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Asymptotic Behavior Analysis of a Coupled Time-Varying System: Application to Adaptive Systems

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Abstract—**Asymptotic behavior of a partial state of a coupled ordinary and/or partial differential equation is investigated. It is specifically shown that if a signal $x(t)$ is a solution to a dynamic system existing for all $t \geq 0$ in an abstract Banach space and is pth ($p \geq 1$) power integrable, then $x(t) \to 0$ as $t \to \infty$. The system is allowed to be nonautonomous and assumes the existence of a Lyapunov function. Since the derivative of the Lyapunov function is negative semidefinite, stability or uniform stability in the sense of Lyapunov would be concluded. However, this paper further asserts that the partial state which remains in the time derivative of the Lyapunov function converges to zero asymptotically.**

Index Terms—**Adaptive systems, convergence, existence, time-varying system, uniqueness.**

I. Introduction

In this paper the asymptotic behavior of a part of the solution of a coupled dynamic system, whose solution exists for all $t \geq 0$, is investigated. The coupled dynamic system is assumed to permit the construction of a Lyapunov function with a negative semidefinite time derivative. The system considered is allowed to be time-varying; therefore, LaSalle's theorem [15, p. 158] for the case of ordinary differential equations would not be applicable. Furthermore, an extension of LaSalle's theorem to infinite-dimensional systems is not yet available. Since the derivative of the Lyapunov function is assumed to be only negative semidefinite, stability or uniform stability in the sense of Lyapunov would be concluded in ordinary cases. However, this paper further addresses the asymptotic convergence to zero of the partial state which remains in the time derivative of the Lyapunov function.

The class of systems considered in this paper originally represented an adaptive control problem. Hence, the applicability of the proposed investigation is already justified. However, the application of the results in Section II below is not limited to adaptive systems. Furthermore, it should be remarked that the form of systems considered in Section II is not critical. The key issue of this paper is what can be said in the Lyapunov analysis about the qualitative behavior of the partial state which remains in the derivative of a Lyapunov function. Since only a partial state remains in the derivative, the asymptotic stability cannot be concluded.

In adaptive identification or control problems [1], [2], [3]–[5], [8]–[11], [13]–[14], [17], the whole closed-loop system involves error dynamics between the plant and model, adaptation rules, and design-dependent normalizing signals. In showing the convergence of error dynamics ($x(t)$) to zero in finite-dimensional adaptive systems, Barbalat's lemma [11, p. 85], [14, p. 19] is used with the fact that $x(t) \in L_2(0, \infty)$. In this paper it will be shown that the adaptive system employing the Lyapunov redesign method intrinsically assures that $x(t) \to 0$ as $t \to \infty$. Therefore, the results obtained in this paper reveal an interesting fact: if the adaptation law is derived in such a way that $\dot{V}(t, x, y, \eta) \leq -\alpha(\|x\|)$, where V is a Lyapunov function for the whole adaptive system, x denotes the error dynamics between the plant and model, y is the adaptation law, η is some normalizing signal, and $\alpha(\cdot)$ is a continuous monotone function, then the trajectory of the plant is guaranteed to follow that of the model.

The main contribution of this paper is Theorem 1, which shows the asymptotic convergence of a partial state of a coupled dynamic system to zero. The application of Theorem 1 is demonstrated in Section III but is not limited to adaptive systems. Noting that some adaptive systems belong to the category of Theorem 1, the second contribution of this paper is in showing that the Lyapunov redesign method itself assures that $x(t) \to 0$ as $t \to \infty$, where $x(t)$ is the error dynamics between the plant and model. In the literature the stability and convergence analysis of a specific adaptive system has been carried out only for the specific system considered. However, this paper clarifies that the converging behavior of the state error dynamics to zero with the Lyapunov redesign method is a general property and is not dependent on the type of the system considered. Furthermore, by using the existence and uniqueness of the solution the convergence analysis adopted in this paper is unique.

