New Conditions for the Exponential Stability of Evolution Equations

Keum-Shik Hong, Jinn-Wen Wu, and Kyo-Il Lee

Abstract—New conditions for the exponential stability for both linear nonautonomous finite and a class of infinite-dimensional systems described by parabolic partial differential equations (PDE's) are derived. The results for the parabolic systems are derived via semigroup approach.

I. Introduction

In adaptive identification and adaptive control of lumped parameter systems, obtaining exponentially stable adaptive systems is very important in the sense that an exponentially stable system can tolerate a certain amount of disturbance and unmodeled dynamics. The exponential stability of nominal adaptive systems is known to be achieved by the persistency of excitation condition of the reference input [12]. Recently, adaptive controls of infinite-dimensional systems were also investigated. Wen [15] proposed an adaptive control algorithm using the command generator tracker approach in Hilbert space, and Hong [7] proposed a direct model reference adaptive control algorithm for parabolic systems and investigated the exponential stability of the closed-loop system and the convergence of the controller parameters to their nominal values.

Consider an initial value problem in a Banach space X

$$\dot{u}(t) = A(t)u(t), \qquad u(s) = u_0, \qquad 0 \le s \le t \le T \tag{1.1}$$

where $A(t)\colon D(A(t)) \subset X \to X$ is a linear operator in X. An X valued function $u\colon [s, T] \to X$ is a (classical) solution of (1.1) if u is continuous on $[s, T]$, $u(t) \in D(A(t))$ for $s < t \le T$, u is continuously differentiable on $s < t \le T$ and satisfies (1.1). The zero solution of (1.1) is said to be exponentially stable if and only if there exist positive constants K and δ such that

$$\|\Phi(t; t_0, u_0)\| \le Ke^{-\delta(t-t_0)}\|u_0\|$$

for all $t \ge t_0 \ge 0$, every $u_0 \in X$, where $\Phi(t; t_0, u_0)$ is the solution of (1.1) at time t starting from u_0 at t_0.

In a special case where $X = R^n$

$$\dot{x}(t) = A(t)x(t), \qquad x(0) = x_0, \qquad x_0 \in R^n, \qquad t \ge 0 \tag{1.2}$$

it is known that if A is a constant matrix such that the real parts of the eigenvalues of A are bounded above by $-\delta$, $\delta > 0$, then the solution of (1.2) tends to zero exponentially as $t \to \infty$. Hence, it is natural to ask whether one might be able to determine the stability of time-varying $A(t)$ by examining the spectrum of $A(t)$ at each time t. However, it has been shown by means of examples [9] that *the frozen time stability* (i.e., the real parts of the spectrum of $A(t)$ for every time instant are bounded above by $-\delta$, $\delta > 0$) does not imply the stability of the time-varying system. The examples in [9] actually show that the solutions grow without bound as $t \to \infty$ even if the spectrum of $A(t)$ remain at fixed locations in the open left-half plane for all $t \ge 0$.

Flatto and Levinson [5] show that if $A(t) = B(\epsilon t)$, where ϵ is a small parameter, and $\dot{B}(t)$ is bounded for all t, then the frozen-time stability is sufficient for all solution of (1.2) to approach to zero exponentially as $t \to \infty$ provided that ϵ is sufficiently small. By choosing ϵ small, we can assure that $\dot{A}(t)$ is small. Other results in the literature for frozen-time analysis for slow-varying systems include Rosenbrock [11] and Desoer [3]. Subsequently, the slow-varying results have been extended to nonlinear systems ([1], [4, p. 125] and [14, p. 218]), and instability analysis [13].

Compared to the finite-dimensional systems the results on infinite-dimensional systems are scarce. Conditions for the exponential stability for parabolic PDE's are found in [8] and [10]. The condition (P4) in [10, p. 173], however, is restrictive since it does not allow even periodicity of the system coefficients.

This note has the following structure. In Section II, we first derive new exponential stability conditions for lumped parameter systems with explicit bounds for the involved constants. In Section III, by using the semigroup approach, we establish the conditions for the exponential stability for parabolic PDE's without the knowledge of the asymptotic behavior of $A(t)$. Conclusions are given in Section IV.

