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Vibrational Control of Nonlinear Time Lag Systems: Vibrational Stabilization and Transient Behavior*

JOSEPH BENTSMAN,†‡ KEUM S. HONG† and JAMEL FAKHFAKH†

Periodic excitations introduced into time lag systems can stabilize unstable equilibria or induce stable periodic solutions with the desired properties. The resulting open-loop technique termed vibrational control can be useful when on-line measurements are not available.

Key Words—Nonlinear systems; time lag systems; stability.

Abstract—This paper addresses two problems of control of nonlinear time lag systems: (i) an existence and a synthesis of parametric vibrations for their stabilization and (ii) the transient behavior analysis of the vibrationally controlled nonlinear systems with time lags. In this work, stabilizability conditions for two vibration types are formulated and procedures for the synthesis of the corresponding stabilizing vibrations are proposed. The method for the transient behavior analysis of vibrationally controlled systems on a finite time interval is developed as well. Several examples are given to support the theory presented.

I. INTRODUCTION

VIBRATIONAL CONTROL is an open-loop technique that utilizes parametric excitation of a dynamical system for achieving control objectives. A well-known example of the vibrational control effect is a stabilization of an inverted pendulum by vertical oscillations of its support. Obviously, in this case there is no interference into the plant structure, and the control objective to keep the pendulum in the upright position is achieved by much simpler means than using feedback. An extensive theoretical and experimental comparison of vibrational control with feedback and feedforward control strategies is given by Meerkov (1980), Cinar *et al.* (1987), Bentsman and Hvostov (1988) and Fakhfakh and Bentsman (1990). These studies show that being an open-loop technique, vibrational control can (1) stabilize the plants when on-line measurements and hence feedback, are impossible, such as in powerful continuous CO_2 lasers and particle accelerators; or (2) under the practical restrictions on sensing and actuation create desired stable operating regimes unattainable by feedback for such plants as chemical reactors and laser illuminated reactions. The mathematical machinery of vibrational control has also found important applications in the synthesis of linear periodic feedback controllers (Lee *et al.*, 1987) that ensure an infinite gain margin in the robust stabilization of the nonminimum phase plants with the right half plane poles, which is not possible with a linear time invariant feedback (cf. Khargonekar *et al.*, 1985).

The theory of vibrational control for systems governed by linear and nonlinear ordinary differential equations has been developed by Meerkov (1980), and Bellman *et al.* (1986a, b) and Bentsman (1987), respectively. However, many physical systems with nonlinear behavior such as chemical reactions and combustion processes have time delayed states (cf. Ray 1981; Kolmanovskii and Nosov, 1986). This motivates studies of oscillatory stabilizing effects in nonlinear systems with time lags.

This paper introduces the first results in vibrational control of nonlinear systems with time delays: conditions for the existence of stabilizing vibrations and a procedure for their synthesis are given for a class of nonlinear time lag systems, a method for the transient behavior analysis of vibrationally controlled systems is proposed, and examples of vibrational stabi-

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† Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, 1206 West Green Street, Urbana, IL 61801, U.S.A.
‡ Author to whom all correspondence should be addressed.

lization of nonlinear time lag systems are presented.

In this paper a class of nonlinear systems with a finite number of constant delays is considered which is described by the equation

$$\dot{x}(t)=\sum_{i=1}^{m} P_i(x(t), x(t-d_i), \lambda),$$

$$P_i: R^n \times R^n \times R^l \to R^n, \quad \dot{x}(t) \triangleq \frac{dx}{dt}, \quad i=1,\ldots,m,$$

$$P_i(x(t), x(t-d_i), \lambda) = [p_{i1}(x(t), x(t-d_i), \lambda), \ldots, p_{in}(x(t), x(t-d_i), \lambda)]^T,$$

$$P_i(0,0,\lambda)=0; \quad i=1,\ldots,m, \tag{1}$$

where $x \in R^n$ is a state, $\lambda=[\lambda_1,\ldots,\lambda_l]^T$ are parameters subjected to vibrations, t is dimensionless time, and d_i, $i=1,\ldots,m$, are time lags of the order $O(\varepsilon)$, $0<\varepsilon \ll 1$. The paper follows the terminology of Bellman *et al.* (1986a, b).