Theorem 1 deals with infinite-dimensional systems, whereas Corollary 1 deals with finite-dimensional systems. Even though the proofs for both cases use the semigroup approach and contradiction argument, the infinite-dimensional case is assumed to be semilinear. This is because in the infinite-dimensional case the proofs for the existence and uniqueness of solutions are readily available for semilinear systems but not for general systems. On the other hand, the conditions for the finite-dimensional case will provide the existence and uniqueness results together with the asymptotic behavior of a portion of the solutions.

The paper is organized as follows. In Section II the asymptotic behavior of the solution of a coupled dynamic system is investigated. In Section III an example of the infinite-dimensional case is given. Conclusions follow in Section IV. $\|\cdot\|$ will be adopted as a generic norm in any Banach space.

II. Asymptotic Behavior of Solution

Theorem 1: Consider a coupled dynamic system as

$$\dot{x}(t) = A(y(t))x(t) + f(t, x, y), \qquad x(0) = x_o \tag{1}$$

$$\dot{y}(t) = g(t, x, y, \eta), \qquad y(0) = y_o \tag{2}$$

$$\dot{\eta}(t) = -\delta_o \eta(t) + h(t, x, y), \qquad \eta(0) = \eta_o \tag{3}$$

where $x \in X$, $y \in Y$, and $\eta \in Z$. X, Y, and Z are Banach spaces. $A(y(t))$ is a family of operators on X where $y(t), t \geq 0$ is to be a time-varying parameter. δ_o is a positive constant. The following assumptions are made.

1) There exists a unique solution to system (1)–(3), and the solution of (1) is given in the variation of constants formula of the following form:

$$x(t) = \Phi(t, 0)x_o + \int_0^t \Phi(t, \tau) f(\tau, x(\tau), y(\tau))\, d\tau \tag{4}$$

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where $\Phi(t,s)$, $0 \le s \le t$ is the evolution operator associated with $A(y(t))$, and $\|\Phi(t,s)\| \le M_1$, where M_1 is a constant, for $0 \le s \le t < \infty$.

2)
$$\|f(t,x,y)\| \le \alpha_0(y)\|x\| + c_0, \quad \forall t \ge 0 \tag{5}$$
where c_0 is a constant and $\alpha_0 : Y \to R^+$ is bounded for finite values of y.

3) There exists a Lyapunov function $V(t,x,y,\eta) : R^+ \times X \times Y \times Z \to R^+$ for system (1)–(3).

4) There exists a continuous nondecreasing function $\alpha(\cdot)$ with $\alpha(0) = 0$ such that
$$\dot{V}(t,x,y,\eta)\big|_{(1)-(3)} \le -\alpha(\|x\|). \tag{6}$$

Then $x(t) \to 0$ as $t \to \infty$ for every $x_o \in X$.

Proof: The conclusion will be proved by a contradiction argument with the following three facts: 1) The existence of a Lyapunov function V, 2) the existence and uniqueness of the solution which includes the existence of the evolution process $\Phi(t,s)$, and 3) (6) which can be rewritten as in (9) below.

The existence of a Lyapunov function V and the fact that $\dot{V} \le 0$ from Condition 4) imply that a set $E_\beta = \{(x,y,\eta) : V \le \beta, \beta \in R^+\}$ is positive invariant [7, p. 82], [16, p. 138]. Hence $\|x(t)\|$, $\|y(t)\|$, $\|\eta(t)\| \le \beta'$, $\forall t \ge 0$, where β' is a constant not depending on time t. Let the unique solution of (1) at time t starting with the initial state $x(s)$ at initial time s be of the form
$$x(t) = \Phi(t,s)x(s) + \int_s^t \Phi(t,\tau)f(\tau,x(\tau),y(\tau))\,d\tau. \tag{7}$$

Denoting (7) as $x(t) = x(t,x(s),s)$, define a two parameter family of map $S(t,s)$ on X as
$$S(t,s)x(s) = x(t,x(s),s), \qquad 0 \le s \le t < \infty. \tag{8}$$

Then, by the uniqueness and continuous dependence of the solution on the triple $(t,x(s),s)$, the mapping $S(t,s)$ on X becomes an evolution process [16, p. 12, p. 49] such that:

1) $S(\cdot,s)x(s) : R^+ \to X$ is continuous (right continuous at $t = s$);
2) $S(t,\cdot)(\cdot) : R^+ \mathrm{x} X \to X$ is continuous;
3) $S(s,s)x(s) = x(s)$;
4) $S(t,s)x(s) = S(t,r)S(r,s)x(s)$, for all $x(s) \in X$ and $0 \le s \le r \le t < \infty$.

