{\Psi}_{110} = P\Delta D_1(t) + \mathcal{N}_1\Delta D_1(t),\\
&\quad \hat{\Psi}_{22} = 0, \quad \hat{\Psi}_{23} = 0, \quad \hat{\Psi}_{24} = 0, \quad \hat{\Psi}_{25} = 0, \quad \hat{\Psi}_{26} = 0,\\
&\quad \hat{\Psi}_{27} = \mathcal{N}_2\Delta W_0(t), \quad \hat{\Psi}_{28} = M_u\mathcal{S}_u + \mathcal{N}_2\Delta W_1(t),\\
&\quad \hat{\Psi}_{29} = \mathcal{N}_2\Delta E(t), \quad \hat{\Psi}_{210} = \mathcal{N}_2\hat{D}_1, \quad \hat{\Psi}_{33} = 0, \quad \hat{\Psi}_{34} = 0,\\
&\quad \hat{\Psi}_{35} = 0, \quad \hat{\Psi}_{36} = 0, \quad \hat{\Psi}_{37} = \mathcal{N}_3\Delta W_0(t),\\
&\quad \hat{\Psi}_{38} = \mathcal{N}_3\Delta W_1(t), \quad \hat{\Psi}_{39} = \mathcal{N}_3\Delta E(t),\\
&\quad \hat{\Psi}_{310} = \mathcal{N}_3\Delta D_1(t), \quad \hat{\Psi}_{44} = 0, \quad \hat{\Psi}_{45} = 0, \quad \hat{\Psi}_{46} = 0,\\
&\quad \hat{\Psi}_{47} = \mathcal{N}_4\Delta W_0(t), \quad \hat{\Psi}_{48} = \mathcal{N}_4\Delta W_1(t),\\
&\quad \hat{\Psi}_{49} = \mathcal{N}_4\Delta E(t), \quad \hat{\Psi}_{410} = \mathcal{N}_4\Delta D_1(t), \quad \hat{\Psi}_{55} = 0,\\
&\quad \hat{\Psi}_{56} = 0, \quad \hat{\Psi}_{57} = 0, \quad \hat{\Psi}_{58} = 0, \quad \hat{\Psi}_{59} = 0, \quad \hat{\Psi}_{510} = 0,\\
&\quad \hat{\Psi}_{66} = 0, \quad \hat{\Psi}_{67} = 0, \quad \hat{\Psi}_{68} = 0, \quad \hat{\Psi}_{69} = 0, \quad \hat{\Psi}_{610} = 0,\\
&\quad \hat{\Psi}_{77} = 0, \quad \hat{\Psi}_{78} = 0, \quad \hat{\Psi}_{79} = 0, \quad \hat{\Psi}_{710} = 0, \quad \hat{\Psi}_{88} = 0,\\
&\quad \hat{\Psi}_{89} = 0, \quad \hat{\Psi}_{810} = 0, \quad \hat{\Psi}_{99} = 0, \quad \hat{\Psi}_{910} = 0, \quad \hat{\Psi}_{1010} = 0,\\
&\quad \overline{C}_2 = \begin{bmatrix} \Delta C(t) + \Delta B_2K(t) & \underbrace{0 \ldots 0}_{8times} & \Delta D_2(t) \end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
\Delta\hat{\Psi} = &\Upsilon_1^T\eta(t)\Upsilon_2 + \Upsilon_2^T\eta(t)\Upsilon_1 + \Upsilon_3^T\eta(t)\Upsilon_4\\
&+ \Upsilon_4^T\eta(t)\Upsilon_3 + \Upsilon_5^T\delta(t)\Upsilon_6 + \Upsilon_6^T\delta(t)\Upsilon_5\\
&+ \Upsilon_7^T\eta(t)\Upsilon_8 + \Upsilon_8^T\eta(t)\Upsilon_7 + \Upsilon_9^T\delta(t)\Upsilon_{10}\\
&+ \Upsilon_{10}^T\delta(t)\Upsilon_9,
\end{aligned} \tag{42}
$$