II. Conditions for Finite-Dimensional Systems

The purpose of this section is to strengthen the exponential stability conditions for (1.2) by replacing the slow-varying condition, i.e., $\sup_{t\ge 0}\|\dot{A}(t)\|$ is sufficiently small (see [1], [3]–[5], [11], [13], [14]), with more general "Holder-type" continuity of $A(t)$.

Theorem 1: Consider the system (1.2). If the function $t \to A(t)$ is a matrix-valued continuous function such that i) there exists $m > 0$ such that $\sup_{t\ge 0}\|A(t)\| = m < \infty$, ii) there exist $K, \delta > 0$ such that $\|e^{A(t)s}\| \le Ke^{-\delta s}$ for every $t, s \ge 0$, and iii) $A(t)$ satisfies

$$\|A(t_1) - A(t_2)\| \le L\mid t_1 - t_2\mid^{\alpha}$$

for all $t_1, t_2 \ge 0$, and $\alpha > 0$, where

$$L < \frac{\delta(\alpha+1)}{2K(2\ln K/\delta)^{\alpha}},$$

then the system (1.2) is exponentially stable.

Proof: Let us choose the initial time $t_0 = 0$, and rewrite (1.2) as

$$\dot{x}(t) = A(0)x(t) + [A(t) - A(0)]x(t), \qquad x(0) = x_0. \tag{2.1}$$

If $x(t)$ is a solution of (2.1), then

$$x(t) = e^{A(0)t}x_0 + \int_0^t e^{A(0)(t-s)}[A(s) - A(0)]x(s)\,ds. \tag{2.2}$$

By taking norms both sides and using the conditions ii) and iii)

$$\|x(t)\| \le Ke^{-\delta t}\|x_0\| + KL\int_0^t e^{-\delta(t-s)}s^{\alpha}\|x(s)\|\,ds. \tag{2.3}$$

Multiply $e^{\delta t}$ both sides and apply the Gronwall's inequality [14], then

$$\begin{aligned}\|x(t)\| &\le Ke^{-\delta t}\exp\left\{KL\int_0^t s^{\alpha}\,ds\right\}\|x_0\| \\ &= e^{\ln K - \delta t + KLt^{\alpha+1}/(\alpha+1)}\|x_0\|.\end{aligned} \tag{2.4}$$

Note also that if the initial time had been t_0 instead of 0, (2.4) becomes

$$\|x(t)\| \le e^{\ln K - \delta(t-t_0) + KL(t-t_0)^{\alpha+1}/(\alpha+1)}\|x_0\|.$$

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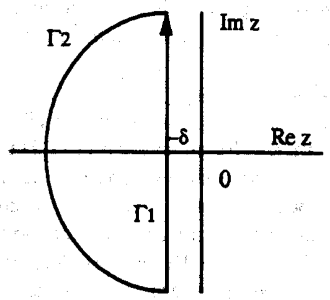

Fig. 1. Closed contour for finite-dimensional systems.

Now choose t in such a way that it satisfies the following inequalities

$$\text{i) } \ln K - \frac{1}{2}\delta t \le 0, \quad \text{and} \quad \text{ii) } -\frac{1}{2}\delta t + \frac{KLt^{\alpha+1}}{\alpha+1} < 0. \qquad (2.5)$$

If $L < \delta(\alpha+1)/2K(2\ln K/\delta)^\alpha$, then there exists $t > 0$ such that the above inequalities are satisfied, i.e.,

$$\frac{2\ln K}{\delta} \le t < \left(\frac{\delta(\alpha+1)}{2KL}\right)^{1/\alpha}. \qquad (2.7)$$

Set t in (2.4) to be $T \triangleq 2\ln K/\delta$, then

$$\begin{aligned}\|x(T)\| &\le \exp\left(\left[\ln K - \delta\frac{2\ln K}{\delta} + KL\left(\frac{2\ln K}{\delta}\right)^{\alpha+1}/(\alpha+1)\right]\frac{\delta}{2\ln K}T\right)\|x_0\| \\ &\le e^{-\beta T}\|x_0\| \qquad (2.8)\end{aligned}$$

where

$$\beta = \frac{\delta}{2} - \frac{KL}{\alpha+1}\left(\frac{2\ln K}{\delta}\right)^\alpha > 0.$$