Finally, note that (6) implies that
$$\int_0^\infty \alpha(\|x(t)\|\,dt = \int_0^\infty \alpha(\|S(t,0)x_o\|)\,dt \le V(0) - V(\infty) < \infty \tag{9}$$
where the initial time and state are chosen to be zero and $x_o \in E_\beta$, respectively.

Indeed, the conclusion of the theorem can be proven by contradiction [12, p. 116]. Suppose that $x(t) = S(t,0)x_o \not\to 0$ as $t \to \infty$. Then there exists a $\varepsilon > 0$ and an infinite sequence $t_j \to \infty$ such that
$$\|S(t_j,0)x_o\| \ge \varepsilon.$$

Now, however small the ε is, there always exist constants $M_2 > 0$ and $\varepsilon_o > 0$ such that
$$M_2 \ge \sup_{\|y(t)\| < \beta} \alpha_0(y(t)) \tag{10}$$
and
$$\frac{\varepsilon}{e} - \frac{c_0}{M_2} \ge \varepsilon_o > 0 \tag{11}$$
where $\alpha_0(\cdot)$ and c_0 are found in (5), and e is the base of natural logarithm. Note that M_2 is a constant related to the Lipschitz-like growth condition on f, and (11) is always satisfied if $c_0 = 0$ in (5). Therefore, taking norms on both sides of (7) yields
$$\begin{aligned}\|x(t)\| &\le \|\Phi(t,s)\|\,\|x(s)\| + \int_s^t \|\Phi(t,\tau)\|\,\|f(\tau,x(\tau),y(\tau))\|\,d\tau \\ &\le M_1\|x(s)\| + \int_s^t M_1(\alpha_0(y(\tau))\|x(\tau)\| + c_0)\,d\tau \\ &\le M_1\|x(s)\| + M_1 M_2 \int_s^t \left(\|x(\tau)\| + \frac{c_0}{M_2}\right)d\tau. \end{aligned} \tag{12}$$

Applying the Bellman–Gronwall's inequality yields
$$\|x(t)\| \le \left(M_1\|x(s)\| + \frac{c_0}{M_2}\right)e^{M_1 M_2 (t-s)} \tag{13}$$
for all $t \ge s \ge 0$. Now, without loss of generality, we can assume the sequence t_j such that $t_{j+1} - t_j > (M_1 M_2)^{-1}$. If we define the intervals Δ_j as $\Delta_j = [t_j - (M_1 M_2)^{-1}, t_j]$, then $m(\Delta_j) > 0$ (m = Lebesgue measure) and the intervals Δ_j do not overlap. For $t \in \Delta_j$
$$\begin{aligned}\varepsilon \le \|x(t_j)\| &= \|S(t_j,0)x_o\| \\ &= \|S(t_j,t)S(t,0)x_o\| \\ &= \|S(t_j,t)x(t)\| \\ &\le \left(M_1\|x(t)\| + \frac{c_0}{M_2}\right)e^{M_1 M_2 (t_j - t)} \\ &\le \left(M_1\|x(t)\| + \frac{c_0}{M_2}\right)e \end{aligned}$$
where the second inequality above is obtained from (13). Therefore we have
$$\|x(t)\| \ge \frac{\varepsilon_o}{M_1} > 0$$
for all $t \in \Delta_j$. Hence
$$\begin{aligned}\int_0^\infty \alpha(\|S(t,0)x_o\|)\,dt &\ge \sum_{j=1}^\infty \int_{\Delta_j} \alpha(\|S(t,0)x_o\|)\,dt \\ &\ge \sum_{j=1}^\infty \int_{\Delta_j} \alpha\left(\frac{\varepsilon_o}{M_1}\right)dt \\ &= \alpha\left(\frac{\varepsilon_o}{M_1}\right)\sum_{j=1}^\infty m(\Delta_j) = \infty \end{aligned}$$
contradicting (9). Thus we must have $x(t) \to 0$ as $t \to \infty$. □

