Pergamon

0005-1098(93)E0034-2

Automatica, Vol. 30, No. 9, pp. 1415–1419, 1994

0005-1098/94 \$7.00 + 0.00

Brief Paper

Application of Averaging Method for Integro-differential Equations to Model Reference Adaptive Control of Parabolic Systems*

KEUM-SHIK HONG† and JOSEPH BENTSMAN‡

Key Words—Model reference adaptive control; averaging method; partial differential equation; convergence analysis; stability.

Abstract—An averaging theorem for integro-differential equations is applied to the convergence analysis of controller parameters of a model reference adaptive control (MRAC) algorithm for a class of parabolic partial differential equations (PDEs) with constant coefficients. The stability of an adaptive control algorithm is proven as well.

1. *Introduction*

THIS PAPER PRESENTS an application of an averaging theorem for nonlinear integro-differential equations to a model reference adaptive control (MRAC) algorithm for linear one-dimensional, parabolic partial differential equations (PDEs). The method of averaging is an asymptotic method which permits the analysis of the dynamic behavior of a nonautonomous system via an autonomous (averaged) system obtained by time-averaging of the original nonautonomous system. Since the first systematic averaging analysis for systems of ordinary differential equations (ODEs) in the standard form was introduced by Bogoliubov and Mitropolsky (1961), this method has been extended to various equations including functional, integro-differential, difference and PDEs. Recently, this method has emerged as a creditable tool for stability analysis in vibrational control and adaptive control (Bellman *et al.*, 1986; Bentsman and Hong, 1991; Bentsman *et al.*, 1991; Hong and Bentsman, 1992). An averaging theorem which will be introduced in Section 2 establishes the asymptotic stability of an attractor of a nonautonomous nonlinear integro-differential equation on the basis of the exponential stability of the corresponding attractor of the autonomous (averaged) nonlinear differential equation.

The adaptive control problem addressed in Section 3 consists of designing a control strategy that achieves the desired objective: either regulation or tracking, for a given class of plants whose structure is known but parameters in the structure are unknown. It is assumed that the unknown parameters in the system equation are constant, and that distributed sensing and actuation are available. The adaptive control problem involves (i) the construction of a control law which is adjusted adaptively using available input and output data from the plant in such a way that the desired control objective is achieved, and (ii) assurance of the global stability of the system when the parameters are tuned according to the adaptation laws.

Besides providing a natural extension of the Lyapunov redesign method of finite dimensional adaptive control (Parks, 1966) to parabolic PDEs, the paper makes the following contributions. (i) The proof for the analysis of the parameter convergence in the finite dimensional MRAC (Anderson *et al.*, 1986; Sastry and Bodson, 1989) is extended to PDEs. This extension consists of representing the closed loop dynamics in the form of an integro-differential equation via the solution of the infinite dimensional error equation. This representation, then, permits the use of the averaging theorem for integro-differential equations to establish the convergence of the controller parameters to their nominal values. (ii) It is demonstrated that the concept of persistency of excitation in the adaptive control of PDEs arises not only with respect to the time variable as in the finite dimensional case but also with respect to the spatial variable, and it is shown that even a constant input can be persistently exciting in MRAC of distributed parameter systems in the sense of ensuring the convergence of the controller parameters to their nominal values.

The paper has the following structure. In Section 2 we introduce an averaging theorem. In Section 3 we consider an adaptive control algorithm for a linear constant coefficient parabolic plant, and show the convergence of a state error to zero and the boundedness of all signals in the closed loop. In Section 4 we use the averaging theorem of Section 2 to demonstrate the convergence of the parameters of the adaptive controller to zero. Conclusions are given in Section 5.

2. *Averaging method for integro-differential equations*

Extensions of Bogoliubov's averaging theorems (Bogoliubov and Mitropolsky, 1961) for systems of ODEs in the standard form to integro-differential equations can be found in several papers. Filatov (1967) first extended the results of the closeness of a solution of the original system to that of the averaged system on a finite, but arbitrarily large, time interval $t \in [0, L/\varepsilon]$, ε, $L > 0$, for the nonlinear integro-differential equation in the following standard form:

$$\dot{x}(t) = \varepsilon X\left(t, x(t), \int_0^t \phi(t, s, x(s))\, ds\right), \quad x(0) = x_0.$$

Subsequently, various forms of averaging theorems (Filatov and Talipova, 1969, 1970; Bakhabov, 1969; Melikidze, 1970;

* Received 1 April 1992; revised 22 December 1992; revised 9 June 1993; received in final form 27 July 1993. The original version of this paper was presented at the 4th IFAC Symposium on Adaptive Systems in Control and Signal Processing which was held in Juillet, France, during 1–3 July 1992. The published Proceedings of this IFAC Meeting may be ordered from Elsevier Science Limited, The Boulevard, Langford Lane, Kidlington, Oxford OX5 1GB, U.K. This paper was recommended for publication in revised form by Associate Editor I. M. Y. Mareels under the direction of Editor C. C. Hang. Corresponding author Professor Keum-Shik Hong. Tel. +82 51 510 2454; Fax: +82 51 512 9835; E-mail: kshong@hyowon.pusan.ac.kr.