Hence for any integer $n \ge 0$

$$\|x(nT)\| \le e^{-\beta nT}\|x_0\|. \qquad (2.9)$$

For $t = nT + \tau$, $0 \le \tau < T$, by taking $t_0 = nT$ (2.4) becomes

$$\|x(t)\| \le Ke^{-\delta\tau}e^{KL\tau^{\alpha+1}/(\alpha+1)}\|x(nT)\|. \qquad (2.10)$$

We suppress τ by T and substitute the upper bound for L into (2.10), then we finally get

$$\begin{aligned}\|x(t)\| &\le K^2e^{-\delta\tau}e^{-\beta nT}\|x_0\| \\ &\le K^2e^{(\beta-\delta)\tau}e^{-\beta t}\|x_0\| \\ &\le K^2e^{-\beta t}\|x_0\| \qquad (2.11)\end{aligned}$$

since $\beta < \delta$. QED

Remark: Results for the case when $\alpha = 1$ in Theorem 1 is found in [2], and its earlier version is found in [6].

Lemma 1: If $A(t)$: $R^+ \to R^{n\times n}$ is a continuous function such that i) there exists $m > 0$ such that $\sup_{t\ge0}\|A(t)\| = m < \infty$, ii) there exists a $\delta > 0$ such that $\text{Re}\,\lambda_i(A(t)) \le -2\delta$, for every i and every $t \ge 0$, then there exists a K such that for all $t \ge 0$

$$\|e^{A(t)s}\| \le Ke^{-\delta s} \qquad (2.12)$$

where

$$K = 1 + \frac{2(2^n-1)}{\pi\delta^n}m(3m)^{n-1}. \qquad (2.13)$$

Proof: The Laplace transform inversion theorem gives

$$e^{A(t)\tau} = \frac{1}{2\pi i}\int_\Gamma (zI - A(t))^{-1}e^{z\tau}\,dz \qquad (2.14)$$

where Γ is a closed contour in the left-half plane consisting of a vertical segment of abscissa at $-\delta$, and an arc of the circle centered at the origin, and of radius $2m$ as in Fig. 1. Note that since $A(t)$ has a uniform bound m for all $t \ge 0$, the contour Γ is chosen to be compact. Taking norms for both sides of (2.14)

$$\|e^{A(t)\tau}\| \le \frac{1}{2\pi}\int_\Gamma \|(zI - A(t))^{-1}\|\,\|e^{z\tau}\|\,|dz|. \qquad (2.15)$$

On Γ_1, we have $z = -\delta + iy$, where $y \in [-2m, 2m]$. Let Ψ_t be the characteristic polynomial of $zI - A(t)$ for each $t \ge 0$, then

$$\begin{aligned}\Psi_t(zI - A(t)) = (zI - A(t))^n + C_1(zI - A(t))^{n-1} + \cdots \\ + C_{n-1}(zI - A(t)) + C_nI = 0 \qquad (2.16)\end{aligned}$$

where

$$\begin{aligned}C_k &= \sum_{i_1i_2\cdots i_k}(-1)^k\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \\ C_n &= \det(zI - A(t))(-1)^n,\end{aligned}$$

and λ_i are the eigenvalues of $(zI - A(t))$. By rearranging (2.16), we obtain

$$\begin{aligned}(zI - A(t))^{-1} = -\frac{1}{C_n}[(zI - A(t))^{n-1} \\ + C_1(zI - A(t))^{n-2} + \cdots + C_{n-1}I]. \qquad (2.17)\end{aligned}$$

Taking norms both sides

$$\begin{aligned}\|(zI - A(t))^{-1}\| \le \frac{1}{|C_n|}[\|zI - A(t)\|^{n-1} \\ + |C_1|\,\|zI - A(t)\|^{n-2} + \cdots + |C_{n-1}|]. \qquad (2.18)\end{aligned}$$

Since $|\lambda_i| \le \|zI - A(t)\| \le |z| + \|A(t)\| \le 2m + m = 3m$ for each i on Γ_1 and

$$|C_k| = \sum_{i_1i_2\cdots i_k}|\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}| \le \binom{n}{k}(3m)^k; \qquad (2.19)$$

