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Stability Criterion For Linear Oscillatory Parabolic Systems[1]

This paper presents a stability criterion for a class of distributed parameter systems governed by linear oscillatory parabolic partial differential equations with Neumann boundary conditions. The results of numerical simulations that support the criterion are presented as well.

1 Introduction

Oscillatory systems arise in many control problems such as vibrational control, adaptive control, and robust control by periodic feedback (vibrational feedback control).

In vibrational control the qualitative change of the global behavior of attractors caused by high frequency zero mean oscillations is used to achieve control objectives (Bellman et al., 1986). This open loop technique can be used as an alternative control method when feedback and/or feedforward fail due to restrictions on sensing and actuation. Examples of vibrational control include experiments with an oscillatory quenching of plasma instabilities (Osovets, 1974), stabilization of the ionization-thermal instabilities of a CO_2 laser (Meerkov and Shapiro, 1976), exothermic reaction in a CSTR (Cinar et al., 1987), (shu et al., 1989), and laser illuminated thermochemical systems (Fakhfakh and Bentsman, 1990). Vibrational control of distributed parameter systems (DPS) has been proposed for linear hyperbolic systems in (Meerkov, 1984), and for nonlinear parabolic systems in (Bentsman, 1990) and (Bentsman and Hong, 1991).

In an adaptive system, which tunes control laws adaptively for plants with unknown parameters via either direct or indirect methods (Sastry and Bodson, 1989), the convergence of the parameters in the control laws to their nominal values is directly related to the persistency of excitation property of a reference input, which is guaranteed by selecting the reference input signal to have a certain number of frequencies (Sastry and Bodson, 1989, p. 90). The parameter convergence is also known to be critical in finite dimensional adaptive systems in terms of tolerating disturbances and unmodeled dynamics. A model reference adaptive control of an infinite dimensional system has recently been investigated, and a direct adaptive algorithm for linear parabolic partial differential equations (PDE's) assuring its exponential stability was proposed in (Hong and Bentsman, 1991).

In vibrational feedback control an additional dynamical element is inserted in the feedback loop to improve robustness. It is known that any linear time-invariant (LTI) feedback controller for plants with at least one pole and zero in the open right half-plane will have a finite gain margin. It was shown that a periodic controller may guarantee an infinite gain margin for unstable nonminimum phase LTI systems for discrete time systems in (Khagonekar et al., 1985), and for continuous time systems in (Lee et al., 1987) using the criterion of (Bellman et al., 1985).

Since many physical plants are distributed parameter systems, the rigorous extension of the strategies described above to DPS is an important problem. At present, however, there are very few results available for stability analysis of oscillatory DPS. The purpose of this paper is to extend the stability criterion of (Bellman et al, 1985) to systems described by linear oscillatory parabolic PDE's with NBC. These equations permit an approximate representation of such physical phenomena as, for example, pulsating combustion in a combustion chamber and oscillatory processes in chemical reactors. The stability criterion developed in (Bellman et al., 1985) has been used as a main tool for the synthesis of the vibrational (Bellman et al., 1986), and vibrational feedback (Lee et al., 1987; Lee and Meerkov, 1991) controllers for finite dimensional systems. Therefore the extension of the criterion of (Bellman et al., 1985) to infinite dimensional systems permits the extension of vibrational and vibrational feedback control theory to more general class of systems. In (Bellman et al., 1985) it was demonstrated that the stability properties of finite dimensional linear systems with oscillations of high amplitudes and frequencies are identical to the stability properties of specially constructed finite dimensional time invariant system. In the present paper this result is extended to DPS and it is shown that under certain conditions the stability properties of oscillatory parabolic DPS with NBC are governed by the stability properties of the linear time invariant parabolic PDE's with NBC. Also a preliminary lemma presented in Section II extends

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the stability criterion of (Nishimura and Kitamura, 1970) for time invariant PDE's with DBC to those with NBC.