† Department of Control and Mechanical Engineering, Pusan National University, Institute of Mechanical Technology, 30 Changjeon-dong, Kumjeong-ku, Pusan, 609-735, Korea.

‡ Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, 1206 W. Green Street, Urbana, IL 61801, U.S.A.

Eshmatov, 1974; Bainov and Sarafova, 1977) for integro-differential equations appeared. Bakhabov (1969) gave a different proof of the results of Filatov (1967), and Filatov and Talipova (1970) proved the proximity of the solutions of the original system and the corresponding averaged system both on a finite and an infinite time interval; however, they did not provide the results on the existence, uniqueness and the exponential stability of an attractor of the time varying system in the vicinity of that of the averaged system. In this section an averaging theorem of Melikidze (1970), which deduced the stability of the original system from that of the corresponding averaged system for a nonlinear system of integro-differential equations in the standard form, is presented.

Consider a system of nonlinear integro-differential equations of the form

$$\dot{x}(t) = \varepsilon X\left(t, x(t), \dot{x}(t), \int_0^t \phi(t, s, x(s), \dot{x}(s))\,ds\right), \quad x(0) = x_0, \tag{1}$$

where x, X, ϕ are n-vector valued functions, $\varepsilon > 0$ is a small parameter, t denotes the time and a dot denotes the derivative with respect to t. System (1) appears in the problems of the dynamics of imperfectly elastic bodies in which derivatives cannot be explicitly solved. $X(t, x, y, z)$ is a vector which is continuous in the domain $Q = \{t \in R, x, y \in D \subset R^n, z \in R^n\}$, where D is assumed to be compact and $0 \in D$. The function $\phi(t, s, x, y)$ is also assumed to be continuous in the domain $Q' = \{(t, s) \in R^2, x, y \in D \subset R^n\}$. Let the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} X\left(t, x, y, \int_0^t \phi(t, s, x, y)\,ds\right) dt \triangleq X_0(x, y) \tag{2}$$

exist uniformly with respect to $(t_0, x, y) \in R \times D$. Notice that the variables x, y in the limiting process of (2) are treated as parameters independent of s. Then the averaged system corresponding to (1) is defined as

$$\dot{\xi} = \varepsilon X_0(\xi, \dot{\xi}), \quad \xi(0) = x_0. \tag{3}$$

Theorem 1 (Melikidze, 1970). Let $X(t, x, y, z)$ and $\phi(t, s, x, y)$ be defined and continuous in the domains Q and Q', respectively. Assume that the following conditions are satisfied. (1) In the domain Q, the function $X(t, x, y, z)$ is bounded, and satisfies the Lipschitz condition, that is,

$$\|X(t, x, y, z)\| \le M,$$
$$\|X(t, x_1, y_1, z_1) - X(t, x_2, y_2, z_2)\| \le \lambda_1\{\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|\}, \tag{4}$$

where M, λ_1 are positive constants. (2) In the domain Q', the function $\phi(t, s, x, y)$ is bounded and satisfies the Lipschitz condition, that is,

$$\|\phi(t, s, x, y)\| \le N,$$
$$\|\phi(t, s, x_1, y_1) - \phi(t, s, x_2, y_2)\| \le \lambda_2\{\|x_1 - x_2\| + \|y_1 - y_2\|\}, \tag{5}$$

where λ_2 is a positive constant. (3) The limit (2) exists uniformly with respect to $(t_0, x, y) \in R \times D$, and equation (3) has a periodic solution such that

$$x = \xi(\omega\tau), \quad \xi(\alpha + 2\pi) = \xi(\alpha). \tag{6}$$

(4) The real parts of all the $(n - 1)$ characteristic exponents of the perturbation equation

$$\frac{d\delta\xi}{d\tau} = X_0'\left(\xi(\omega\tau), \frac{d\xi}{dt}\right)\delta\xi \tag{7}$$

corresponding to the periodic solution (6) are different from zero. (5) It is possible to find some ρ-neighborhood U_ρ of the orbit of this periodic solution, such that the function $X(t, x, y, z)$ and its partial derivatives with respect to x, y, z up to $(m + 2)$th order are bounded and uniformly continuous with respect to x, y in the domain $\{t \in R, x, y \in U_\rho \subset D\}$. (6) $X(t, x, y, z)$ is an almost periodic function of t, uniformly with respect to $x, y \in U_\rho$. Then there exist positive numbers ε_0 and σ_0 such that for any $\varepsilon < \varepsilon_0$ the following assertions are valid: (i) (1) has a unique integral manifold S_t which belongs to the domain U_ρ for every t; (ii) S_t admits a parametric representation of the form $x = F(t, \theta)$, where $F(t, \theta)$ is defined for every real t and θ, and has period 2π with respect to the angle θ. In addition, it is possible to find functions $\delta(\varepsilon)$, $\eta(\varepsilon) \to 0$ for $\varepsilon \to 0$ such that $\|F(t, \theta) - \xi(\theta)\| \le \delta(\varepsilon)$, and $\|F(t, \theta_1) - F(t, \theta_2)\| \le \eta(\varepsilon)|\theta_1 - \theta_2|$ for any real t, θ_1, θ_2. It is possible to construct a function $G(t, \theta)$ defined for every real t, θ satisfying the inequalities $\|G(t, \theta)\| \le \delta^*(\varepsilon)$, $\|G(t, \theta_1) - G(t, \theta_2)\| \le \eta^*(\varepsilon)|\theta_1 - \theta_2|$, where $\delta^*(\varepsilon)$, $\eta^*(\varepsilon) \to 0$ for $\varepsilon \to 0$ such that any solution of (1) belonging to the manifold S_t is representable in the form $x = F(t, \theta(t))$, where $\theta(t)$ is some solution to the equation $d\theta/dt = \varepsilon G(t, \theta)$. Furthermore, let that $x(t)$ be any solution of (1) which satisfies the relation for some t_0 as $x(t_0) \in U_{\sigma_0}$, $\sigma_0 < \rho$, where U_{σ_0} is the σ_0-neighborhood of the orbit of (6). Then if the real part of each of the $(n - 1)$ characteristic exponents of the system (3) is negative, the distance $d(x(t), S_t)$ from the point to the set S_t tends exponentially to zero as $t \to \infty$.