(2.18) becomes

$$\begin{aligned}\|(zI - A(t))^{-1}\| &\le \frac{1}{\delta^n}\left[\binom{n}{0}(3m)^{n-1} + \binom{n}{1}(3m)^{n-1} + \cdots + \binom{n}{n-1}(3m)^{n-1}\right] \\ &= \frac{1}{\delta^n}\left[\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1}\right](3m)^{n-1} \\ &= \frac{2^n-1}{\delta^n}(3m)^{n-1}. \qquad (2.20)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2\pi}\int_{\Gamma_1}\|(zI - A(t))^{-1}\|\,\|e^{\tau z}\|\,|dz| \\ \le \frac{1}{2\pi}\frac{2^n-1}{\delta^n}(3m)^{n-1}e^{-\delta\tau}4m \\ = K_1e^{-\delta\tau} \qquad (2.21)\end{aligned}$$

where

$$K_1 = \frac{2}{\pi}\frac{2^n-1}{\delta^n}m(3m)^{n-1}.$$

On the other hand, on Γ_2, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\Gamma_2} \|(zI - A(t))^{-1}\| \|e^{\tau z}\| \, |dz| &\le \frac{1}{2\pi}\left(\frac{1}{|z| - \|A(t)\|}\right) e^{-\delta\tau} 2m\pi \\ &\le \frac{1}{2\pi}\frac{1}{m} 2m\pi \, e^{-\delta\tau} \\ &= e^{-\delta\tau}. \end{aligned} \tag{2.22}$$

Therefore (2.15) with (2.21) and (2.22) becomes

$$\|e^{A(t)\tau}\| \le \frac{1}{2\pi}\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right) \le Ke^{-\delta\tau} \tag{2.23}$$

where K is defined in (2.13). Note that K depends on only m and δ and does not depend on t. QED

Theorem 2: Consider the system (1.2). If $A(t)$ is continuous such that i) there exists $m > 0$ such that $\sup_{t\ge 0} \|A(t)\| = m < \infty$, ii) there exists a $\delta > 0$ such that $\operatorname{Re} \lambda_i(A(t)) \le -2\delta$, for every i and every $t \ge 0$, and iii) $A(t)$ satisfies

$$\|A(t_1) - A(t_2)\| \le L \, |t_1 - t_2|^{\alpha}$$

for all $t_1, t_2 \ge 0$, and any $\alpha > 0$, where $L < \delta(\alpha + 1)/2K(2\ln K/\delta)^{\alpha}$, then the system (1.2) is exponentially stable.

Proof: The proof directly follows from Theorem 1 and Lemma 1.

Example 1: Consider the following second-order ODE used as models in the areas of micro waves, acoustics, and random vibrations [16].

$$\ddot{x}(t) + \mu\dot{x}(t) + (1 + 0.5\cos(2t))x(t) = 0$$

or in a state variable form

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 - 0.5\cos(2t) & -\mu \end{bmatrix} x(t) \triangleq A(t)x(t). \tag{2.24}$$

Specifically, if $\mu = 0.2$, then $\operatorname{Re}\sigma(A(t)) = -0.1$, and $\|A(t)\| \le \sqrt{3.29}$ for all $t \ge 0$. Also for $t, s \ge 0$

$$\begin{aligned}\|A(t) - A(s)\| &= \left\| \begin{bmatrix} 0 & 0 \\ 0.5[\cos(2t) - \cos(2s)] & 0 \end{bmatrix} \right\| \\ &= 0.5\,|\cos(2t) - \cos(2s)| \\ &= 0.5\,|2\sin(2\tau)|\,|t - s| \end{aligned}$$

for some $\tau \in [s, t]$. Hence $L = 1$ and $\alpha = 1$. Therefore, the condition for L in Theorem 2 is not satisfied. The approximate monodromy (cf. [16]) of (2.24) defined as $\tilde{\Phi}(\pi, 0) = \prod_{k=1}^{N}(I + hA(kh))$, $hN = \pi$, for $N = 10^3$ and $\mu = 0.2$ is given by numerical analysis (MATLAB) as

$$\tilde{\Phi}(\pi, 0) = \begin{bmatrix} -0.9073 & 0.3297 \\ 0.3575 & -0.9429 \end{bmatrix}$$

and the Floquet multipliers are -0.5813, and -1.2689. Therefore, (2.24) is unstable when $\mu = 0.2$. On the other hand, if we increase μ up to 0.5, then

$$\tilde{\Phi}(\pi, 0) = \begin{bmatrix} -0.4349 & 0.2283 \\ 0.1304 & -0.5506 \end{bmatrix}$$

and the Floquet multipliers are -0.3108, and -0.6747. So (2.24) is exponentially stable.