A class of DPS considered in the paper is described by linear parabolic PDE's

$$u_t = Au_{xx} + Bu_x + \left(C + \frac{1}{\epsilon}F\left(\frac{t}{\epsilon}\right)\right)u,$$

$$u_x(0,t) = u_x(l,t) = 0, t \geq 0, u, (x,0) = u_0(x) \qquad (1)$$

where $u = u(x,t): R(0,1) \times R_+ \rightarrow R^n$; $x \epsilon (0,1)$; $A,B,C \epsilon R^{n \times n}$ are constant matrices; $F(\cdot)$ is a T-periodic zero average (PAZ) $n \times n$ matrix valued function; $0 < \epsilon < < 1$; subscripts of u denote corresponding partial derivatives with respect to t and x; the Neumann boundary conditions are given by

$$u_x(0,t) = u_x(1,t) = 0, t \geq 0; \qquad (2)$$

and the initial condition by $u(x,0) = u_0(x)$.

We first introduce a stability criterion for parabolic PDE's with NBC and then use this criterion and the results of (Henry, 1981) on averaging to derive a stability criterion for system (1) for sufficiently small ϵ. Finally, we present numerical examples that demonstrate the application of the criterion and the evaluation of the smallness of parameter ϵ.

Throughout the paper it is assumed that for a given initial condition $u(x, 0) = u_0(x)$ and boundary conditions (2), system (1) is well-posed in the Sobolev space $H^{n1}(0,1)$ of vector functions $v(x) = [v_1(x), \ldots, v_n(x)]^T$ with components $v_i(x)$ in $L_2(0,1)$ which have the first distributional derivatives, or $u(x,t) \epsilon L^2(0,T: H^1(0,1))$. Norm on $H^{n1}(0,1)$ is defined as

$$\|v\|_1 \triangleq \left(\int_0^1 v_x^T(x)v_x(x) + v^T(x)v(x)dx\right)^{1/2}. \qquad (3)$$

Superscripts "n" and "1" in H^{n1} (0,1) indicate the dimension of the vector $v(x)$ and the order of the highest derivative with respect to x in the definition of norm (3).

II Stability Criterion

A. Preliminary Lemma

Lemma : The null solution of the linear system

$$u_t = Au_{xx} + Bu_x + Cu,\ A,B,C \epsilon R^{n \times n},\ u \epsilon H^{n1}(0,1), \qquad (4)$$

with Neumann boundary conditions (2) is asymptotically stable with respect to the norm (3) if there exists a positive definite matrix $M \epsilon R^{n \times n}$ such that (i) $A^TM + MA$ is a positive definite matrix; (ii) $B^TM = MB$; (iii) $C^TM + MC$ is a negative definite matrix; (iv) C is a Hurwitz matrix.

Proof: Proof is given in the Appendix.

Remark: Under no restrictions on matrix M, the existence of positive definite matrix M that satisfies condition (iii) is implied by condition (iv). However, under the restrictions imposed by conditions (i) and (ii) this is not necessarily the case. Therefore condition (iii) can not be omitted. Also, condition (iii) does not imply condition (iv) since there might exist another positive matrix M_1 such that $M_1 \neq M$ but $C^TM_1 + M_1C = C^TM + MC$, which implies that C is not Hurwitz.

B. Main Results

Theorem: Assume that (1) is parabolic and $u \epsilon H^{n1}(0,1)$. Then there exists a constant $\epsilon_0 > 0$ such that for any $\epsilon \epsilon (0, \epsilon_0]$, the null solution $u = 0$ of (1) is asymptotically stable if (i) there exists a PAZ matrix $F(t)$ such that a fundamental matrix $\Phi(t), t \epsilon (-\infty, \infty)$, of $\dot{y} = F(t)y$, $y: R \rightarrow R^n$, is periodic, and (ii) there exsits a positive definite matrix M such that 1) $\overline{A}^TM + M\overline{A}$ is positive definite; 2) $M\overline{B} = \overline{B}^TM$; 3) $\overline{C}$ is a Hurwitz matrix; 4) $\overline{C}^TM + M\overline{C}$ is negative definite; where

$$\overline{A} \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi^{-1}(t)A\Phi(t)dt,$$

$$\overline{B} \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi^{-1}(t)B\Phi(t)dt,$$