and

$$
\begin{aligned}
&\Upsilon_1 = \begin{bmatrix} F_1^TP^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\\
&\Upsilon_2 = \begin{bmatrix} -G_1 & 0 & 0 & 0 & 0 & 0 & G_2 & G_3 & G_4 & G_5 \end{bmatrix},\\
&\Upsilon_3 = \begin{bmatrix} F_1^T\mathcal{N}_1^T & F_1^T\mathcal{N}_2^T & F_1^T\mathcal{N}_3^T & F_1^T\mathcal{N}_4^T & \underbrace{0 \ldots 0}_{6times} \end{bmatrix},\\
&\Upsilon_4 = \begin{bmatrix} -G_1 & 0 & 0 & 0 & 0 & 0 & G_2 & G_3 & G_4 & G_5 \end{bmatrix},\\
&\Upsilon_5 = \begin{bmatrix} H_1^T(\mathcal{N}_1^T + P^T) & H_1^T\mathcal{N}_2^T & H_1^T\mathcal{N}_3^T \\ H_1^T\mathcal{N}_4^T & 0 \; 0 \; 0 \; 0 \; 0 \; 0 \end{bmatrix},\\
&\Upsilon_6 = \begin{bmatrix} -B_1H_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\\
&\Upsilon_7 = \begin{bmatrix} F_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\\
&\Upsilon_8 = \begin{bmatrix} -G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_5 \end{bmatrix},\\
&\Upsilon_9 = \begin{bmatrix} H_1^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\\
&\Upsilon_{10} = \begin{bmatrix} -B_2H_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Using Lemma 4, the above inequality 42 is equivalent to the following inequality

$$\begin{aligned}\boldsymbol{\Psi} =& \bar{\Psi} + \varrho_1\Upsilon_1\Upsilon_1^T + \varrho_1^{-1}\Upsilon_2^T\Upsilon_2 + \varrho_2\Upsilon_3\Upsilon_3^T \\ &+ \varrho_2^{-1}\Upsilon_4^T\Upsilon_4 + \varrho_3\Upsilon_5\Upsilon_5^T + \varrho_3^{-1}\Upsilon_6^T\Upsilon_6 + \varrho_4\Upsilon_7\Upsilon_7^T \\ &+ \varrho_4^{-1}\Upsilon_8^T\Upsilon_8 + \varrho_5\Upsilon_9\Upsilon_9^T + \varrho_5^{-1}\Upsilon_{10}^T\Upsilon_{10} < 0.\end{aligned} \tag{43}$$

By using Schur complement Lemma,

$$\begin{bmatrix} \bar{\Psi} & \bar{C}_1 & \Upsilon_1 & \varrho_1\Upsilon_2 & \Upsilon_3 & \varrho_2\Upsilon_4 & \Upsilon_5 & \varrho_3\Upsilon_6 & \Upsilon_7 & \varrho_4\Upsilon_8 & \Upsilon_9 & \varrho_5\Upsilon_{10} \\ * & -I & -\varrho_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\varrho_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varrho_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\varrho_2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\varrho_3 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\varrho_3 I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\varrho_4 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\varrho_4 I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\varrho_5 I & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\varrho_5 I \end{bmatrix} < 0, \tag{44}$$

pre and post multiplying 44 by diag $\{X, I, I, I, I, I, I, I, I, I, I, I, I, I, I, I, I, I, I, I\}$ respectively, we get 38. By Definition 3, the system 37 is finite-time bounded, which completes the proof.

4 | NUMERICAL EXAMPLE

In this section we present a numerical example to illustrate the correctness and feasibility of the obtained results.