Remark 1: Even though the existence and uniqueness of the solution have been assumed, which is to assert the asymptotic convergence of the partial state x to zero, the class of systems that permits application of the theorem is not empty, as will be shown in Section III. The specific conditions for the existence and uniqueness of solutions will be different, depending on the types of systems in the considered Banach spaces. Also, there is no general existence theorem available yet for general infinite-dimensional systems. By defining $B(t) = A(y(t))$, the conditions [with respect to $B(t)$] for the existence of the evolution operator $\Phi(t,s)$ are found in [12, p. 134] for the hyperbolic system and in [12, p. 149] or [6, p. 108] for the parabolic system. Furthermore, with the same conditions $\|\Phi(t,s)\| \le Me^{\omega(t-s)}$ (ω is a stability constant) for the hyperbolic case [12, p. 135], and $\|\Phi(t,s)\| \le M$ for the parabolic case [12, p. 150], are achieved. Also, the existence, stability, and smoothness of invariant manifolds of the hybrid system (1), (2) have been studied in [7, p. 275]. For more detailed results refer to [6], [7], [12], [16], and the references therein.

Remark 2: In adaptive control, the state x represents the error dynamics between the closed-loop plant with filters and the model. The state y denotes the estimated parameter vector which is referred to as the adaptation law. η is a design variable known as the normalizing signal.

On the other hand, the conditions in Corollary 1 will assure the existence and uniqueness of the solution together with the asymptotic convergence to zero.

Corollary 1: Consider a finite-dimensional system as

$$\dot{x} = f(t,x,y), \quad x(0) = x_o \tag{14}$$

$$\dot{y} = g(t,x,y,\eta), \quad y(0) = y_o \tag{15}$$

$$\dot{\eta} = -\delta_o\eta + h(t,x,y), \quad \eta(0) = \eta_o \tag{16}$$

where $x \in R^n$, $y \in R^m$, $\eta \in R^1$, and $\delta_o > 0$ is a constant. The following assumptions are made.

1) $f(t,0,0) = 0$ and $g(t,0,0,\eta) = 0$. f, g, and h are piecewise continuous in t and are continuous in other variables. Furthermore, f and h are locally Lipschitz in x and y. g is locally Lipschitz in x, y, and η.

2) a)

$$\|f(t,x,y)\| \le \alpha_0(y)\|x\| + c_0, \qquad \forall t \ge 0;$$

b)

$$|h(t,x,y)| \le \alpha_1(y)\|x\|^2 + \alpha_2(y)\|x\| + c_1, \qquad \forall t \ge 0;$$

where c_0, c_1 are constants and $\alpha_0, \alpha_1, \alpha_2 : R^m \to R^+$ are bounded for finite values of y.

3) There exists a functional $V : R^+ \times R^{n+m} \to R^+$ such that

$$k_1\|x\|^2 + k_2\|y\|^2 \le V(t,x,y) \le k_3\|x\|^2 + k_4\|y\|^2$$

where k_1, k_2, k_3, k_4 are positive constants.

4)

$$\dot{V}(t,x,y)\big|_{(14)-(15)} \le -\alpha(\|x\|)$$

where $\alpha(\cdot)$ is a continuous monotone function with $\alpha(0) = 0$.

Then, $x(t) \to 0$ as $t \to \infty$.

Proof: The method of proof is exactly the same as that of Theorem 1. The only difference is that (14) is nonlinear and (1) is semilinear.