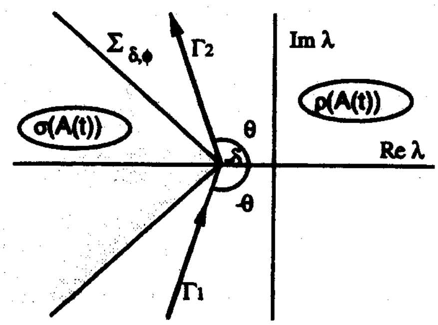

Fig. 2. Contour for parabolic systems.

III. Conditions for Parabolic Systems

In this section, we derive conditions for the exponential stability for PDE's. We first define a sector in the complex plane such that

$$\Sigma_{\delta,\phi} \triangleq \{\lambda: |\arg(\lambda + \delta)| < \pi/2 + \phi\} \cup \{-\delta\} \tag{3.1}$$

where $\delta \ge 0$, and $0 < \phi < \pi/2$. A densely defined operator A in X satisfying i) $\Sigma_{\delta,\phi} \subset \rho(A)$, where $\rho(A)$ is the resolvent set of A which is open in the complex plane, and ii) $\|R(\lambda: A)\| \le M/|\lambda|$ for $M > 0$, and $\lambda \in \Sigma_{\delta,\phi}$ is the infinitesimal generator of an (uniformly bounded) analytic semigroup [10].

Lemma 2: Consider $\dot{u}(t) = A(t)u(t)$, $t \ge 0$, $u \in X$. If i) the domain $D(A(t)) = D$, $t \ge 0$, is dense in X and independent of t, ii) there exist constants $\delta > 0$, $\phi \in (0, \pi/2)$, and $M > 0$ such that the resolvent $R(\lambda: A(t))$ of $A(t)$ exists for all $\lambda \in \Sigma_{\delta,\phi} \subset \rho(A(t))$ uniformly in $t \ge 0$, and

$$\|R(\lambda: A(t))\| \le \frac{M}{|\lambda + \delta| + 1}, \qquad \text{for } \lambda \in \Sigma_{\delta,\phi},\ t \ge 0 \tag{3.2}$$

then there exists $K > 0$ such that for all $s \in [0, \infty)$

i) $$\|\Phi_s(t)\| \le Ke^{-\delta t}, \qquad t > 0 \tag{3.3}$$

ii) $$\|A(s)\Phi_s(t)\| \le \frac{Ke^{-\delta t}}{t}, \qquad t > 0 \tag{3.4}$$

where $\Phi_s(t)$ is the semigroup generated by $A(s)$, $s \ge 0$, and K depends only on M and ϕ.

Proof:

Set for $t \ge 0$

$$\Phi_\tau(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda: A(\tau))\, d\lambda \tag{3.5}$$

where Γ as in Fig. 2 is the path composed of two rays $\Gamma_1 = \{-\delta + re^{-i\theta}: 0 \le r < \infty\}$, and $\Gamma_2 = \{-\delta + re^{i\theta}: 0 \le r < \infty\}$, where $\theta \in (\pi/2, \pi/2 + \phi)$. Note that Γ is oriented so that $\operatorname{Im}\lambda$ increases along Γ. From (3.5) it follows easily that for $t > 0$ the integral in (3.5) converges in the uniform topology. Moreover since $R(\lambda: A(t))$ is analytic in $\Sigma_{\delta,\phi}$, we may shift the path of integration in (3.5) to Γ_t, $t > 0$, where $\Gamma_t = \Gamma_1' \cup \Gamma_2' \cup \Gamma_3'$ and $\Gamma_1' = \{-\delta + re^{-i\theta}: 1/t \le r < \infty\}$, $\Gamma_2' = \{-\delta + (1/t)e^{i\varphi}: -\theta \le \varphi \le \theta\}$, $\Gamma_3' = \{-\delta + re^{i\theta}: 1/t \le r < \infty\}$ without changing the value of the integral in (3.5). Hence