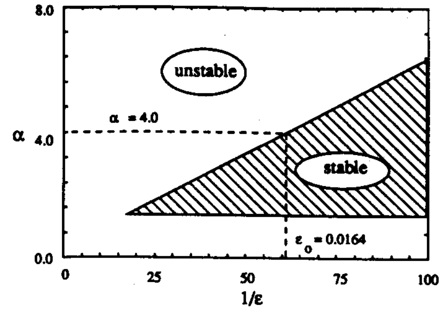

Fig. 1 Stability-instability boundary of system (8) in the $\alpha - 1/\epsilon$ plane

$$\overline{C} \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi^{-1}(t)C\Phi(t)dt. \qquad (6)$$

Proof: Proof of the theorem is given in the Appendix.

III Numerical Examples

Example 1. Consider (1) with

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 7 \\ 3 & -3 \end{bmatrix}, F(t) = \begin{bmatrix} 0 & 0 \\ \alpha\cos & 0 \end{bmatrix}, \qquad (7)$$

the initial conditions $u_1(x,0) = \cos\pi x$, $u_2(x,0) = -\cos\pi x$, and boundary conditions $u_1(0,t) = u_1(1,t) = 0$, $u_2(0,t) = u_2(1,t) = 0$. Clearly $F(t)$ is a PAZ matrix and a fundamental matrix $\Phi(t)$ in the theorem can be taken as

$$\Phi(t) = \begin{bmatrix} 1 & 0 \\ \alpha\sin t & 1 \end{bmatrix}. \qquad (8)$$

Since A and B are diagonal matrices, $\overline{A} = A$ and $\overline{B} = B$. Hence the conditions 1), 2) of the theorem are satisfied without any restriction on M. The matrix $\overline{C}$ is given as

$$\overline{C} = \begin{bmatrix} 2 & 7 \\ 3 - 3.5\alpha^2 & -3 \end{bmatrix}. \qquad (9)$$

Therefore, $\overline{C}$ is Hurwitz for every $\alpha > 1.050$. The existence of M satisfying condition iii) is guaranteed by the Lyapunov equation (Vidyasagar, 1978, Theorem 55, p. 175). Now by the theorem given above for every $\alpha > 1.050$ there exists an ϵ_0 such that the system (7) will be asymptotically stable for any $\epsilon \epsilon (0, \epsilon_0]$. This is indeed demonstrated in Fig. 1 which shows the stability-instability boundary in the α versus $1/\epsilon$ plane. For example, for $\alpha = 4.0$, the dotted line in Fig. 1 indicates that $\epsilon_0 = 0.0164$, Fig. 2 shows the asymptotically stable solution $u_1(x,t)$ for $\alpha = 2.0$ and $\epsilon = 0.016$. For the purpose of comparison Fig. 3 demonstrates the unstable behavior of $u_1(x,t)$ when $\alpha = 0$.

Example 2. Consider (1) with

$$A = \begin{bmatrix} 1 & 0.5 \\ 0.3 & 2 \end{bmatrix}, B = 0, C = \begin{bmatrix} 1 & 45 \\ 16 & -30 \end{bmatrix}, F(t) = \begin{bmatrix} 0 & 0 \\ \alpha\sin t & 0 \end{bmatrix}, \qquad (10)$$

the initial conditions $u_1(x,0) = 1$, $u_2(x,0) = -1$, and the same boundary conditions as Example 1. The matrix $\Phi(t)$ can be taken as

$$\Phi(t) = \exp\left\{\alpha \begin{bmatrix} 0 & 0 \\ \cos t & 0 \end{bmatrix}\right\}. \qquad (11)$$