3. *Direct adaptive control of parabolic systems*

In this section we derive and analyze an adaptive control algorithm for a class of DPSs described by linear, one-dimensional, parabolic PDEs with unknown constant coefficients. As in adaptive control of finite dimensional systems, we will focus on MRAC under the assumption that the structure of the plant is known and only parameters in the system equation (not in the boundary conditions) are unknown.

Consider a DPS described by a linear parabolic PDE with constant coefficients

$$u_t(x, t) = a u_{xx}(x, t) + b u(x, t) + f(x, t), \quad u(0, t) = \beta_1(t),$$
$$u(1, t) = \beta_2(t), \quad u(x, 0) = u_0(x), \tag{8}$$

where $x \in [0, 1]$, $t > 0$, a and b are constants, subscripts t and x stand for partial derivatives with respect to t and x, respectively, and $f(x, t)$ is a control input function. The output y of (8) in general is given by $y(x, t) = Hu(x, t)$, where $H: C([0, 1] \times R^+) \to C((\Omega \subset [0, 1]) \times R^+)$ is a linear bounded time-invariant operator with the form depending on the characteristics of the sensor. The following assumptions are made.

Assumptions. (i) The structure of the plant is *a priori* known. (ii) Boundary conditions are *a priori* known, and $\beta_1(\cdot), \beta_2(\cdot) \in C^\infty[0, \infty)$. (iii) Distributed sensing and actuation are available, and the observation operator H is *a priori* known (we may assume that $H = I$, where I denotes the identity operator from $C([0, 1] \times R^+)$ onto itself). (iv) Coefficients a and b are unknown; however, $a > 0$ (due to parabolicity).

Now the reference model can be introduced as

$$\hat{u}_t(x, t) = \hat{a}\hat{u}_{xx}(x, t) + \hat{b}\hat{u}(x, t) + r(x, t), \quad \hat{u}(0, t) = \beta_1(t),$$
$$\hat{u}(1, t) = \beta_2(t), \quad \hat{u}(x, 0) = \hat{u}_0, \tag{9}$$

where a circumflex indicates variables and parameters related to the reference model, and $r(x, t)$ is the reference input, which is analytic on $[0, 1] \times [0, \infty)$. It is assumed that $\hat{a} > 0$, $\hat{b} < 0$. It is known that if $r(\cdot, \cdot)$ is analytic in $[0, 1] \times [0, \infty)$, then the solution of (9) is analytic in $[0, 1] \times \{0 < t < T < \infty\}$ (Friedman, 1969, p. 212). The control objective of the MRAC can now be stated as follows: find a bounded control signal f that drives u to $\hat{u}$ asymptotically and keeps all signals in the closed loop uniformly bounded.

Now consider the following control law with adjustable controller parameters $\phi_a(t)$ and $\phi_b(t)$ such that

$$f(x, t) = \phi_a(t)u_{xx}(x, t) + \phi_b(t)u(x, t) + r(x, t), \tag{10}$$

$$\dot{\phi}_a(t) = \varepsilon\langle e_x(x, t), e_x(x, t) + \hat{u}_x(x, t)\rangle, \quad \phi_a(0) = \phi_{a0} > 0, \tag{11a}$$

$$\dot{\phi}_b(t) = -\varepsilon\langle e(x, t), e(x, t) + \hat{u}(x, t)\rangle, \quad \phi_b(0) = \phi_{b0}, \tag{11b}$$

where $e(x, t) \triangleq u(x, t) - \hat{u}(x, t)$, the dot represents derivative with respect to time, and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(0, 1)$, which is the space of square integrable functions on $[0, 1]$.

Theorem 2. Let the parabolic plant (8) satisfy the above assumptions, and the reference model be given by (9). Let the feedback control law f be given as (10) with tuning laws (11a) and (11b). Then all the signals in the closed loop system are bounded, and $\|e(x, t)\| \to 0$ as $t \to \infty$.

We first introduce the following Lemmas to be used subsequently for proving Theorem 2.