Introduce into (1) parametric vibrations according to the law

$$\lambda(t)=\lambda_0+f(t) \tag{2}$$

where λ_0 is a constant vector and $f(t)$ is a periodic zero average (PAZ) vector. Then, (1) takes the form

$$\dot{x}(t)=\sum_{i=1}^{m} P_i(x(t), x(t-d_i), \lambda_0+f(t)). \tag{3}$$

Throughout the paper it will be assumed that (3) can be represented as

$$\dot{x}(t)=\sum_{i=1}^{m} P_i(x(t), x(t-d_i), \lambda_0)+Q(f(t), x(t)) \tag{4}$$

where $Q(\cdot,\cdot)$ is a vector function linear with respect to its first argument.

Following Bellman *et al.* (1986a, b), if $Q(f(t), x(t))=l(t)$, where $l(t)$ is a PAZ vector, the introduced vibrations are referred to as *vector additive*, if $Q(f(t), x(t))=D(t)x(t)$, where $D(t)$ is an $n\times n$ PAZ matrix, the vibrations are called *linear multiplicative*, and if $Q(f(t), x(t))=D(t)X(x(t))$, where $X: R^n \to R^n$ is a nonlinear map, the vibrations are termed *nonlinear multiplicative*. In the present paper, we consider vibrational stabilization and transient behavior of a class of nonlinear systems (4) with time delays d_i, $i=1,\ldots,m$, of the same order of magnitude as the period of vibrations and with linear multiplicative and vector additive vibrations. The proofs of all formal statements are given in the Appendix.

II. VIBRATIONAL STABILIZATION

Assume that (1) has an equilibrium point $x_s(t)=x_s=\text{const.}$ for a fixed λ_0 (note that $x_s(t)=x_s(t-d_i)=x_s$).

Definition 1. An equilibrium point x_s of (1) is said to be *vibrationally stabilizable* (*v-stabilizable*) if for any $\delta>0$ there exists a PAZ vector $f(t)$ such that (3) has an asymptotically stable almost periodic solution $x^s(t)$, $-\infty<t<\infty$, characterized by

$$\|\bar{x}^s - x_s\|<\delta, \quad \bar{x}^s \equiv \overline{x^s(t)} \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T x^s(t)\,dt. \tag{5}$$

Definition 2. An equilibrium point x_s of (1) is said to be *totally vibrationally stabilizable* (*t-stabilizable*) if it is v-stabilizable and in addition $x^s(t)=\text{const.}=x_s$, $-\infty<t<\infty$.

The problem of vibrational stabilization consists of: (1) Finding the conditions for the *existence* of stabilizing vibrations (v- and t-stabilizability) and (2) finding the actual *parameters* of vibrations that ensure the desired stabilization.

In order to address this problem for zero equilibrium of system (1) introduce system

$$\dot{x}(t)=\sum_{i=1}^{m} P_i(x(t), x(t-\varepsilon r_i), \lambda) \tag{6}$$

which is obtained from (1) by replacing d_i by εr_i, $r_i=O(1)$, $i=1,\ldots,m$. In the discussion below, vibrational stabilization of (6) will be first considered and then related to that of (1).

A. *Linear multiplicative vibrations*

Define

$$A \triangleq \sum_{i=1}^{m} \left.\frac{\partial P_i(\xi,\eta,\lambda_0)}{\partial \xi}\right|_{\xi=\eta=x_s=0}$$

and

$$B_i \triangleq \left.\frac{\partial P_i(\xi,\eta,\lambda_0)}{\partial \eta}\right|_{\xi=\eta=x_s=0}. \tag{7}$$