And the matrix $\overline{C}$ is given as

$$\overline{C} = \begin{bmatrix} 1 & 45 \\ 16 - 22.5\alpha^2 & -30 \end{bmatrix}. \qquad (12)$$

Therefore for $\alpha > 0.861$, the system (10) is asymptotically stable for sufficiently small ϵ_0. Fig. 4 shows the stability-instability boundary of the system (10). Figure 5 depicts an asymptotically

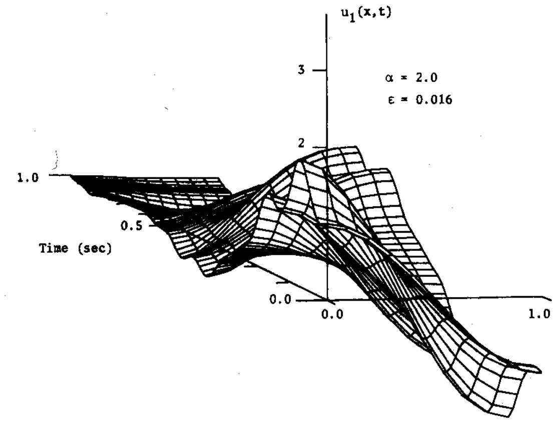

Fig. 2 Asymptotically stable solution $u_1(x,t)$ of system (8) for $\alpha=2.0$ and $\epsilon=0.016$

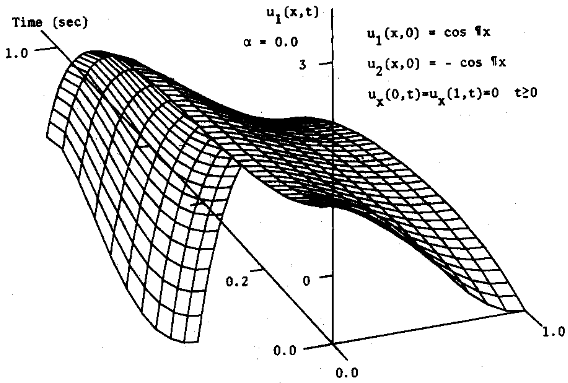

Fig. 3 Unstable solution $u_1(x,t)$ of system (8) with $\alpha=0.0$

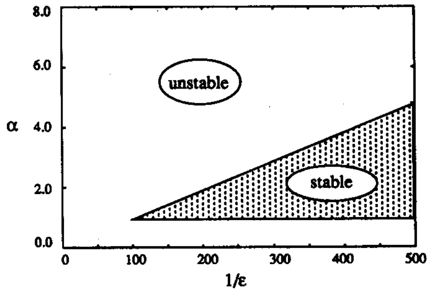

Fig. 4 Stability-instability boundary of system (12) in the $\alpha-1/\epsilon$ plane

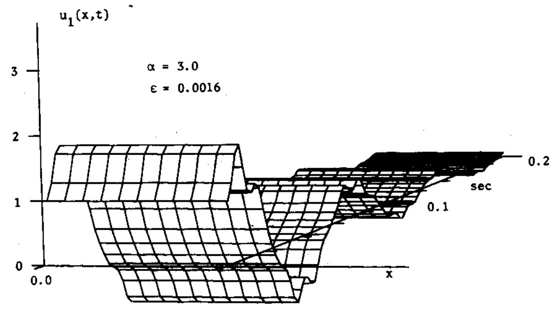

Fig. 5 Asymptotically stable solution $u_1(x,t)$ of system (12) for $\alpha=3.0$ and $\epsilon=0.016$

