Transient Behavior Analysis of Vibrationally Controlled Nonlinear Parabolic Systems with Neumann Boundary Conditions

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Abstract—In the first part [1] of this work the conditions for the existence of the stabilizing vibrations for a class of distributed parameter systems governed by parabolic partial differential equations with Neumann boundary conditions were derived, and the guidelines for the choice of the vibration parameters that ensure stabilization were given. The present note addresses the transient behavior analysis of vibrationally controlled systems of the same class.

I. Introduction

In [1] it was shown that appropriately chosen parametric excitations are capable of asymptotically stabilizing unstable equilibria or inducing asymptotically stable oscillatory regimes in the vicinity of unstable equilibria in a class of distributed parameter systems (DPS) described by nonlinear parabolic partial differential equations (PDE) with Neumann boundary conditions (NBC). Since the resulting open-loop technique, termed vibrational control [2], has found applications in the stabilization of such distributed plants as plasma pinches [3] and powerful continuous CO_2 lasers [4], not easily stabilizable by feedback, it is of interest to develop a tool for the transient behavior analysis of the class of systems considered in [1]. This note develops such a tool which consists of a certain mapping and a time invariant PDE whose trajectories under this mapping yield the approximate moving averages along the trajectories of the vibrationally controlled system. This tool considerably enhances the understanding of the behavior of vibrationally controlled DPS and facilitates a selection of the parameters of stabilizing vibrations that ensure transient behavior with more desirable properties. This note also represents an extension of the results of [5] and [6] on the transient behavior of the vibrationally controlled ordinary and delay differential equations, respectively, to PDE's with NBC.

II. Statement of the Problem

A class of DPS considered in the note is described by a nonlinear parabolic PDE

$$u_t = Au_{xx} + Bu_x + C(u,\lambda), \qquad u_x(0,t) = u_x(1,t) = 0, \qquad t \ge 0, \qquad u(x,0) = u_0(x), \tag{1}$$

where $u = u(x,t)$: $R(0,1) \times R_+ \to R^n$; $A, B \in R^{n\times n}$ are constant matrices; $\lambda \in R^m$ is a vibratile parameter; C: $R^n \times R^m \to R^n$ is a nonlinear vector function such that $C(0,\lambda) = 0$; subscripts of u denote corresponding partial derivatives with respect to t and x; the Neumann boundary conditions are given by

$$u_x(0,t) = u_x(1,t) = 0, \qquad t \ge 0; \tag{2}$$

and initial condition by $u(x,0) = u_0(x)$.

Assuming λ fixed, introduce in (1) parametric vibrations as

$$\lambda \to \lambda + f(t) \tag{3}$$

where $f(t)$ is a *periodic vector function with average value equal to zero* (PAZ vector). As a result, (1) becomes

$$u_t = Au_{xx} + Bu_x + C(u, \lambda + f(t)), \qquad u_x(0,t) = u_x(1,t) = 0, \qquad t \ge 0, \qquad u(x,0) = u_0(x). \tag{4}$$

Throughout this note, it will be assumed that (4) has the form

$$u_t = Au_{xx} + Bu_x + C(u) + C_1(f(t),u),$$
$$u_x(0,t) = u_x(1,t) = 0, \qquad t \ge 0, \qquad u(x,0) = u_0(x),$$
$$C_1\colon R^m \times R^n \to R^n, \qquad C(u) \equiv C(u,\lambda), \tag{5}$$

where $C_1(\cdot,\cdot)$ is a vector function linear with respect to its first argument. If $C_1(f(t),u) = l(t)$, where $l(t)$ is a PAZ vector, the introduced vibrations are referred to as *vector additive*, if $C_1(f(t),u) = D(t)u$, where $D(t)$ is an $n \times n$ PAZ matrix, the vibrations are called *linear multiplicative*, and if $C_1(f(t),u) = D(t)X(u)$, where X: $R^n \to R^n$ is a nonlinear map, the vibrations are termed *nonlinear multiplicative*.