The existence and uniqueness of the solution to system (14)–(16) has been established in [13]. Let the solution of (14) with the initial state $x(s)$ at initial time s be of the form

$$x(t) = x(s) + \int_s^t f(\tau, x(\tau), y(\tau))\, d\tau. \tag{17}$$

Taking norms using Condition 2)-a) yields

$$\|x(t)\| \le \|x(s)\| + \int_s^t M_2\left(\|x(\tau)\| + \frac{c_0}{M_2}\right) d\tau \tag{18}$$

where M_2 is chosen to satisfy (11). Applying the Bellman–Gronwall's inequality

$$\|x(t)\| \le \left(\|x(s)\| + \frac{c_0}{M_2}\right)e^{M_2(t-s)} \tag{19}$$

for $t \ge s \ge 0$. Therefore, choosing $\Delta_j = [t_j - (M_2)^{-1}, t_j]$

$$\|x(t)\| \ge \varepsilon_o \quad \text{for all } t \in \Delta_j.$$

The rest of the proof is the same as Theorem 1. □

Remark 3: Comparing (12) and (18), the difference is that one is from the semilinear (1) and the other is from the nonlinear (14). M_2 comes from the Lipschitz-like growth condition on f for both systems (1) and (14), respectively. Equations (13) and (19) imply that the solutions existing for all $t \ge 0$ cannot grow faster than some exponential function.

Remark 4: Note that (9) must hold for all initial conditions $x_o \in B_\beta, \beta \in R^+$, due to the positive invariance of B_β. Therefore, (9) will exclude the typical situation where f in (14) is a function of only t and y. Indeed, if f were of the form

$$\dot{x} = f(t,y), \qquad x(0) = x_o \tag{20}$$

(this will never happen in adaptive control since x denotes the error dynamics), the solution would be of the form

$$x(t) = x_o + \int_0^t f(\tau, y(\tau))\, d\tau. \tag{21}$$

Then, (9) is never achieved for any $x_o \ne 0$ because (9) can be satisfied for only one particular x_o by offsetting the second term in (21) but not for all initial conditions.

Remark 5: Corollary 1 also concludes the following. In general, $x(t) \in L_p(0,\infty)$ does not imply that $x(t) \to 0$ as $t \to \infty$. The uniform continuity of $x(t)$ is needed as is shown in Barbalat's lemma. However, besides the fact that $x(t) \in L_p$, if the signal comes through a dynamical system as $\dot{x} = f(t,x,y)$, where a unique solution exists for all $t \ge 0$ and y is a bounded parameter, then $x(t) \to 0$ as $t \to \infty$. Let us consider a pathological signal $x(t)$ which belongs to L_p but does not tend to zero (this signal will violate the uniform continuity condition). Also, let the derivative of $x(t)$ be $\zeta(t)$. Then $x(t)$ can be considered as a signal generated through a dynamical system of the form

$$\dot{x}(t) = \zeta(t), \qquad x(0) = 0$$

which is exactly the form of (20), and only the zero initial condition will provide $x(t) \in L_p$.

Remark 6: The above results may suggest the following design procedure for a model following adaptive system: 1) derive an adaptive control law which permits exact equation matching between the plant and the model when the adjustable parameters in the controller converge to some values; 2) assure the existence and uniqueness of solutions; and 3) assure the existence of a Lyapunov function for the whole adaptive system and that the derivative of the Lyapunov function is of the form

$$\dot{V} \le -\alpha(\|x\|)$$

where x is the state error between plant and model and α is monotonic. Then $x(t) \to 0$ as $t \to \infty$.

Remark 7: For the finite-dimensional case, the conditions in Corollary 1 are equivalent to the boundedness of $\dot{x}$. Hence $x(t) \in L_2(0,\infty)$ together with Barbalat's lemma implies that $x(t) \to 0$ as $t \to \infty$. The use of Barbalat's lemma is standard in adaptive control literature.

The above observation is summarized in the following remark.

Remark 8: Let $x(t) \in L_p(0,\infty), p \ge 1$ be the unique solution for all $t \ge 0$, of $\dot{x} = f(t,x,y)$, where $x \in R^n$, and $y \in R^m$, and let y be a bounded parameter. Then, $\lim_{t\to\infty} x(t) = 0$.