Lemma 1 (Popov, 1973, p. 211). If $f(t): R^+ \to R$ is uniformly continuous for $t \geq 0$, and $\lim_{t\to\infty} \int_0^t |f(\tau)|\,d\tau$ exists and is finite, then $f(t) \to 0$ as $t \to \infty$.

The next Lemma is an extension of Lemma 1 to functions more than one independent variables.

Lemma 2. If $e(x, t): (\Omega \subseteq R^n) \times R^+ \to R$ is bounded, $\{e(x, t)\}_{x\in\Omega}$ is equicontinuous in t, and $\lim_{t\to\infty} \int_0^t \|e(x, \tau)\|^2_{L^2(\Omega)}\,d\tau$ exists and is finite, then $\lim_{t\to\infty} \|e(x, t)\|_{L^2(\Omega)} = 0$.

Proof. Let $\int_\Omega dx = A < \infty$ and the bound for $e(x, t)$ be M. From the equicontinuity for any $\varepsilon < 0$ there exists $\delta(\varepsilon) > 0$ such that whenever $|t_1 - t_2| < \delta$ for every $x \in \Omega$ we have $|e(x, t_1) - e(x, t_2)| < \varepsilon/2AM$. Now,

$$|e^2(x, t_1) - e^2(x, t_2)| = |e(x, t_1) + e(x, t_2)|\,|e(x, t_1) - e(x, t_2)| \leq \varepsilon/A.$$

Therefore $\{e^2(x, t)\}_{x\in\Omega}$ is equicontinuous in t. Define $f(t) \triangleq \|e(x, t)\|^2$. Then

$$|f(t_1) - f(t_2)| = \big|\|e(x, t_1)\|^2 - \|e(x, t_2)\|^2\big| \leq \int_\Omega |e^2(x, t_1) - e^2(x, t_2)|\,dx \leq \varepsilon.$$

Hence $f(t)$ is uniformly continuous. Since $f(t)$ satisfies both hypotheses in Lemma 1, $\|e\| \to 0$ as $t \to \infty$.

Corollary 1. If $e(x, t) \in L^2(\Omega \times R^+) \cap L^\infty(\Omega \times R^+)$, and $e_1(x, t)$ is bounded, then $\|e(x, t)\| \to 0$ as $t \to \infty$.

Proof. The proof follows directly from Lemma 2.

Proof of Theorem 2. Define the nominal values of the controller parameters as $\phi_a^* \triangleq (\hat{a} - a)$, $\phi_b^* \triangleq (\hat{b} - b)$, and controller parameter errors as

$$\psi_a(t) \triangleq \phi_a(t) - \phi_a^*, \quad \psi_b(t) \triangleq \phi_b(t) - \phi_b^*. \tag{12}$$

Note that when $\phi_a(t)$ and $\phi_b(t)$ in (11a) and (11b) converge to ϕ_a^* and ϕ_b^*, respectively, the closed loop equation of the plant with the control law (10) matches the reference model equation exactly, that is,

$$u_t = (a + \phi_a(t))u_{xx} + (b + \phi_b(t))u + r \tag{13}$$

becomes the exact reference model when $\phi_a(t) = \phi_a^*$ and $\phi_b(t) = \phi_b^*$. Also note that $\dot{\phi}_a(t) = \dot{\psi}_a(t)$ and $\dot{\phi}_b(t) = \dot{\psi}_b(t)$ from (12). Subtracting (9) from (13) yields the state error equation as

$$e_t = (\hat{a} + \psi_a(t))e_{xx} + (\hat{b} + \psi_b(t))e + \psi_a(t)\hat{u}_{xx} + \psi_b(t)\hat{u}, \tag{14}$$

becomes the exact reference model when $\psi_a(t) = \psi_a$ and
$\phi_b(t) = \phi_b^*$. Also note that $\dot{\phi}_a(t) = \dot{\psi}_a(t)$ and $\dot{\phi}_b(t) = \dot{\psi}_b(t)$
Now consider a Lyapunov functional as

$$V(e, \psi_a, \psi_b) = \frac{1}{2}\int_0^1 e^2(x, t)\,dx + \frac{1}{2\varepsilon}(\psi_a^2(t) + \psi_b^2(t)). \tag{15}$$

Differentiating V with respect to t along the trajectories of (14) employing integration by parts, using boundary conditions, and utilizing (11a) and (11b) yields

$$\dot{V} \leq \int_0^1 (-\hat{a}e_x^2 + \hat{b}e^2)\,dx \leq \hat{b}\int_0^1 e^2\,dx \leq 0. \tag{16}$$

Since $V(e, \psi_a, \psi_b)$ is nonincreasing and bounded below, e, ψ_a and ψ_b are bounded with respect to their norms. Also, from $\dot{V} \leq \hat{b}\,\|e\|^2$, we have $e \in L^2([0, 1] \times [0, \infty))$. Furthermore, from the first inequality in (16), $e_x \in L^2([0, 1] \times [0, \infty))$. Therefore $e \in L^\infty([0, 1] \times [0, \infty))$, which is seen from the linear structure of (14) with homogeneous Dirichlet boundary condition. Since $\hat{u}$ is bounded for bounded input r, so is u. ϕ_a and ϕ_b are all bounded from (12) due to the boundedness of ψ_a and ψ_b. Finally, since the separation of variables holds for (14) and all the derivatives appearing in the right-hand side of (14) are with respect to the spatial variable, $\partial e(x, t)/\partial t$ is bounded. Hence from Corollary 1 it follows that $\|e\| \to 0$ as $t \to \infty$. Q.E.D.