Throughout the note, it is also assumed that for a given initial condition $u(x,0) = u_0(x)$ and boundary conditions (2), systems (1) and (4) are well-posed in the Sobolev space $H^{n1}(0,1)$ of vector functions $v(x) = [v_1(x),\cdots,v_n(x)]^T$ with components $v_i(x)$ in $L_2(0,1)$ which have the first distributional derivatives. Norm on $H^{n1}(0,1)$ is defined as

$$\|v\|_1 \triangleq \left(\int_0^1 v_x^T(x)v_x(x) + v(x)^T v(x)\,dx\right)^{1/2}. \tag{6}$$

Superscripts "n" and "1" in $H^{n1}(0,1)$ indicate the dimension of the vector $v(x)$ and the order of the highest derivative with respect to x in the definition of norm (6).

Let in (3) a T-periodic $f(t)$ be given as

$$f(t) = \frac{1}{\epsilon}\phi\left(\frac{t}{\epsilon}\right), \tag{7}$$

where ϵ is a positive constant and $\phi(\cdot)$ is a PAZ vector and introduce a *moving average along the trajectory* $u(t) \equiv u(x,t,u_0(x),0)$, $u(0) = u_0(x)$, of (5) defined as

$$\bar{u}(t) \triangleq \frac{1}{T}\int_t^{t+T} u(x,s,u_0(x),0)\,ds. \tag{8}$$

The system (5) is time varying and therefore difficult to analyze. If, however, ϵ is sufficiently small, then the trajectories of (5) with T-periodic $f(t) = (1/\epsilon)\phi(t/\epsilon)$, $0 < \epsilon \ll 1$, are usually composed of a fast oscillatory part and a slow "evolutionary" part. Consequently, if we represent the slow part by a moving average along a trajectory of (5), and construct a time invariant system whose trajectories, possibly under a time invariant map, approximate the moving averages along the trajectories of (5), the behavior of this time invariant system will reveal the global "evolutionary" behavior of the vibrationally controlled system unobscured by the fast oscillatory component. In [7], this approach has been used for the analysis of system (5) with Dirichlet boundary conditions and linear multiplicative vibrations. In this note this approach is applied to the analysis of system (5) with Neumann boundary conditions and a more general class of vibrations $C_1(f(t),u)$ with $C_1(\cdot,\cdot)$ linear in its first argument.

For this purpose consider an ODE

$$\frac{d\xi}{dt} = C_1(\phi(t),\xi) \tag{9}$$

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where $C_1(\cdot,\cdot)$ and $\phi(\cdot)$ are defined in (5) and (7), respectively. Assume that (9) has a unique solution defined by every initial condition $\xi_0 \in \Omega \subset R^n$, $\forall t \geq 0$. Denote the general solution of (9) as

$$\xi(t) = h(t, q), \qquad h: R \times R^n \to R^n, \tag{10}$$

where $q \in R^n$ is a constant uniquely defined for every pair of initial conditions (ξ_0, t_0) and assume that $h(t, q)$ is almost periodic in t for any q. Introduce into (5) a substitution of the form

$$u(x,t) = h(t, v(x,t)), \qquad v: R(0,1) \times R_+ \to R^n. \tag{11}$$

Assuming that $C_1(\cdot, u)$ is differentiable with respect to u, (5) takes the form

$$\begin{aligned} v_t &= [\partial h/\partial v]^{-1}[Ah(t,v)_{xx} + Bh(t,v)_x + C(h(t,v))] \\ &= F_1(t,v) + F_2(t,v) + F_3(5,v) \end{aligned} \tag{12}$$

where $[\partial h/\partial v]^{-1}$ always exists (cf. [8, Section II]) and

$$F_1(t,v) \triangleq [\partial h/\partial v]^{-1} Ah(t,v)_{xx},$$
$$F_2(t,v) \triangleq [\partial h/\partial v]^{-1} Bh(t,v)_x,$$
$$F_3(t,v) \triangleq [\partial h/\partial v]^{-1} C(h(t,v)).$$