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_{\Gamma_1'} e^{\lambda t} R(\lambda: A(\tau))\, d\lambda\right\| &\le \frac{1}{2\pi}\int_{\Gamma_1'} |e^{\lambda t}| \, \|R(\lambda: A(\tau))\| \, |d\lambda| \\ &\le \frac{e^{-\delta t}}{2\pi}\int_{1/t}^{\infty} e^{tr\cos\theta} \frac{M}{|\lambda + \delta| + 1}\, dr \\ &\le \frac{Me^{-\delta t}}{2\pi}\int_{-\cos\theta}^{\infty} \frac{e^{-s}}{s}\, ds \\ &= K_1 e^{-\delta t} \end{aligned} \tag{3.6}$$

where $K_1 = (M/2\pi)\int_{-\cos\theta}^{\infty}(e^{-s}/s)\, ds$. A substitution $s = -tr\cos\theta$ and $\cos\theta < 0$ have been used in getting the third inequality in (3.6). The integral on Γ_3' is established similarly as on Γ_1'. On Γ_2', we have

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_2'} e^{\lambda t}R(\lambda: A(\tau))\,d\lambda\right\| \le \frac{Me^{-\delta t}}{2\pi}\int_{-\infty}^{\infty} e^{\cos\eta}\,d\eta = K_2e^{-\delta t} \tag{3.7}$$

where $K_2 = (M/2\pi)\int_{-\theta}^{\theta} e^{\cos\eta}\,d\eta$. Therefore there is a constant such that

$$\|\Phi_\tau(t)\| \le (2K_1 + K_2)e^{-\delta t} = K_3e^{-\delta t}, \qquad t \ge 0. \tag{3.8}$$

Now differentiating (3.6) formally with respect to t yields

$$\frac{d}{dt}(\Phi_\tau(t)) = A(\tau)\Phi_\tau(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}R(\lambda: A(t))\,d\lambda \tag{3.9}$$

where the contour is $\Gamma = \Gamma_1 \cup \Gamma_2$. On Γ_1 by using the fact that $|\lambda| \le r + \delta$, we have

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_1} \lambda e^{\lambda t}R(\lambda: A(\tau))\,d\lambda\right\| \le \frac{e^{-\delta t}}{2\pi}\int_0^\infty e^{tr\cos\theta}\frac{M|\lambda|}{|\lambda+\delta|+1}\,dr$$
$$\le \frac{Me^{-\delta t}}{2\pi}\int_0^\infty e^{tr\cos\theta}\frac{r+\delta}{r+1}\,dr.$$

Also if $\delta > 1$, then $r+\delta \le \delta(r+1)$. If $\delta < 1$, then $(r+\delta)/(r+1) < 1$. Therefore, we have

$$\begin{aligned}
&\left\|\frac{1}{2\pi i}\int_{\Gamma_1} \lambda e^{\lambda t}R(\lambda: A(\tau))\,d\lambda\right\| \\
&\le \frac{Me^{-\delta t}\max\{1,\delta\}}{2\pi}\int_0^\infty e^{tr\cos\theta}\,dr \\
&= \frac{Me^{-\delta t}\max\{1,\delta\}}{2\pi}\int_0^\infty \frac{e^{\eta\cos\theta}}{t}\,d\eta \\
&= \frac{Me^{-\delta t}\max\{1,\delta\}}{2\pi}\,\frac{1}{t\,|\cos\theta|} \\
&= K_4\frac{e^{-\delta t}}{t}
\end{aligned} \tag{3.10}$$

where $K_4 = M\max\{1,\delta\}/2\pi\,|\cos\theta|$. Let $K = \max\{K_3, 2K_4\}$. Then (3.3) and (3.4) are obtained for all $\tau \in [0,\infty)$. QED