Remark. Equations (14) and (11a) and (11b) represent the overall adaptive system, where $\hat{u}$ is an exogenous signal. By substituting (11a) and (11b) into (14) the state error system (14) has the form

$$e_t(x, t) = \alpha(e_x, t)e_{xx}(x, t) + g(e, e_x, t), \tag{17}$$

where

$$\alpha(e_x, t) = \hat{a} + \psi_{a0} + \varepsilon\int_0^t\int_0^1 (e_x e_x + e_x \hat{u}_x)\,dx\,dt, \tag{18}$$

$$g(e, e_x, t) = \hat{b}e + \left(\psi_{b0} - \varepsilon\int_0^t\int_0^1 e(e + \hat{u})\,dx\,dt\right)(e + \hat{u}) + \left(\psi_{a0} + \varepsilon\int_0^t\int_0^1 e_x(e_x + \hat{u}_x)\,dx\,dt\right)\hat{u}_{xx}. \tag{19}$$

The initial condition of (11a) needs to be chosen such that $\hat{a} + \psi_{a0} > 0$. Since the exogeneous signal $\hat{u}$ is smooth, there exists a $t_0 > 0$ such that the principal coefficient $\alpha(e_x, t)$ is strongly elliptic for all $t \in [0, t_0]$. Therefore (17) is parabolic (Friedman, 1969, p. 134). Hence, the results of Friedman (1969, pp. 169–181) are applicable for the existence of a unique solution of (17) for $t \in [0, t_0]$. Specifically, the A_0 on p. 169 of Friedman (1969) is $(\hat{a} + \psi_{a0})\,\partial^2/\partial x^2$, and satisfaction of the conditions F2–F4 of Friedman (1969, pp. 169–170) is easily seen by choosing those α, σ, ρ on p. 170 of Friedman (1969) as $\alpha = 1/2$ and $\sigma = \rho = 1$ in our case. Finally, the Lyapunov function defined as in (15) ensures that all solutions belong to a closed bounded set; their existence for all $t \geq 0$ guaranteed as well.

4. *Analysis of parameter convergence*

In this section, we demonstrate the convergence of the controller parameters to their nominal values using the results on averaging in Section 2. A new finding from the analysis given below is that unlike the finite dimensional case, even the constant input can be persistently exciting in the infinite dimensional setting in the sense of ensuring convergence of the controller parameters to their nominal values, and that the concept of the persistency of excitation arises in DPS not only with respect to the time variable but also the spatial variable.

Following Anderson *et al.* (1986), our analysis utilizes the linearization. The difference in comparison to Anderson *et al.* (1986) is in using the explicit solution of the parabolic PDE which represents a linearized state error equation in the form of an integro-differential equation and then applying averaging for integro-differential systems. The linearized error equations (Frechet differentials) of (11a) and (11b) and (14) around zero are

$$\dot{\psi}_a(t) = -\varepsilon\langle e(x, t), \hat{u}_{xx}(x, t)\rangle, \quad \psi_a(0) = \psi_{a0}, \tag{20}$$

of an integro-differential equation and then applying

$$e_t(x, t) = \hat{a}e_{xx}(x, t) + \hat{b}e(x, t) + \psi_a(t)\hat{u}_{xx}(x, t) + \psi_b(t)\hat{u}(x, t), \tag{22}$$

$$e(0, t) = e(1, t) = 0, \quad e(x, 0) = u_0 - \hat{u}_0.$$

Theorem 3. Consider the parabolic plant (8) with homogeneous boundary condition and the reference model (9). Let the feedback control law f be given as (10) with tuning laws (20) and (21). Let the reference input r be of the form (i) $r = \phi(x)$, and (ii) $r = \phi(x)\sin(t)$. Then $|\psi_a(t)|$, $|\psi_b(t)| \to 0$ exponentially as $t \to \infty$, if $\phi(x) \neq 0$ on at least one interval of nonzero measure in x for all t.

Proof. Noting that (8), (9) and (22) have the same form, the solution of (9) with homogeneous boundary conditions is given as

$$\hat{u}(x,t)=\sum_{n=1}^{\infty}\hat{u}_{0n}e^{-k_n t}\varphi_n(x)+\sum_{n=1}^{\infty}\left[\int_0^t e^{-k_n(t-\tau)}r_n(\tau)\,d\tau\right]\varphi_n(x),\quad n=1,2,\ldots,\tag{23}$$

where $k_n=\hat{a}(n\pi)^2-\hat{b}$, and $\varphi_n(x)=\sin(n\pi x)$, $\hat{u}_{0n}=2\langle\hat{u}_0(x),\varphi_n(x)\rangle$, $r_n(t)=2\langle r(x,t),\varphi_n(x)\rangle$. The first and second series in (23) reflects the influence of the initial state $\hat{u}_0(x)$ and the forcing term r, respectively.