Introduce an averaged equation

$$w_t = P_1(w) + P_2(w) + P_3(w), \qquad w_x(0,t) = w_x(1,t) = 0, \quad t \geq 0, \qquad w(x,0) = w_0(x), \tag{13}$$

where

$$w: R(0,1) \times R_+ \to R^n,$$
$$P_1(v) \triangleq \overline{F_1(t,v)} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T F_1(t,v)\,dt,$$
$$P_2(v) \triangleq \overline{F_2(t,v)}, \qquad P_3(v) \triangleq \overline{F_3(t,v)},$$
$$\text{and} \qquad u_0(x) = h(0, w_0(x)), \tag{14}$$

and a time invariant map

$$\bar{h}(\cdot) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T h(s,\cdot)\,ds, \qquad \bar{h}: R^n \to R^n. \tag{15}$$

The problem consists in giving the conditions for the closeness of the exact moving averages (8) along the trajectories of (5) with the trajectories of (13) under map (15) on finite as well as infinite time interval. These conditions, if found, indicate when the trajectories of (13) under map (15) are approximate moving averages along the trajectories of (5), and therefore can serve as a tool for the simplified analysis of the transient behavior of (5). This problem is addressed next.

III. Approximate Moving Averages Along the Trajectories of a Vibrationally Controlled DPS with NBC

Theorem 1: Assume that i) the general solution $h(t,q)$ of (9) is T^*-periodic in t, where T^* is the period of $\phi(t)$, and is linear or affine in q; ii) $C(\xi, \lambda)$ is continuously differentiable for all $\xi \in \Omega$ in a sufficiently large open set $\Omega \subset R^n$. Let $u(t) \equiv u(x, t, u_0(x), 0)$ and $w(t) \equiv w(x, t, w_0(x), 0)$, with $h(0, w_0(x)) = u_0(x)$, be solutions of (5) with T-periodic $f(t) = (1/\epsilon)\phi(t/\epsilon)$ and (13), respectively, where (13) is assumed to be parabolic. Then for any positive δ as small as desired and κ as large as desired there exists $\epsilon_0 = \epsilon_0(\delta, \kappa)$ such that for system (5) with $f(t) = (1/\epsilon)\phi(t/\epsilon)$, $0 < \epsilon \leq \epsilon_0$, the following holds:

i) $$\|\bar{u}(t) - \bar{h}(w(t))\|_1 < \delta, \qquad \forall t \in [0, \kappa], \tag{16}$$

where

$$\bar{h}(w(t)) = \frac{1}{T^*}\int_0^{T^*} h(s, w(t))\,ds; \tag{17}$$

ii) whenever the null solution of the linearization of (13) at zero is exponentially stable, there exists a domain $\Omega_1 \subset H^{n1}(0,1)$, $0 \in \Omega_1$, such that (16) holds for all $t \in [0, \infty)$ provided $\bar{h}(w_0(x))$, $u_0(x) \in \Omega_1$, $\forall x \in (0,1)$.

Proof: The proof of Theorem 1 is given in the Appendix.

Remark 1: Inequality (16) demonstrates that (17) is indeed an approximate moving average along the trajectories of (5) with $f(t) = (1/\epsilon)\phi(t/\epsilon)$ and ϵ sufficiently small.

Remark 2: A comparison of the trajectories of the system without vibrations with the approximate moving averages along the trajectories of the corresponding vibrationally controlled system reveals the nonlocal changes in the system behavior caused by oscillations.

Remark 3: Although a derivation of the rigorous vibrational stabilizability conditions for parabolic systems with Dirichlet boundary conditions (DBC) is an open problem unlike those for systems with Neumann boundary conditions (NBC), simulations show that vibrational stabilizability is not significantly affected by the choice of boundary conditions. Indeed, [7] demonstrates vibrational stabilization of a parabolic system with DBC which is also vibrationally stabilizable when DBC are replaced by NBC.