Theorem 3: Let $A(t)$ be an infinitesimal generator of analytic semigroup $\Phi_t(s)$, $s \ge 0$, defined for all $t \ge 0$ such that i) the domain $D(A(t)) = D$, $t \ge 0$, is dense in X and independent of t, ii) there exist constants $\delta > 0$, $\phi \in (0, \pi/2)$, and $M > 0$ such that the resolvent $R(\lambda: A(t))$ of $A(t)$ exists for all $\lambda \in \Sigma_{\delta,\phi} \subset \rho(A(t))$ uniformly in $t \ge 0$, and

$$\|R(\lambda: A(t))\| \le \frac{M}{|\lambda+\delta|+1}, \qquad \text{for } \lambda \in \Sigma_{\delta,\phi}, \qquad t \ge 0,$$

iii) there exist constants $L, \beta > 0$ and α, $0 < \alpha < 1$ such that

$$\|(A(t_1) - A(t_2))A(t_3)^{-\alpha}\| \le L\,|t_1 - t_2|^\beta$$

for $t_1, t_2, t_3 \ge 0$. Then (1.1) is exponentially stable.

Proof: Let us take the initial time to be 0 for simplicity. For any fixed $T \ge 0$, we can rewrite (1.1) in the form

$$\dot u(t) = A(T)u(t) + (A(t) - A(T))u(t). \tag{3.11}$$

Let $\Phi_T(t)$ be the semigroup generated by $A(T)$. Then the solution of (3.11) is of the form [10, p. 105]

$$u(t) = \Phi_T(t)u(0) + \int_0^t \Phi_T(t-s)(A(s) - A(T))u(s)\,ds$$

Let $\Phi_T(t)$ be the semigroup generated by $A(T)$. Then the solution

$$\begin{aligned}
&= \Phi_T(t)u(0) + \int_0^t (A(s) - A(T))A^{-\alpha}(T)A^{\alpha}(T) \\
&\quad\cdot \Phi_T(t-s)u(s)ds
\end{aligned} \tag{3.12}$$

where $\alpha < 1$ is chosen. Taking norms both sides

$$\begin{aligned}
\|u(t)\| &\le \|\Phi_T(t)u(0)\| + \int_0^t \|(A(s) - A(T)A^{-\alpha}(T)\| \\
&\quad\cdot \|A^\alpha(T)\Phi_T(t-s)\|\,\|u(s)\|ds \\
&\le Ke^{-\delta t}\|u(0)\| + \int_0^t KL|T-s|^\beta\frac{e^{-\delta(t-s)}}{(t-s)^\alpha}\|u(s)\|ds \\
&\le Ke^{-\delta t}\|u(0)\| + KLT^\beta e^{-\delta t}\int_0^t \frac{1}{(t-s)^\alpha}e^{\delta s}\|u(s)\|ds.
\end{aligned} \tag{3.13}$$

Now by multiplying $e^{\delta t}$ both sides of (3.13) and applying an extended form of the Gronwall's inequality [17, pp. 190], then we obtain

$$\|u(t)\| \le KCe^{-\delta t}\|u(0)\| \tag{3.14}$$

where $C = C(\alpha, KLT^\beta) < \infty$ is constant. We now rewrite (3.14) as

$$\|u(t)\| \le e^{-\delta t/2}e^{(\ln(KC)-\delta t/2)}\|u(0)\|.$$

Then for any given K, δ, and C there exists $T_0 > 0$ such that $\ln(KC) - \delta T_0/2 < 0$. Hence for some fixed $T \ge T_0$

$$\begin{aligned}
\|u(T)\| &\le e^{-\delta T/2}\|u(0)\| \\
&\le e^{-\beta T}\|u(0)\|
\end{aligned}$$

where $\beta = \delta/2$. Therefore for any integer $n \ge 0$

$$\|u(nT)\| \le e^{-\beta nT}\|u(0)\|. \tag{3.15}$$

For an arbitrary $t = nT + \tau$, $n \ge 0$, $0 \le \tau < T$, by taking τ as the initial time, and using the same argument as above we obtain

$$\|u(t)\| \le e^{-\beta nT}\|u(\tau)\|. \tag{3.16}$$

Now by Theorem 6.1 [10, p. 150] under the conditions i)–iii), there exists a unique evolution system $\Phi(t, s)$ such that $t \to \Phi(t, s)$ is strongly differentiable in X and