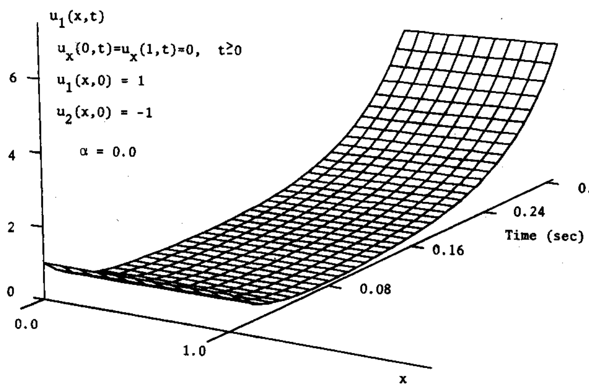

Fig. 6 Unstable solution $u_1(x,t)$ of system (12) with $\alpha=0.0$

stable solution for $\alpha=3.0$ and $\epsilon=0.0016$. Instability is shown in Fig. 6 for $\alpha=0$.

IV Conclusions

This paper presents a stability criterion for a class of linear oscillatory parabolic DPS with Neumann boundary conditions. The examples that demonstrate the application of the criterion are given as well. The criterion might be useful in the synthesis of periodic feedback and vibrational controllers for distributed parameter systems.

APPENDIX

Proof of Lemma: Let

$\tilde{u}(x,t) \triangleq u(x,t)-p(t)$,

$$p(t)=[p_1(t),\ldots,p_n(t)]^T,\ p(t)\triangleq\int_0^1 u(x,t)dx. \quad \text{(A.1)}$$

Consider a functional with a positive definite matrix M such that

$$V(t)=\int_0^1 \tilde{u}_x^T M\tilde{u}_x(x,t)dx=\int_0^1 u_x^T M u_x dx. \quad \text{(A.2)}$$

Differentiating (A.2) with respect to t, integrating by parts, and making use of the given Neumann boundary conditions yields

$$\frac{dV}{dt}=-\int_0^1 u_{xx}^T(A^TM+MA)u_{xx}dx+\int_0^1 u_{xx}^T(B^TM-MB)u_x dx$$
$$+\int_0^1 u_x^T(C^TM+MC)u_x dx. \quad \text{(A.3)}$$

For the linear set of functions $\phi(x)$ continuous with their first derivatives in the closed interval [0,1] with $\phi(0)=\phi(1)=0$, the following Poincaré inequality holds (Reddy, 1986)

$$\int_0^1\left(\frac{d\phi}{dx}\right)^2 dx\geq\left(\frac{\pi}{4}\right)^2\int_0^1\phi^2 dx. \quad \text{(A.4)}$$

If we consider $d\phi/dx$ to be a generalized derivative of $\phi(\cdot)$, then inequality (A.4) also holds for $\phi\in H_0^1(0,1)$ and consequently, when the condition (i) of Lemma 1 is satisfied, with $\phi\rightarrow u_x$ it follows from (A.4) and NBC in (1) that

$$\int_0^1 u_{xx}^T(A^TM+MA)u_{xx}dx\geq\left(\frac{\pi}{4}\right)^2\int_0^1 u_x^T(A^TM+MA)u_x dx. \quad \text{(A.5)}$$

Further, if the conditions (ii) and (iii) of the Lemma hold, from (A.3) and (A.5) it follows that

$$\frac{dV}{dt} \leq \int_0^1 u_x^T [C^T M + MC - \left(\frac{\pi}{4}\right)^2 (A^T M + MA)] u_x dx < 0. \quad (A.6)$$

Therefore, $u_x(x,t)$ and hence $\tilde{u}_x(x,t)$ converges to 0 almost everywhere, which in turn implies that $\tilde{u}(x,t)$ converges to 0 pointwise. Further, since $p_x(t) = p_{xx}(t) \equiv 0$, convergence properties of $p(t)$ are defined by the spectrum of matrix C, therefore, if condition (iv) of the Lemma is satisfied, $p(t)$ converges to 0, and consequently $u(x,t)$ converges to zero pointwise. Thus, if all the conditions of the Lemma hold, the null solution of (7) is globally asymptotically stable with respect to the norm (5). Q.E.D.