III. An Example

In this section we demonstrate the applicability of Theorem 1 by considering a model reference adaptive control of a parabolic partial differential equation (PDE) and show the convergence of state error dynamics to zero by applying Theorem 1.

Briefly surveying infinite-dimensional adaptive systems, Baumeister and Scondo [2] investigated the parameter estimation problems of elliptic and parabolic equations in order to generalize the finite-dimensional adaptive control technique to the infinite-dimensional case (their work is also found in [1, p. 256]). Demetriou and Rosen [3]–[5] investigated on-line parameter estimation for parabolic and hyperbolic systems, approximation theory, and persistence of excitation of adaptive systems. The error equations appearing in their work are identical to (25) and (26) below. The existence and uniqueness of solutions of those error equations has been established, and the convergence of state error to zero was proven using the contradiction argument. More interesting questions such as identifiability are also referred to in [1]–[5].

Parabolic PDE's arise in many physical, biological and engineering problems, for instance, in heat transfer, nuclear reactor dynamics, chemical reactions, crystal growth, population genetics, flow of electrons and holes in a semiconductor, nerve axon equations, hydrology, petroleum recovery area, and fluid mechanics. For more examples refer to [1], [7], [16], and the references therein.

Consider model reference adaptive control of a linear parabolic PDE with spatially varying coefficients as

$$\dot{\xi}(p,t) = (a(p)\xi'(p,t))' + b(p)\xi(p,t) + u(p,t), \qquad t > 0 \tag{22}$$

where t is time and $p \in \Omega \subset R$ denotes the spatial variable. The symbols $\cdot$ and $'$ denote the derivatives with respect to t and p, respectively. $u(p,t)$ is a control input function. $a(p)$ and $b(p)$ are unknown but $a(p) > 0$ for system (22) to be parabolic. Boundary and initial conditions are given as

$$\xi(p,t) = \beta(t), \quad p \in \partial\Omega$$
$$\xi(p,0) = \xi_o(p).$$

It is assumed that $a(p)$, $b(p)$, and the boundary data $\beta(t)$ are analytic in their appropriate domains. It is also assumed that $\beta(t)$ is known *a priori*, and distributed sensing and actuation are available. A reference model is defined as

$$\begin{aligned}
&\dot{\xi}_m(p,t) = (a_m(p)\xi_m'(p,t))' + b_m(p)\xi_m(p,t) + r(p,t), \quad t > 0\\
&\xi_m(p,t) = \beta(t), \quad p \in \partial\Omega\\
&\xi_m(p,0) = \xi_{mo}(p)
\end{aligned} \tag{23}$$

where $r(p,t)$ is a bounded reference input. It is assumed that $a_m(p) \geq a_o > 0, b_m(p) < 0, |b_m(p)| \geq b_o > 0$ and that $a_m(p)$, $b_m(p)$ are analytic in Ω. Now consider the following control law $u(p,t)$ with adjustable parameters $\phi_a(p,t)$ and $\phi_b(p,t)$ such that

$$u(p,t) = (\phi_a(p,t)\xi'(p,t))' + \phi_b(p,t)\xi(p,t) + r(p,t). \tag{24}$$

The closed-loop plant equation becomes identical to the equation of the reference model when $\lim_{t\to\infty}\phi_a(p,t) = \phi_a^*$ and $\lim_{t\to\infty}\phi_b(p,t) = \phi_b^*$, where $\phi_a^*(p)$ and $\phi_b^*(p)$ are nominal functions defined as $\phi_a^*(p) = a_m(p) - a(p)$ and $\phi_b^*(p) = b_m(p) - b(p)$. Define the state error e as $e(p,t) = \xi(p,t) - \xi_m(p,t)$ and the controller parameter errors ψ_a and ψ_b as $\psi_a(p,t) = \phi_a(p,t) - \phi_a^*(p)$ and $\psi_b(p,t) = \phi_b(p,t) - \phi_b^*(p)$, respectively. Subtracting (23) from (22) yields the state error equation with homogeneous boundary conditions as