$$u(\tau) = \Phi(\tau, 0)u(0) \tag{3.17}$$

where $0 \le \tau < T$. Therefore

$$\begin{aligned}
\|u(\tau)\| &\le \|\Phi(\tau, 0)\|\|u(0)\| \\
&\le \max_{t\in[0,T]}\|\Phi(\tau, 0)\|\,\|u(0)\| \triangleq \gamma\|u(0)\|
\end{aligned}$$

where $\gamma = \max_{t\in[0,T]}\|\Phi(\tau,0)\| < \infty$. Hence

$$\begin{aligned}
\|u(t)\| &\le \gamma e^{-\beta nT}\|u(0)\| = \gamma e^{\beta\tau}e^{-\beta t}\|u(0)\| \\
&\le \gamma e^{\beta T}e^{-\beta t}\|u(0)\| = He^{-\beta t}\|u(0)\|
\end{aligned}$$

where $H = \gamma e^{\beta T}$. QED

Example 2: Consider (1.1) with

$$\begin{aligned}
A(t) &= (1 + 0.5\sin(t))\frac{\partial^2}{\partial x^2} \triangleq a(t)D^2, \\
u(t, 0) &= u(t, 1) = 0, \\
u(0) &= u_0
\end{aligned} \tag{3.18}$$

where $D(A(t)) = W_0^2(J)$, $J = [0, 1]$, $a(t)$ is differentiable, and $\sigma(A(t)) = \{a(t)n^2\pi^2,\ n = 1, 2, 3, \cdots\}$. $A(t)$ is a closed and densely defined operator which generates an analytic semigroup for each $t \ge 0$, and the semigroup generated by $A(t)$ is given as

$$\Phi_t(s) = \sum_{n=1} e^{-a(t)n^2\pi^2 s}\langle R_n, \cdot\rangle R_n$$

where $R_n = \sin(n\pi x)$. Since $a(t) \geq 1/2$, $\|\Phi_t(s)\| \leq Ke^{-a(t)\pi^2 s} \leq Ke^{-\delta s}$, where $\delta = \pi^2/2$ and K is independent of t. Therefore

$$\|R(\lambda: A(t))\| \leq K \int_0^\infty e^{-(\mathrm{Re}\lambda)s - \delta s}\, ds = \frac{K}{Re\lambda + \delta}$$

where $\mathrm{Re}\lambda > -\pi^2/4$. Now, if we let $A(t_3)^{-1}v = u$ for $v \in L_0^2(J)$

$$\begin{aligned} &\|(A(t_1) - A(t_2))A(t_3)^{-1}\| \\ &= \sup_{\|v\|=1,\, v\in L_0^2(J)} \|(A(t_1) - A(t_2))A(t_3)^{-1}v\| \\ &= \sup_{\|v\|=1,\, v\in L_0^2(J)} \left\| \frac{a(t_1) - a(t_2)}{a(t_3)} v \right\| \\ &\leq 2 \mid t_1 - t_2 \mid^\alpha, \qquad 0 < \alpha < 1. \end{aligned} \tag{3.19}$$

The inequality in (3.19) is from that i) if $| t_1 - t_2 | \geq 1$, $| 0.5(\sin t_1 - \sin t_2)/(1 + 0.5 \sin t_3) | \leq 2 \leq 2 | t_1 - t_2 |^\alpha$, and ii) if $| t_1 - t_2 | < 1$, then $| 0.5(\sin t_1 - \sin t_2)/(1 + 0.5 \sin t_3) | \leq 2 \sup_{t\in[0,\infty)} | a'(t) | | t_1 - t_2 | \leq | t_1 - t_2 |^\alpha$. Therefore (3.18) is exponentially stable.

IV. Conclusions

New conditions for the exponential stability for a linear finite-dimensional system as well as a class of infinite-dimensional systems described by parabolic partial differential equations are derived. It is shown for finite-dimensional systems that the frozen time analysis is justifiable for the systems with Holder-type continuity which is broader than the class of slow-varying systems. For parabolic systems the restrictive condition such as the existence of $A(\infty)$ has been removed. The proofs are carried out using the semigroup theory and variation of constant formulas, and specific bounds for K and L are

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