Example 1. Consider the neural networks with time-varying delays 37 with the following values.

$$A = \begin{bmatrix} 12.5 & 0 \\ 0 & 10 \end{bmatrix}, \quad W_0 = \begin{bmatrix} -0.2 & 4.5 \\ 0.4 & -0.1 \end{bmatrix},$$
$$W_1 = \begin{bmatrix} 0.2 & 0.2 \\ 1 & 0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1.8 & 0 \\ 0 & 2.8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.01 \\ 0.1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.02 \\ 0.03 \end{bmatrix},$$
$$F_1 = F_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & 0.1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.5 & 0.2 \end{bmatrix},$$
$$G_3 = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}, \quad G_4 = 0.2, \quad G_5 = 0.1, \quad \bar{\tau} = 0.6,$$
$$\bar{\rho} = 0.9, \quad \bar{\tau}_D = 0.3, \quad \bar{\rho} = 0.2, \quad d = 0.03, \quad T = 5,$$
$$c_1 = 1.3, \quad c_2 = 12.3 \quad \alpha = 0.02.$$

TABLE 1 Calculated $\gamma_{\min}$ for different values of $\bar{\tau}$

$\bar{\tau}$	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
$\gamma_{\min}$	0.5610	0.6942	0.7380	0.7946	0.8651

TABLE 2 Calculated maximum upper bound of $\bar{\tau}$ for different γ

$\gamma_{\min}$	**1.0**	**2.0**	**3.0**	**4.0**	**5.0**
$\bar{\tau}$	0.8651	1.4215	2.0316	2.8744	3.2145

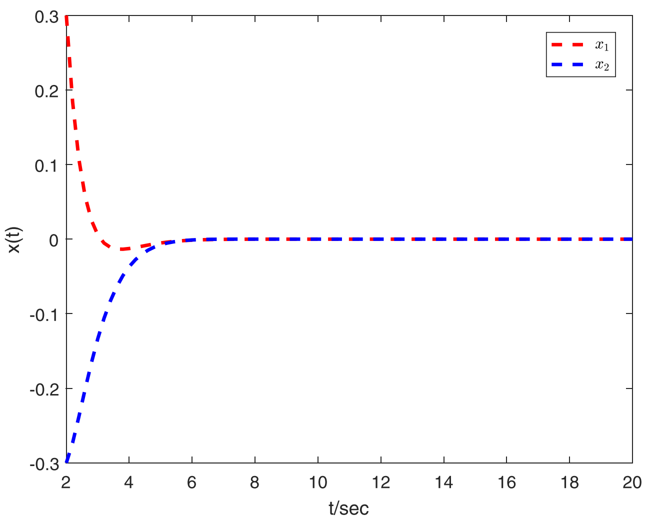

FIGURE 1 State trajectories of neural networks in Example 1 [Color figure can be viewed at wileyonlinelibrary.com]

The activation function and the time-varying delay are taken as $f(x) = (1)/(4)(|x+1| - |x-1|)$ and $\tau(t) = 0.3 + 0.3\cos(t)$, respectively. Then, it is obvious that, $\bar{\tau} = 0.6$ and $\bar{\tau}_D = 0.3$.

Then, by solving LMI 38, the control gain matrix can be found as

$$K = P^{-1}L = \begin{bmatrix} -0.8721 & -0.2346 \end{bmatrix}.$$

Our purpose is to design a uncertain neutral-type neural networks is finite-time bounded with H_∞ prescribed attenuation level $\gamma = 0.5610$. The guaranteed optimal H_∞ performance level $\gamma > 0$ for different values of fixed time-delay upper bound $\bar{\tau}$ is shown in Table 1. Also for different values of $\gamma > 0$, by solving the LMI condition, we obtain values of time-delay upper bound $\bar{\tau}$, which are given in Table 2. Figure 1 and Figure 2 represents different value of the time response of the state vector $x(t)$.

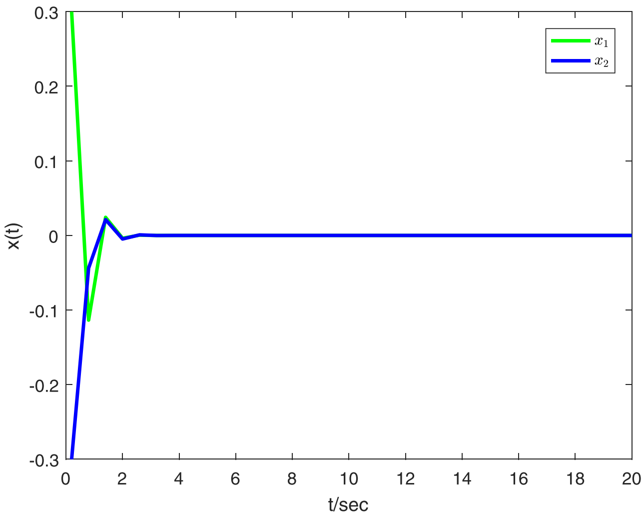

FIGURE 2 State trajectories of neural networks in Example 1 [Color figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSIONS

In this paper, the problem of resilient H_∞ performance for finite-time boundedness of uncertain neutral-type neural networks has been investigated. By constructing an appropriate Lyapunov-Krasovskii functional, a sufficient condition is derived such that the closed-loop system is finite-time bounded and satisfies the given level. The H_∞ performance can be obtained by using the exiting LMI optimization techniques. Finally, a numerical example has been provided to show the usefulness of the proposed method. Our future work will focus on finding the new methods or integral inequalities to reduce the conservativeness of the stability criteria for time-delay systems.

ACKNOWLEDGEMENTS

This work was financially supported by the 2019 Post-Doc Development Program of Pusan National University and the National Research Foundation (NRF) of Korea under the auspices of the Ministry of Science and ICT, Korea (grant no. NRF-2017R1A2A1A17069430).

ORCID

Saravanan Shanmugam https://orcid.org/0000-0001-5016-7560
M Syed Ali https://orcid.org/0000-0003-3747-3082
Keum-Shik Hong https://orcid.org/0000-0002-8528-4457
Quanxin Zhu https://orcid.org/0000-0003-3130-4923

REFERENCES

1. O. Kwon et al., *Analysis on delay-dependent stability for neural networks with time-varying delays*, Neurocomputing **103** (2013), 114–120.
2. O. Kwon et al., *New augmented Lyapunov–Krasovskii functional approach to stability analysis of neural networks with time-varying delays*, Nonlinear Dyn. **76** (2014), no. 1, 221–236.
3. X. Li, J.-A. Fang, and H. Li, *Exponential synchronization of memristive chaotic recurrent neural networks via alternate output feedback control*, Asian J. Control **20** (2018), no. 1, 469–482.
4. Q. Song, *Exponential stability of recurrent neural networks with both time-varying delays and general activation functions via LMI approach*, Neurocomputing **71** (2008), no. 13, 2823–2830.
5. J. Tian and S. Zhong, *Improved delay-dependent stability criterion for neural networks with time-varying delay*, Appl. Math. Comput. **217** (2011), no. 24, 10278–10288.
6. J. Wang, X. Liu, J. Bai, and Y. Chen, *A new stability condition for uncertain fuzzy Hopfield neural networks with time-varying delays*, Int. J. Control Autom. Syst. **17** (2019), no. 5, 1322–1329.
7. Z.-G. Wu et al., *Stability and dissipativity analysis of static neural networks with time delay*, IEEE Trans. Neural Netw. Learn. Syst. **23** (2012), no. 2, 199–210.
8. M Syed Ali, P. Balasubramaniam, and Q. Zhu, *Stability of stochastic fuzzy BAM neural networks with discrete and distributed time-varying delays*, Int. J. Mach. Learn. Cybern. **8** (2017), no. 1, 263–273.
9. Q. Zhu and J. Cao, *Robust exponential stability of Markovian jump impulsive stochastic Cohen-Grossberg neural networks with mixed time delays*, IEEE Trans. Neural Netw. **21** (2010), no. 8, 1314–1325.
10. Q. Zhu and J. Cao, *Exponential stability of stochastic neural networks with both Markovian jump parameters and mixed time delays*, IEEE Trans. Syst. Man. Cybern. Part B-Cybern. **41** (2011), no. 2, 341–353.
11. Q. Zhu and J. Cao, *Stability of Markovian jump neural networks with impulse control and time varying delays*, Nonlinear Anal. Real World Appl. **13** (2012), no. 5, 2259–2270.
12. Q. Zhu et al., *Robust stability of Markovian jump stochastic neural networks with time delays in the leakage terms*, Neural Process. Lett. **41** (2015), no. 1, 1–27.
13. X.-M. Zhang et al., *An overview of recent developments in Lyapunov–Krasovskii functionals and stability criteria for recurrent neural networks with time-varying delays*, Neurocomputing **313** (2018), 392–401.
14. X.-M. Zhang et al., *Overview of recent advances in stability of linear systems with time-varying delays*, IET Control Theory Appl. **13** (2018), no. 1, 1–16.
15. K.-S. Hong and P.-T. Pham, *Control of axially moving systems: A review*, Int. J. Control Autom. Syst. **17** (2019), no. 12, 2983–3008.
16. Q. C. Nguyen, M. Piao, and K.-S. Hong, *Multivariable adaptive control of the rewinding process of a roll-to-roll system governed by hyperbolic partial differential equations*, Int. J. Control Autom. Syst. **16** (2018), no. 5, 2177–2186.
17. A. K. Pamosoaji, M. Piao, and K.-S. Hong, *Pso-based minimum-time motion planning for multiple vehicles under acceleration and velocity limitations*, Int. J. Control Autom. Syst. **17** (2019), no. 10, 2610–2623.