(i) $r=\phi(x)$. Equation (23) with $r=\phi(x)$ reduces to

$$\hat{u}(x,t)=\sum_{n=1}^{\infty}\hat{u}_{0n}e^{-k_n t}\varphi_n(x)+\sum_{n=1}^{\infty}\frac{\phi_n}{k_n}(1-e^{-k_n t})\varphi_n(x),\tag{24}$$

where $\phi_n=2\langle\phi(x),\varphi_n(x)\rangle$. Similarly, the solution of (22) has the form

$$e(x,t)=\sum_{n=1}^{\infty}(u_{0n}-\hat{u}_{0n})e^{-k_n t}\varphi_n(x)+\sum_{n=1}^{\infty}\left[\int_0^t e^{-k_n(t-\tau)}F_n(\tau)\,d\tau\right]\varphi_n(x),\tag{25}$$

where

$$F_n(t)=2\langle\psi_a\hat{u}_{xx}+\psi_b\hat{u},\varphi_n(x)\rangle=-\frac{\phi_n(n\pi)^2}{k_n}(1-e^{-k_n t})\psi_a(t)+\frac{\phi_n}{k_n}(1-e^{-k_n t})\psi_b(t).\tag{26}$$

By substituting (24) and (25) into (20) and (21), respectively, the adaptation law becomes

$$\begin{aligned}\dot{\psi}_a(t)=&\ \varepsilon\sum_{n=1}^{\infty}\tfrac{1}{2}\hat{u}_{0n}(u_{0n}-\hat{u}_{0n})(n\pi)^2e^{-2k_n t}\\&+\varepsilon\sum_{n=1}^{\infty}\frac{(u_{0n}-\hat{u}_{0n})\phi_n(n\pi)^2}{2k_n}e^{-k_n t}(1-e^{-k_n t})\\&-\varepsilon\sum_{n=1}^{\infty}\frac{\hat{u}_{0n}\phi_n(n\pi)^4}{2k_n}e^{-k_n t}\int_0^t e^{-k_n(t-\tau)}(1-e^{-k_n\tau})\psi_a(\tau)\,d\tau\\&+\varepsilon\sum_{n=1}^{\infty}\frac{\hat{u}_{0n}\phi_n(n\pi)^2}{2k_n}e^{-k_n t}\int_0^t e^{-k_n(t-\tau)}(1-e^{-k_n\tau})\psi_b(\tau)\,d\tau\\&-\varepsilon\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^4}{2k_n^2}(1-e^{-k_n t})\int_0^t e^{-k_n(t-\tau)}(1-e^{-k_n\tau})\psi_a(\tau)\,d\tau\\&+\varepsilon\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^2}{2k_n^2}(1-e^{-k_n t})\int_0^t e^{-k_n(t-\tau)}(1-e^{-k_n\tau})\psi_b(\tau)\,d\tau\end{aligned}\tag{27a}$$

$$\begin{aligned}\dot{\psi}_b(t)=&-\varepsilon\sum_{n=1}^{\infty}\tfrac{1}{2}\hat{u}_{0n}(u_{0n}-\hat{u}_{0n})e^{-2k_n t}\\&-\varepsilon\sum_{n=1}^{\infty}\frac{(u_{0n}-\hat{u}_{0n})\phi_n}{2k_n}e^{-k_n t}(1-e^{-k_n t})\\&+\varepsilon\sum_{n=1}^{\infty}\frac{\hat{u}_{0n}\phi_n(n\pi)^2}{2k_n}e^{-k_n t}\int_0^t e^{-k_n(t-\tau)}(1-e^{-k_n\tau})\psi_a(\tau)\,d\tau\\&-\varepsilon\sum_{n=1}^{\infty}\frac{\hat{u}_{0n}\phi_n}{2k_n}e^{-k_n t}\int_0^t e^{-k_n(t-\tau)}(1-e^{-k_n\tau})\psi_b(\tau)\,d\tau\\&+\varepsilon\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^2}{2k_n^2}(1-e^{-k_n t})\int_0^t e^{-k_n(t-\tau)}(1-e^{-k_n\tau})\psi_a(\tau)\,d\tau\\&-\varepsilon\sum_{n=1}^{\infty}\frac{\phi_n^2}{2k_n^2}(1-e^{-k_n t})\int_0^t e^{-k_n(t-\tau)}(1-e^{-k_n\tau})\psi_b(\tau)\,d\tau.\end{aligned}\tag{27b}$$