Example 1: Consider (1) with

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$C(u) = C_0 u + R(u), \qquad C_0 = \begin{bmatrix} 2 & 7 \\ 3 & -3 \end{bmatrix},$$
$$R(u) = \begin{bmatrix} u_2 \\ -u_1 + \mu u_2 - \mu u_1^2 u_2 \end{bmatrix}, \qquad \mu = \text{constant}, \tag{18}$$

and initial conditions $u_1(x,0) = \cos \pi x$, $u_2(x,0) = -\cos \pi x$, further referred to as system (18). The null solution of (18) is unstable. Introduce vector additive vibrations as

$$l(t) = \frac{1}{\epsilon} m\left(\frac{t}{\epsilon}\right) = \begin{bmatrix} \frac{\alpha}{\epsilon} \sin \frac{t}{\epsilon} \\ \frac{\alpha}{\epsilon} \sin \frac{t}{\epsilon} \end{bmatrix}. \tag{19}$$

In [1] it was shown that the linearization at zero of the averaged equation (13) corresponding to (18) with vibrations (19) is asymptotically stable for $\alpha = 1.593$ when $\mu = 1.0$ in (18). In [1], it was also demonstrated that vibrations (19) induce in (18) an asymptotically stable oscillatory regime with the average located in the vicinity of the trivial solution of (18). Fig. 1 shows such a soultion $u_2(x,t)$ for $\mu = 1.0$, $\alpha = 3$, and $\epsilon = 0.008$. Here, the general solution of (9) is $h_1(t,q) = -\alpha \cos t + q_1$, $h_2(t,q) = -\alpha \cos t + q_2$. Equation (13) has the form

$$w_t = Aw_{xx} + Bw_x + P_4(w), \qquad w_x(1,t) = w_x(0,t) = 0, \quad t \geq 0,$$

with A and B of (18) and

$$P_4(w) = \begin{bmatrix} 2w_1 + 8w_2 \\ 2\left(1 - \mu\frac{\alpha^2}{2}\right)w_1 \\ +\left(-3 + \mu\left(1 - \frac{\alpha^2}{2}\right)\right)w_2 - \mu w_1^2 w_2 \end{bmatrix},$$

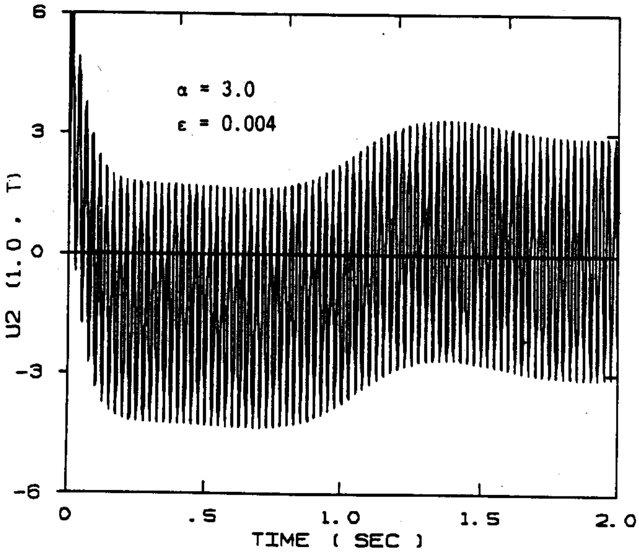

Fig. 1. Solution $u_2(x,t)$ of (18) with vector additive vibrations at $x = 1$.

and the approximate moving average along $u(t)$ is given as $\bar{h}_1(w(t)) = w_1(t)$, $\bar{h}_2(w(t)) = w_2(t)$, with mapping (15) being a unity matrix.

Fig. 2 shows exact and approxiamte moving averages $\bar{u}_2(t)$ and $\bar{h}_2(w(t))$ at $x = 1$. Similar closeness is observed in simulations for any $x \in (0,1)$ and for $u_1(x,t)$.