18. S. Zhang, W.-Y. Cui, and F. E. Alsaadi, *Adaptive backstepping control design for uncertain non-smooth strictfeedback nonlinear systems with time-varying delays*, Int. J. Control Autom. Syst. **17** (2019), no. 9, 2220–2233.
19. X. Zhang et al., *Robust adaptive neural control for a class of time-varying delay systems with backlash-like hysteresis input*, Asian J. Control **18** (2016), no. 3, 1087–1101.
20. A. Seuret and F. Gouaisbaut, *Wirtinger-based integral inequality: application to time-delay systems*, Automatica **49** (2013), no. 9, 2860–2866.
21. H.-B. Zeng et al., *Free-matrix-based integral inequality for stability analysis of systems with time-varying delay*, IEEE Trans. Automat. Control **60** (2015), no. 10, 2768–2772.
22. C.-K. Zhang et al., *Stability analysis of systems with time-varying delay via relaxed integral inequalities*, Syst. Control Lett. **92** (2016), 52–61.
23. C.-K. Zhang et al., *Delay-dependent stability analysis of neural networks with time-varying delay: A generalized free-weighting-matrix approach*, Appl. Math. Comput. **294** (2017), 102–120.
24. A. Seuret and F. Gouaisbaut, *Stability of linear systems with time-varying delays using Bessel–Legendre inequalities*, IEEE Trans. Automat. Control **63** (2017), no. 1, 225–232.
25. P. Park et al., *Reciprocally convex approach to stability of systems with time-varying delays*, Automatica **47** (2011), no. 1, 235–238.
26. C.-K. Zhang, Y. He, L. Jiang, Q.-G. Wang, and M. Wu, *Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality*, IEEE Trans. Cybern. **47** (2017), no. 10, 3040–3049.
27. C.-K. Zhang et al., *An extended reciprocally convex matrix inequality for stability analysis of systems with time-varying delay*, Automatica **85** (2017), 481–485.
28. M. S. Mahmoud and N. B. Almutairi, *Resilient decentralized stabilization of interconnected time-delay systems with polytopic uncertainties*, Int. J. Robust Nonlinear Control **21** (2011), no. 4, 355–372.
29. M. Mahmoud, *Resilient L_2-L_∞ filtering of polytopic systems with state delays*, IET Contr. Theory Appl. **1** (2007), no. 1, 141–154.
30. H. Shen et al., *Delay-dependent robust stabilization for uncertain stochastic switching systems with distributed delays*, Asian J. Control **11** (2009), no. 5, 527–535.
31. M Syed Ali and P. Balasubramaniam, *Stability analysis of uncertain fuzzy Hopfield neural networks with time delays*, Commun. Nonlinear Sci. Numerical Simul. **14** (2009), no. 6, 2776–2783.
32. M. Zhu and S. Martínez, *On the performance analysis of resilient networked control systems under replay attacks*, IEEE Trans. Automat. Control **59** (2014), no. 3, 804–808.
33. Y.-Y. Cao and P. M. Frank, *Analysis and synthesis of nonlinear time-delay systems via fuzzy control approach*, IEEE Trans. Fuzzy Syst. **8** (2000), no. 2, 200–211.
34. Y.-Y. Cao and P. M. Frank, *Stability analysis and synthesis of nonlinear time-delay systems via linear Takagi–Sugeno fuzzy models*, Fuzzy Sets Syst. **124** (2001), no. 2, 213–229.