Noting that (27a) and (27b) is of the form of an integro-differential equation (1), it is easily seen that assumptions (4) and (5) of Theorem 1 are satisfied due to the linearity of the equations. Furthermore, $X(t,0,0,0)=0$, and $\phi(t,s,0,0)=0$. Averaging the right-hand side of (27a) and (27b), that is,

$$\lim_{T\to\infty}\frac{1}{T}\int_\tau^{\tau+T}\begin{bmatrix}\text{R.H.S. of (27a)}\\\text{R.H.S. of (27b)}\end{bmatrix}dt,\tag{28}$$

where the limit exist uniformly, yields an averaged system such that

$$\frac{d}{dt}\begin{bmatrix}\xi_a(t)\\\xi_b(t)\end{bmatrix}=-\varepsilon\begin{bmatrix}\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^4}{2k_n^3} & -\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^2}{2k_n^3}\\-\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^2}{2k_n^3} & \sum_{n=1}^{\infty}\frac{\phi_n^2}{2k_n^3}\end{bmatrix}\begin{bmatrix}\xi_a(t)\\\xi_b(t)\end{bmatrix}\equiv\varepsilon A\xi(t),$$
$$\begin{bmatrix}\xi_a(0)\\\xi_b(0)\end{bmatrix}=\begin{bmatrix}\psi_{a0}\\\psi_{b0}\end{bmatrix},\tag{29}$$

where ϕ_n is the Fourier coefficient of $\phi(x)$ and $k_n=\hat{a}(n\pi)^2-\hat{b}$. Note that $\operatorname{tr}A<0$, and

$$\det A=\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^4}{2k_n^3}\sum_{n=1}^{\infty}\frac{\phi_n^2}{2k_n^3}-\sum_{n=1}^{\infty}\frac{\phi_n(n\pi)^2}{(2k_n^3)^{1/2}}\frac{\phi_n}{(2k_n^3)^{1/2}}\times\sum_{n=1}^{\infty}\frac{\phi_n(n\pi)^2}{(2k_n^3)^{1/2}}\frac{\phi_n}{(2k_n^3)^{1/2}}>0.\tag{30}$$

The strict inequality in (30) is achieved by applying to the second term of (30) the Cauchy–Schwarz inequality (Royden, 1988) such that $\sum_{i=0}^{\infty}|x_iy_i|\le\left(\sum_{i=0}^{\infty}|x_i|^2\right)^{1/2}\left(\sum_{i=0}^{\infty}|y_i|^2\right)^{1/2}$, where the equality is achieved only when $x=\lambda y$ for some $\lambda\ge0$. Therefore, (29) is exponentially stable if $\phi(x)\ne0$ on at least one interval of nonzero measure. Further, we note that the periodic solution (6) mentioned in Theorem 1 is in fact the trivial solution of (29), and that the corresponding solution of the original system (27a) and (27b) is its trivial solution as well. Hence by applying Theorem 1 to (27a) and (27b) the parameter errors $\psi_a(t)$, $\psi_b(t)$ converge to zero exponentially as long as there exists at least one $\phi_n\ne0$, which is one of the Fourier coefficients of $\phi(x)$, which is the case if $\phi(x)\ne0$ on at least one interval of nonzero measure. Now, due to the almost periodicity of (27a) and (27b) the trivial solution $\psi_a(t)=\psi_b(t)=0$ of (27a) and (27b) is uniformly exponentially stable. Combined with the results of Theorem 2, this implies that the zero equilibrium of (11a) and (11b) and (14) is uniformly asymptotically stable, and that there is a neighborhood of zero equilibrium where both $\psi_a(t)$ and $\psi_b(t)$ have exponential convergence to zero.

(ii) $r=\phi(x)\sin(t)$. First we note that the first term on the right-hand side of (23), which tends to zero exponentially, did not affect the form of (29) obtained through averaging. Therefore, we choose all initial conditions to be zero. Through a similar development as in case (i) we finally obtain the adaptation law as

$$\begin{aligned}\dot{\psi}_a(t)=&-\varepsilon\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^4}{2(1+k_n^2)^2}(k_n\sin t-\cos t+e^{-k_n t})\times\int_0^t e^{-k_n(t-\tau)}(k_n\sin\tau-\cos\tau+e^{-k_n\tau})\psi_a(\tau)\,d\tau\\&+\varepsilon\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^2}{2(1+k_n^2)^2}(k_n\sin t-\cos t+e^{-k_n t})\times\int_0^t e^{-k_n(t-\tau)}(k_n\sin\tau-\cos\tau+e^{-k_n\tau})\psi_b(\tau)\,d\tau\end{aligned}\tag{31a}$$

$$\begin{aligned}\dot{\psi}_b(t)=&\ \varepsilon\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^2}{2(1+k_n^2)^2}(k_n\sin t-\cos t+e^{-k_n t})\times\int_0^t e^{-k_n(t-\tau)}(k_n\sin\tau-\cos\tau+e^{-k_n\tau})\psi_a(\tau)\,d\tau\\&-\varepsilon\sum_{n=1}^{\infty}\frac{\phi_n^2}{2(1+k_n^2)^2}(k_n\sin t-\cos t+e^{-k_n t})\times\int_0^t e^{-k_n(t-\tau)}(k_n\sin\tau-\cos\tau+e^{-k_n\tau})\psi_b(\tau)\,d\tau.\end{aligned}\tag{31b}$$