Example 2: Consider (1), (2) with A, B, C_0 as in (18), $C(u) = C_0 u + R(u)$, and

$$R(u) = \begin{bmatrix} 0.1u_1^3 & 0.1u_2^3 \end{bmatrix}^T, \tag{20}$$

further referred to as system (20). The null solution of (20) is unstable. Introducing into (20) linear multiplicative vibrations

$$D(t)u = \frac{1}{\epsilon}F\left(\frac{t}{\epsilon}\right)u = \begin{bmatrix} 0 & 0 \\ \frac{\alpha}{\epsilon}\cos\frac{t}{\epsilon} & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{21}$$

yields stabilization of the trivial solution of (20) as shown in [1]. Fig. 3 demonstrates solution $u_2(x,t)$ of (20) with zero equilibrium stabilized by linear multiplication vibrations when $\alpha = 3.0$ and $\epsilon = 0.004$. In this case, the general solution of (9) is

$$h(t,q) = \begin{bmatrix} 1 & 0 \\ \sin t & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \tag{22}$$

(13) has the form

$$w_t = Aw_{xx} + Bw_x + P_3(w), \quad w_x(1,t) = w_x(0,t) = 0, \quad t \geq 0,$$

with A and B of (20) and

$$P_3(w) = \begin{bmatrix} 2w_1 + 7w_2 + 0.1w_1^3 \\ [3 - (\alpha^2/2)7]w_1 - 3w_2 + 0.1w_2^3 \end{bmatrix},$$

and the approximate moving average along $u(t)$ is given by $\bar{h}_1(w(t)) = w_1(t)$, $\bar{h}_2(w(t)) = w_2(t)$, with $u_0(x) = h(0, w_0(x)) = w_0(x)$.

Fig. 4 presents exact and approximate moving averages $\bar{u}_2(t)$ and $\bar{h}_2(w(t))$ along $u_2(x,t)$ at $x = 1$. Simulation shows similar behavior for any $x \in (0,1)$ and for $u_1(x,t)$ as well. Thus, it is seen that evolutionary component of $u(x,t)$ is indeed described very closely by the approximate moving average $\bar{h}(w(t))$ for small ϵ. Simulations also demonstrate that as ϵ becomes smaller the difference between the exact and approximate moving averages becomes vanishingly small on the entire time interval.

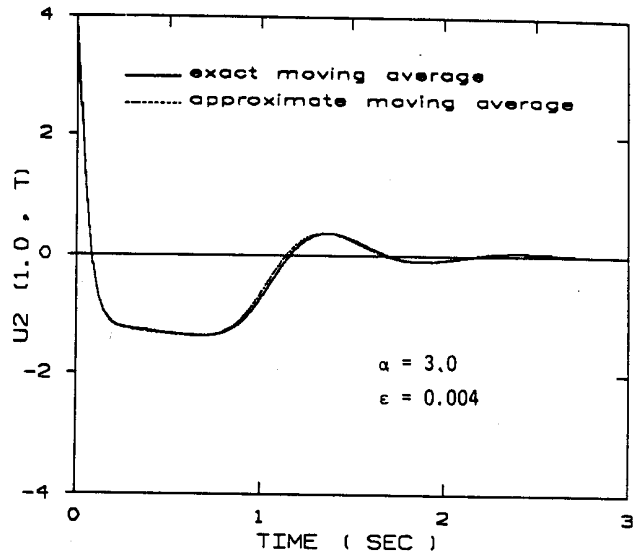

Fig. 2. Comparison of exact and approximate moving averages of solution $u_2(x,t)$ of (18) at $x = 1$.