Proof of the Theorem: System (1) can be rewritten in time $\tau = t/\epsilon$ as

$$u_\tau = \epsilon[Au_{xx} + Bu_x + Cu] + F(\tau)u. \quad (A.7)$$

Fundamental solution $\Phi(\cdot)$ of the equation $\dot{y} = F(t)y$ yields the following substitution with t replaced by τ.

$$u(x,\tau) = \Phi(\tau)v(x,\tau). \quad (A.8)$$

Substituting (A.8) into (A.7) yields

$$v_\tau = \epsilon[\Phi^{-1}(\tau)A\Phi(\tau)v_{xx} + \Phi^{-1}(\tau)B\Phi(\tau)v_x + \Phi^{-1}(\tau)C\Phi(\tau)v]$$

$$\triangleq \epsilon P(\tau)v. \quad (A.9)$$

Assuming that the condition (i) of the Theorem holds and averaging the right-hand side of (A.9) with respect to τ yields.

$$w_\tau = \overline{A}w_{xx} + \overline{B}w_x + \overline{C}w, \quad (A.10)$$

where $\overline{A}$, $\overline{B}$, and $\overline{C}$ are defined in (6). Defining

$$\overline{P}\psi(x) \triangleq \left(\overline{A}\frac{\partial^2}{\partial x^2} + \overline{B}\frac{\partial}{\partial x} + \overline{C}\right)\psi(x), \psi(x) \epsilon H^{n1}(0,1), \quad (A.11)$$

and noting that (A.10) is parabolic, operator $\overline{P}$: $H^{n1}(0,1) \rightarrow L_2(0,1)$ is m-sectorial in the sense of Kato (1976, p.280) or sectorial in the sense of Henry (1981, p.18). Equation (A.9) can be rewritten in time t as

$$v_t = \overline{A}v_{xx} + \overline{B}v_x + \overline{C}v + P\left(\frac{t}{\epsilon}, v\right), \quad (A.12)$$

where

$$P\left(\frac{t}{\epsilon}, v\right) \triangleq P\left(\frac{t}{\epsilon}\right)v - \overline{P}v$$

Representing (A.12) as an evolution equation in Sobolev space $H^{n1}(1,0)$, assertion (ii) on p. 222 of (Henry, 1981) is applicable to (A.12). Indeed, if the conditions (1), (2), and (3) of the Theorem hold, then all the conditions of the Lemma are satisfied as well and, consequently, the null solution of (A.10) is asymptotically stable. Therefore spectrum of the linear operator P in the evolution lies in the open left half-plane. Identifying $P(t/\epsilon, v)$ in (A.12) with $f(t,x)$ on p. 222 of (Henry, 1981) and following the notation of (Henry, 1981) we have $\partial f_0(0)/\partial x = 0$ where

$$f_0(x) = \frac{1}{T^*}\int_0^{T^*} f(t,x)dt, T^* = T\epsilon \quad (A.13)$$

where T is the period of $\Phi(\tau)$. Consequently, defining $\epsilon \triangleq 1/\omega$, due to assertions (i) and (ii) on p. 222 of (Henry, 1981) for any given $\eta > 0$ there exists ω_0 and, hence $\epsilon_0 = 1/\omega_0 = \epsilon_0(\eta)$ such that for any $0 < \epsilon < \epsilon_0$ or $\omega > \omega_0$ (A.12) has a unique T^* – periodic asymptotically stable solution $v^s(t,x)$ for which

$$\sup \| v^s(x,t) \|_1 < \eta. \quad (A.14)$$

Finally asymptotic stability of 0 in (1) follows from asymptotic stability of $v_s(x,t)$, stability reserving maping $\Phi(t/\epsilon)$ $v^s(x,t)$ and the uniqueness of the asymptotically stable solution of (A.12) in the vicinity of 0 which must be the null solution itself, since it is already known to exist. Q.E.D.