$$\begin{aligned}
\dot{e}(p,t) &= (a_m(p)e'(p,t))' + b_m(p)e(p,t)\\
&\quad + (\psi_a(p,t)\xi'(p,t))' + \psi_b(p,t)\xi(p,t)\\
&= ((a_m(p) + \psi_a(p,t))e'(p,t))'\\
&\quad + (b_m(p) + \psi_b(p,t))e(p,t)\\
&\quad + (\psi_a(p,t)\xi_m'(p,t))' + \psi_b(p,t)\xi_m(p,t)\\
e(p,t) &= 0, \quad p \in \partial\Omega\\
e(p,0) &= \xi_o(p) - \xi_{mo}(p)
\end{aligned} \tag{25}$$

where ξ_m is an exogenous signal at our disposal. Consider the adaptation laws given by

$$\dot{\phi}_a(p,t) = \varepsilon e'(p,t)\xi'(p,t), \phi_a(p,0) = \phi_{ao} \tag{26a}$$
$$\dot{\phi}_b(p,t) = -\varepsilon e(p,t)\xi(p,t), \phi_b(p,0) = \phi_{bo} \tag{26b}$$

where $\varepsilon > 0$ is the adaptation gain. Then by considering a functional $V : (L_2(\Omega))^3 \to R^+$ as

$$V(e,\psi_a,\psi_b) = \frac{1}{2}\int_\Omega \left(e^2(p,t) + \frac{1}{\varepsilon}(\psi_a^2(p,t) + \psi_b^2(p,t))\right)dp \tag{27}$$

and differentiating (27) with respect to t and substituting (25)

$$\dot{V} = \int_\Omega \Big(e(a_m e')' + b_m e^2 + e(\psi_a\xi')' + \psi_b e\xi + \frac{1}{\varepsilon}(\psi_a\dot{\psi}_a + \psi_b\dot{\psi}_b)\Big)\, dp.$$

If we integrate the first and third terms by parts, then

$$\int_\Omega e(a_m e')'\, dp = e(a_m e')|_{\partial\Omega} - \int_\Omega a_m (e')^2\, dp$$

and

$$\int_\Omega e(\psi_a\xi')'\, dp = e(\psi_a\xi')|_{\partial\Omega} - \int_\Omega \psi_a e'\xi'\, dp.$$

Since the state error equation has homogeneous boundary conditions, the first two terms in the above expressions are zero. Hence $\dot{V}$ becomes

$$\dot{V} = \int_\Omega \Big(-a_m(e')^2 + b_m e^2 - \psi_a e'\xi' + \psi_b e\xi + \frac{1}{\varepsilon}(\psi_a\dot{\psi}_a + \psi_b\dot{\psi}_b)\Big)\, dp. \tag{28}$$

Now applying (26a) and (26b) to (28) yields

$$\begin{aligned}
\dot{V} &= \int_\Omega (-a_m(e')^2 + b_m e^2)\, dp\\
&\leq \int_\Omega b_m e^2\, dp\\
&\leq -b_o\|e\|^2.
\end{aligned} \tag{29}$$

Note that (25), (26), and (29) correspond precisely to (1), (2), and (6), respectively, in Theorem 1. There is no signal corresponding to (3) in the example. From the results in [1, p. 258], it follows that the coupled system (25) and (26) has a unique solution (e,ϕ_a,ϕ_b) with $(e,\phi_a,\phi_b) \in L^2(0,T; H_0^1 \times L^2 \times L^2)$ for any $T > 0$. And (27) is a Lyapunov function for (25) and (26). The specific forms of the solutions to (25) and (26) when plant (22) and reference model (23) have constant coefficients rather than spatially varying coefficients are referred to in [8]. Hence, by applying Theorem 1, $e(p,t) \to 0$ as $t \to \infty$.

IV. Conclusion

The asymptotic behavior of a part of the solution of a hybrid dynamic system was investigated. Since the system is nonautonomous and the time derivative of the Lyapunov function is negative semidefinite, only stability or uniform stability would be concluded using Lyapunov's second method. However, further investigation in this paper has shown that the partial state which remains in the derivative of the Lyapunov function converges to zero asymptotically. The results were applied to an adaptive system.

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