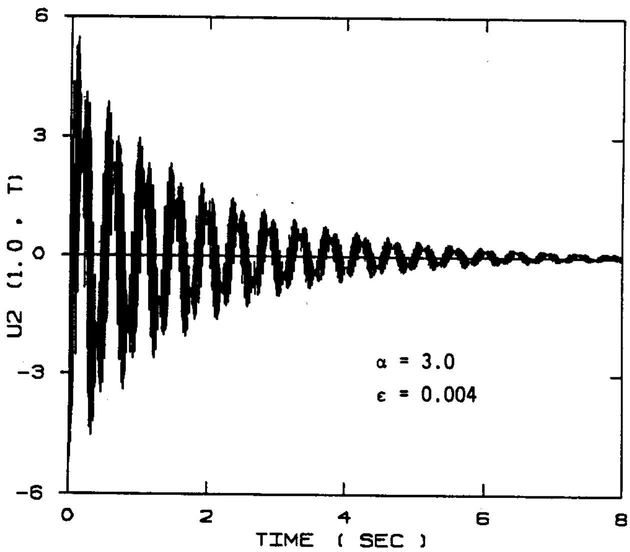

Fig. 3. Solution $u_2(x,t)$ of (20) with linear multiplicative vibrations at $x = 1$.

IV. Conclusions

This note shows that a time invariant system can be constructed whose trajectories under a time invariant coordinate transformation represent the approximate moving averages along the trajectories of the vibrationally controlled nonlinear parabolic PDE with Neumann boundary conditions. This yields a convenient tool for the analysis and improvement of the transient behavior of the vibrationally controlled DPS with NBC.

Appendix

Proof of Theorem 1—Proof of Assertion i): Equation (5) with $f(t) = (1/\epsilon)\phi(t/\epsilon)$ takes the form

$$u_t = Au_{xx} + Bu_x + C(u) + C_1\left(\frac{1}{\epsilon}\phi\left(\frac{t}{\epsilon}\right), u\right). \tag{A.1}$$

Since $C_1(\cdot,\cdot)$ is linear with respect to its first argument, (A.1) can be written in time $\tau = t/\epsilon$ as

$$u_\tau = \epsilon[Au_{xx} + Bu_x + C(u)] + C_1(\phi(\tau), u). \tag{A.2}$$

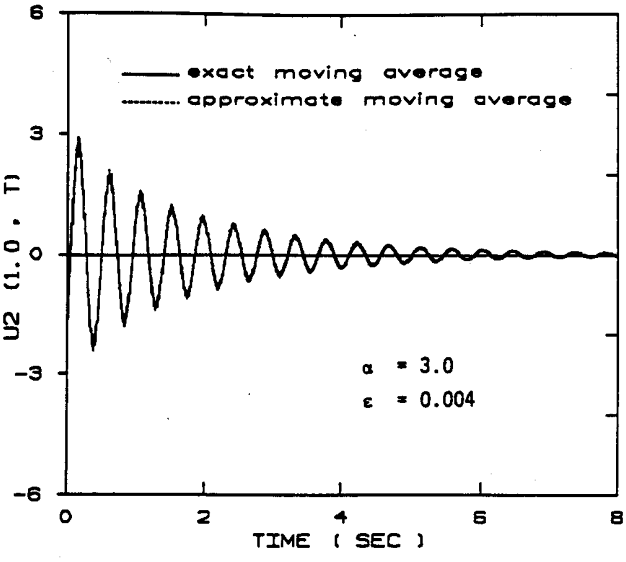

Fig. 4. Comparison of exact and approximate moving averages of solution $u_2(x, t)$ of (20) at $x = 1$.

Equation (9) with t replaced by τ yields a substitution

$$u(x,\tau) = h(\tau, v(x,\tau)) \tag{A.3}$$

where $h(\cdot,\cdot)$ is given in (10). Introducing into (A.2) substitution (A.3) yields

$$v_\tau = \epsilon[F_1(\tau,v) + F_2(\tau,v) + F_3(\tau,v)], \tag{A.4}$$

where $F_i(\cdot,\cdot)$, $i = 1,2,3$, are defined in (12). Averaging the right-hand side of (A.4) with respect to τ yields