35. Z. Liu, C. Gao, and Y. Kao, *Robust H_∞ control for a class of neutral-type systems via sliding mode observer*, Appl. Math. Comput. **271** (2015), 669–681.
36. K. Mathiyalagan, R. Sakthivel, and S. Marshal Anthoni, *Robust exponential stability and H_∞ control for switched neutral-type neural networks*, Int. J. Adapt. Control Signal Process. **28** (2014), no. 3-5, 429–443.
37. M Syed Ali and S. Saravanan, *Robust finite-time H_∞ control for a class of uncertain switched neural networks of neutral-type with distributed time varying delays*, Neurocomputing **177** (2016), 454–468.
38. S. Lakshmanan et al., *Neutral-type of delayed inertial neural networks and their stability analysis using the LMI approach*, Neurocomputing **230** (2017), 243–250.
39. Q. Zhu and J. Cao, *Stability analysis for stochastic neural networks of neutral type with both Markovian jump parameters and mixed time delays*, Neurocomputing **73** (2010), no. 13, 2671–2680.
40. F. Amato and M. Ariola, *Finite-time control of discrete-time linear systems*, IEEE Trans. Automat. Control **50** (2005), no. 5, 724–729.
41. E. Moulay et al., *Finite-time stability and stabilization of time-delay systems*, Syst. Control Lett. **57** (2008), no. 7, 561–566.
42. J. Cheng et al., *Finite-time H_∞ control for a class of Markovian jump systems with mode-dependent time-varying delays via new Lyapunov functionals*, ISA Trans. **52** (2013), no. 6, 768–774.
43. M Syed Ali, S. Saravanan, and S. Arik, *Finite-time H_∞ state estimation for switched neural networks with time-varying delays*, Neurocomputing **207** (2016), 580–589.
44. Z. Xiang, Y.-N. Sun, and M. S. Mahmoud, *Robust finite-time H_∞ control for a class of uncertain switched neutral systems*, Commun. Nonlin. Sci. Numerical Simul. **17** (2012), no. 4, 1766–1778.
45. X. Zhang et al., *Robust finite-time H_∞ control of a class of disturbed systems using LMI-based approach*, Asian J. Control **19** (2017), no. 2, 575–586.
46. F. Amato, M. Ariola, and P. Dorato, *Finite-time control of linear systems subject to parametric uncertainties and disturbances*, Automatica **37** (2001), no. 9, 1459–1463.
47. X. Yang, Y. Tian, and X. Li, *Finite-time boundedness and stabilization of uncertain switched delayed neural networks of neutral type*, Neurocomputing **314** (2018), 468–478.
48. M. Park et al., *Stability of time-delay systems via Wirtinger-based double integral inequality*, Automatica **55** (2015), 204–208.
49. P. Park, W. I. Lee, and S. Y. Lee, *Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems*, J. Franklin Inst. **352** (2015), no. 4, 1378–1396.
50. I. R. Petersen, *A stabilization algorithm for a class of uncertain linear systems*, Syst. Control Lett. **8** (1987), no. 4, 351–357.

How to cite this article: Shanmugam S, Syed Ali M, Hong K-S, Zhu Q. Robust resilient H_∞ performance for finite-time boundedness of neutral-type neural networks with time-varying delays. *Asian J Control*. 2020;1–10 https://doi.org/10.1002/asjc.2361