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References

Bellman, R. E., Bentsman, J., and Meerkov, S. M., 1985, "Stability of Fast Periodic Systems," *IEEE Transaction on Automatic Control*, Vol. AC-30, pp. 289–291.

Bellman, R. E., Bentsman, J., and Meerkov, S. M., 1986, "Vibrational Control of Nonlinear Systems: Vibrational Stabilizability, "*IEEE Transaction on Automatic Control*, Vol. AC-31, pp. 710–716.

Bentsman, J., 1990, "Oscillations-Induced Transitions and Their Application in Control of Dynamical Systems," ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 112, pp. 313–319.

Bentsman, J., and Hong, K. S., 1991, "Vibrational Stabilization of Nonlinear Parabolic Systems with Neumann Boundary Conditions," *IEEE Transaction on Automatic Control*, Vol. AC-36, No. 4, pp. 501–507.

Cinar, A., Deng, J., Meerkov, S. M., and Shu, X., 1987, "Vibrational Control of an Exothermic Reaction in a CSTR: Theory and Experiment," *AIChE Journal*, Vol. 33, No. 3, pp. 353–365.

Fakhfakh, J., and Bentsman, J., 1990, "Experiment with Vibrational Control of a Laser Illuminated Thermochemical System," ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 112, pp. 42–47.

Henry, D., 1981, *Geometric Theory of Semilinear Parabolic Equations*, Berlin, Springer-Verlag.

Hong, K. S., and Bentsman, J., 1991, "Direct Adaptive Control of Parabolic Systems: Algorithm Synthesis, and Convergence and Stability Analysis," Submitted to *IEEE Transaction on Automatic Control*.

Kato, T., 1976, *Perturbation Theory for Linear Operators*, Berlin, Springer-Verlag.

Khargonekar, P., Poolla, K., and Tannenbaum, A., 1985, "Robust Control of Linear Time-Invariant Plants using Periodic Compensation," *IEEE Transaction on Automatic Control*, Vol. AC-30, pp. 1088–1096.

Lee, S., Meerkov, S. M., and Runolfsson, T., 1987, "Vibrational-Feedback Control: Zero Placement Capabilities," *IEEE Transaction on Automatic Control*, Vol. AC-32, No. 7, pp. 604–611.

Lee, S., and Meerkov, S. M., 1991, "Vibrational Feedback Control in the Problem of Absolute Stability," *IEEE Transaction on Automatic Control*, Vol. AC-36, pp. 482–485.

Meerkov, S. M., 1984, "Vibrational Stabilizability of Distributed Parameter Systems, *J. Math. Anal. Appl.*, Vol. 92, pp. 408–418.

Meerkov, S. M., and Shapiro, G. I., 1976, "Method of Vibrational Control in the Problem of Stabilization of Ionization-Thermal Instability of a Powerful Continuous CO2 Laser," *Automatic Remote Control*, Vol. 37, pp. 821–830.

Nishimura, M., and Kitamura, S., 1970, "Stability Criterions for a Class of Distributed Parameter Systems," *IEEE Transaction on Automatic Control*, pp. 145–146.

Osovets, S. M., 1974, "Dynamic Methods of Retention and Stabilization of Hot Plasma," *Soviet Physics Uspdkhi*, Vol. 112, pp. 637–683.

Reddy, J. N., 1986, *Applied Functional Analysis and Variational Methods in Engineering*, McGraw-Hill, New York.

Sastry, S. S., and Bodson, M., 1989, *Adaptive Control: Stability, Convergence and Robustness*, Prentice Hall, Englewood Cliffs, New Jersey.

Shu, X., Rigopoulos, K., and Cinar, A., 1989, "Vibrational Control of an Exothermic CSTR: Productivity Improvement by Multiple Input Oscillations," *IEEE Transaction on Automatic Control*, Vol. 34, pp. 193–195.

Vidyasagar, M., 1978, *Nonlinear System Analysis*, Prentice-Hall, Englewood Cliffs, NY.