$$w_\tau = \epsilon[P_1(w) + P_2(w) + P_3(w)] \tag{A.5}$$

where $P_i(\cdot)$, $i = 1,2,3$, are defined in (14). Since by assumption $h(\cdot,\cdot)$ is linear or affine with respect to the second argument, $P_1(w)$ and $P_2(w)$ can be represented as

$$P_1(w) = P_1' w_{xx}, \quad \text{and} \quad P_2(w) = P_2' w_x,$$

where P_1' and P_2' are constant $n \times n$ matrices. Therefore, denoting the linearization of $P_3(w)$ at $w = 0$ as $P_{30}w$, defining

$$P\psi(x) \triangleq \left(P_1'\frac{\partial^2}{\partial x^2} + P_2'\frac{\partial}{\partial x} + P_{30}\right)\psi(x), \quad \psi(x) \in H^{n1}(0,1), \tag{A.6}$$

and noting that (13) is assumed to be parabolic, operator P: $H^{n1}(0,1) \to L_2(0,1)$ is m-sectorial in the sense of Kato (cf. [9, p. 280]) or sectorial in the sense of Henry (cf. [10, p. 18]). Now, representing $P_3(w)$ in terms of linear part P_{30} at $w = 0$ and high order terms as $P_3(w) = P_{30}w + P_{3h}(w)$, (A.4) can be rewritten in time t as

$$v_t = P_1' v_{xx} + P_2' v_x + P_{30} v + P_4\left(\frac{t}{\epsilon}, v\right), \tag{A.7}$$

where

$$P_4\left(\frac{t}{\epsilon}, v\right) = \sum_{i=1}^{3} F_i\left(\frac{t}{\epsilon}, v\right) - \sum_{i=1}^{3} P_i(v) + P_{3h}(v).$$

Now by [10, theorem 3.4.9], for any $\kappa > 0$ and $\eta > 0$ there exists $\epsilon_1 > 0$ such that

$$\|v(t) - w(t)\|_1 < \eta, \quad \forall \in [0,\kappa], \tag{A.8}$$

where $v(t) = v(x,t,w_0(x),0)$ and $w(t) = w(x,t,w_0(x),0)$ are solutions of (A.7) and (A.5) in time t, respectively. Since

$$u(x,t) = h\left(\frac{t}{\epsilon}, v(x,t)\right), \tag{A.9}$$

$T = O(\epsilon)$, and u_t and v_t are of the order $O(1)$ in time $t = t/\epsilon$, we have

$$\bar{u}(t) = \frac{1}{T}\int_t^{t+T} h\left(\frac{t}{\epsilon}, v(t)\right) dt = \frac{1}{T}\int_t^{t+T} h\left(\frac{s}{\epsilon}, v(t)\right) ds + K_1(\epsilon)$$

$$= \frac{1}{T^*}\int_t^{T^*} h(s, v(t))\, ds + K_1(\epsilon), \tag{A.10}$$

where $\|K_1(\epsilon)\|_1 = O(\epsilon)$ so that $\|K_1(\epsilon)\|_1$ uniformly approaches 0 as $\epsilon \to 0$. Now, for the left-hand side of inequality (16) from (A.8) and (A.10) for any given η and κ we have

$$\|\bar{u}(t) - \bar{h}(w(t))\|_1$$

$$= \left\|\frac{1}{T^*}\int_0^{T^*} h(s,v(t))\, ds - \frac{1}{T^*}\int_0^{T^*} h(s,w(t))\, ds + K_1(\epsilon)\right\|_1$$

$$\le \frac{1}{T^*}\int_0^{T^*} \|h(s,v(t)) - h(s,w(t))\|_1\, ds + \|K_1(\epsilon)\|_1$$

$$\le \mu\frac{1}{T^*}\int_0^{T^*} \|v(t) - w(t)\|_1\, ds + \|K_1(\epsilon)\|_1$$

$$\le \mu\eta + \|K_1(\epsilon)\|_1, \quad 0 < \epsilon < \epsilon_1$$

where μ is some bounded number which exists since $h(s,q)$ is a periodic solution of (9) defined on $s \in [0,\infty)$.