Therefore, the averaged system corresponding to the above integro-differential equation becomes

$$\frac{d}{dt}\begin{bmatrix}\xi_a(t)\\ \xi_b(t)\end{bmatrix} = -\varepsilon\begin{bmatrix}\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^4}{4(1+k_n^2)^2} & -\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^2}{4(1+k_n^2)^2}\\ -\sum_{n=1}^{\infty}\frac{\phi_n^2(n\pi)^2}{4(1+k_n^2)^2} & \sum_{n=1}^{\infty}\frac{\phi_n^2}{4(1+k_n^2)^2}\end{bmatrix}\begin{bmatrix}\xi_a(t)\\ \xi_b(t)\end{bmatrix} \equiv \varepsilon B\xi(t),$$

$$\begin{bmatrix}\xi_a(0)\\ \xi_b(0)\end{bmatrix} = \begin{bmatrix}\psi_{a0}\\ \psi_{b0}\end{bmatrix}, \quad (32)$$

where $\operatorname{tr} B < 0$, $\det B > 0$. Consequently, again by Theorem 1, the parameter errors $\psi_a(t)$, $\psi_b(t)$ converge to zero exponentially if at least one $\phi_n \neq 0$, that is, if $\phi(x) \neq 0$ for at least one interval of nonzero measure. Therefore, considerations similar to those at the end of the proof of part (i) complete the proof.

Remark. Although the same conclusions may be arrived at by using the theorem of Filatov and Talipova (1970), we have chosen to quote the theorem of Melikidze (1970) due to its wider applicability and the explicit statements regarding the existence, uniqueness and the exponential stability of an attractor of the time-varying system.

5. *Conclusions*

In this paper we considered an application of an averaging theorem for integro-differential equations to the convergence analysis of parameters of an MRAC algorithm for parabolic systems with constant coefficients. The stability of an adaptive algorithm for a specific control law obtained through the Lyapunov redesign method in Section 3 was proved under the assumption that distributed measurement and control were possible. The convergence analysis of controller parameters for a constant coefficient parabolic system in Section 4 shows that if the reference input is not zero for at least one interval in x of nonzero measure, in which case at least one of the Fourier coefficients of the reference input is not zero, then the controller parameters converge to their nominal values. The averaging theorem given in Section 2 is applicable to a much more general class of equations than considered here, and therefore it is likely to permit extensions of the results presented in this paper to a broader class of systems.

Acknowledgements—The first author is supported by the Korea Science and Engineering Foundation under KOSEF grant 941-1000-012-2. The second author was supported in part by NSF grant MSS 89-57198 PYI and in part by EPRI contract EPRI RP-8010-19.

References

Anderson, B. D. O., R. R. Bitmead, C. R. Johnson, Jr, P. V. Kokotovic, R. L. Kosut, I. M. Y. Mareels, L. Praly and B. D. Ridele (1986). *Stability of Adaptive Systems: Passivity and Averaging Analysis.* MIT Press, Cambridge, MA.

Bainov, D. D. and G. H. Sarafova (1977). On the application of the averaging method for nonlinear systems of integro differential equations. *Publ. RIMS, Kyoto Univ.*, **13**, 583–588.

Bakhabov, G. (1969). On the problem of the validity of a method of averaging for integro-differential equations. *Ukrainian Math. J.*, **21**, 675–678.

Bellman, R. E., J. Bentsman and S. M. Meerkov (1986). Vibrational control of nonlinear systems: vibrational stabilizability. *IEEE Trans. Automat. Contr.*, **AC-31**, 710–716.

Bentsman, J. and K. S. Hong (1991). Vibrational stabilization of nonlinear parabolic systems. *IEEE Trans. Automat. Contr.*, **AC-36**, 501–507.

Bentsman, J., K. S. Hong and J. Fakhfakh (1991). Vibrational control of nonlinear time lag systems: vibrational stabilization and transient behavior. *Automatica*, **27**, 491–500.

Bogoliubov, N. N. and Y. A. Mitropolsky (1961). *Asymptotic Methods in the Theory of Nonlinear Oscillators.* Gordon and Breach, New York.

Eshmatov, Kh. (1974). Averaging in some systems of integro-differential equations. *Differential Equations*, **10**, 868–872.

Filatov, A. N. (1967). Averaging in systems of integro-differential equations. *Differential Equations*, **3**, 896–899.

Filatov, A. N. and L. I. Talipova (1969). An averaging variant in systems of integro-differential equations. *Differential Equations*, **5**, 701–704.

Filatov, A. N. and L. I. Talipova (1970). On a variant of averaging in integro-differential equations. *Soviet Math. Dokl.*, **11**, 737–739.

Friedman, A. (1969). *Partial Differential Equations.* Holt, Reinhart and Winston, New York.

Hong, K. S. and J. Bentsman (1992). Stability criterion for linear oscillatory parabolic systems. *ASME J. Dyn. Syst. Measurement Contr.*, **114**, 175–178.

Melikidze, T. V. (1970). On the averaging of systems of nonlinear integro-differential equations on the infinite line. *Ukrainian Math. J.*, **22**, 598–600.

Parks, P. C. (1966). Lyapunov redesign of model reference adaptive control systems. *IEEE Trans. Automat. Contr.*, **AC-11**, 362–367.

Popov, V. M. (1973). *Hyperstability of Control Systems.* Springer, New York.

Royden, H. L. (1988). *Real Analysis* (3rd Edn). Macmillan, New York.

Sastry, S. S. and M. Bodson (1989). *Adaptive Control: Stability, Convergence and Robustness.* Prentice-Hall, Englewood Cliffs, NJ.