Finally, since $\epsilon_0 < \epsilon_1$ and η can always be chosen so that

$$\mu\eta + \|K_1(\epsilon_0)\|_1 \le \delta, \quad \forall t \in [0,\kappa], \tag{A.11}$$

assertion i) of Theorem 1 is proven.

Proof of Assertion ii): Let $v(t) \equiv v(x,t,w_0(x),t_0)$ and $w(t) = w(x,t,w_0(x),t_0)$ be solutions of (A.4) and (A.5), respectively, in time t, with $w_0(x) \in \Omega_1$. Since by assumption $C(0,\lambda) = 0$, $F_3(\tau,0) = 0$ as well, hence (A.4) has zero equilibrium $v_s = 0$, and by uniqueness stated in assertion i), [10, p. 222], there exists ϵ_2 such that for any $0 < \epsilon \le \epsilon_2$ there is no periodic solution in the vicinity of zero other than zero itself. Then by assertion ii), [10, pp. 222] there exist positive constants, r, ϵ_3, β such that for $0 < \epsilon \le \min(\epsilon_2, \epsilon_3)$ if at some time t_1

$$\|v(t_1)\|_1 < r \tag{A.12}$$

then

$$\|v(t)\|_1 \le re^{-\beta(t-t_1)}, \quad B\forall t \ge t_1. \tag{A.13}$$

Since $P_3(0) = 0$, (A.5) has equilibrium $w_s = 0$ which must be (uniformly) asymptotically stable due to the assumption of exponentially stable linearization of (A.5) at zero. Let $D \subset H^{n1}(0,1)$ be the domain of attraction of $w_s = 0$ in (A.5), and let $\Omega_1 \subset D$. Then due to uniform asymptotic stability of $w_s = 0$ we can always choose $\kappa > 0$ such that for any given $\theta > 0$, no matter how small, a solution $w(t)$ satisfies

$$\|w(t)\|_1 \le \theta/2 \quad \text{for } t \ge \kappa/2,$$

or taking $\theta = r$, where r is that of (A.12)

$$\|w(t)\|_1 \le r/2 \quad \text{for } \kappa/2 \le t < \infty. \tag{A.14}$$

By [10, theorem 3.4.9] choose $\epsilon_1 \le \min(\epsilon_2, \epsilon_3)$ such that for $0 < \epsilon \le \epsilon_1$ and previously chosen κ

$$\|v(t) - w(t)\|_1 \le r/2, \quad t_0 \le t \le \kappa. \tag{A.15}$$

For any t we have

$$\|v(t)\|_1 \le \|v(t) - w(t)\|_1 + \|w(t)\|_1,$$

which due to (A.14) and (A.15) yields

$$\|v(t)\|_1 \le r, \qquad \kappa/2 \le t \le \kappa. \tag{A.16}$$

Thus, (A.16) shows that for every $t = t_1 \in [\kappa/2, \kappa]$ inequality (A.12) is satisfied. Consequently, taking $t_1 = \kappa/2$, from (A.13) we obtain

$$\|v(t)\|_1 < r, \qquad \kappa/2 \le t < \infty. \tag{A.17}$$

Since

$$\|v(t) - w(t)\|_1 \le \|v(t)\|_1 + \|w(t)\|_1, \qquad \forall t \in [0, \infty),$$

from (A.17) and (A.14) we have

$$\|v(t) - w(t)\|_1 \le r + \frac{r}{2} = \frac{3}{2}r, \qquad \kappa/2 \le t < \infty. \tag{A.18}$$

Combining (A.18) and (A.15) and noting a time overlap, we obtain

$$\|v(t) - w(t)\|_1 \le (3/2)r, \qquad t_0 \le t < \infty.$$

Finally, choosing $r = (2/3)\eta$, we extend the validity of inequality (16) to infinite time interval. Q.E.D.

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