Denote by $\Phi(t)$ a fundamental matrix solution of the equation

$$\dot{x}(t)=F(t)x(t) \tag{8}$$

and introduce an ordinary differential equation

$$\dot{z}(t)=Rz(t) \tag{9}$$

where R is defined as follows:

$$R \triangleq \bar{A}+\sum_{i=1}^{m}\bar{B}_i \tag{10}$$

and $\bar{A}$ and $\bar{B}_i$ are computed as

$$\bar{A} = \overline{\Phi^{-1}(t)A\Phi(t)}$$

$$\triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi^{-1}(t)A\Phi(t)\,dt, \qquad (11)$$

and

$$\bar{B}_i = \overline{\Phi^{-1}(t)B_i\Phi(t-r_i)}, \quad i=1,\ldots,m. \qquad (12)$$

Theorem 1. Assume that there exists a sufficiently large set $\Omega \subset R^n$, $0 \in \Omega$, such that $P_i(\xi, \eta, \lambda_0)$ introduced in (1) is continuously differentiable for all $\xi, \eta \in \Omega$; $\forall i = 1, \ldots, m$; then

(i) The zero equilibrium $x_s = 0$ of (6) is t-stabilizable if there exists a PAZ matrix $F(t)$ such that a fundamental matrix $\Phi(t)$, $t \in (-\infty, \infty)$, of $\dot{x} = F(t)x$ is almost periodic, and R defined in (10) is a Hurwitz matrix;

(ii) There exists positive $\varepsilon_0 = \text{const.}$ such that $x_s = 0$ of (6) is t-stabilizable by linear multiplicative vibrations $D(t)x(t) = (1/\varepsilon)F(t/\varepsilon)x(t)$, $0 < \varepsilon \leq \varepsilon_0$, where $F(t)$ satisfies all the conditions of assertion (i) above.

Corollary 1. Let an assumption of Theorem 1 hold for system (1) with fixed delays d_i, $i = 1, \ldots, m$. Suppose that there exists a set of constants r_i, $i = 1, \ldots, m$, such that $d_i/r_i = d_j/r_j$, $i, j = 1, \ldots, m$, and $x_s = 0$ of system (6) with these constants r_i, $i = 1, \ldots, m$, is t-stabilizable by linear multiplicative vibrations $(1/\varepsilon)F(t/\varepsilon)x(t)$, $0 < \varepsilon \leq \varepsilon_0$. Then the zero equilibrium of (1) is t-stabilizable by linear multiplicative vibrations $(1/\varepsilon_1)F(t/\varepsilon_1)x(t)$, $\varepsilon_1 \triangleq d_i/r_i$, if $\varepsilon_1 \leq \varepsilon_0$.

Remark 1. The significance of Corollary 1 is in that it relates the vibrational stabilization of system (1) with fixed delays d_i to the vibrational stabilization of system obtained from (1) by replacing delays d_i by quantities εr_i, $d_i/r_i = d_j/r_j$, $i, j = 1, \ldots, m$, with fixed r_is and varying ε. This leads to a constructive method for the synthesis of the stabilizing vibrations, since for fixed r_i, $i = 1, \ldots, m$, Theorem 1 decomposes this synthesis into a two-stage procedure. First, PAZ matrix $F(t)$ is sought which makes R Hurwitz.

Once the desired matrix $F(t)$ is found, the second stage consists of a one-parameter (ε) computer search where equation (4) with linear multiplicative vibrations $D(t)x(t) = (1/\varepsilon)F(t/\varepsilon)x(t)$ and $d_i \to \varepsilon r_i$, $i = 1, \ldots, m$, $d_i/r_i = d_j/r_j$, $\forall i, j = 1, \ldots, m$, is simulated until ε_0 is found for which stability is achieved. Such ε_0 will necessarily exist for fixed r_i, $i = 1, \ldots, m$, due to assertion (ii) of Theorem 1. If $d_i/r_i \equiv \varepsilon_1 \leq \varepsilon_0$, then the stabilizing vibrations for (4) with the original delays d_i, $i = 1, \ldots, m$, are given by $D(t)x(t) = (1/\varepsilon_1)F(t/\varepsilon_1)x(t)$.

Remark 2. Analytical estimates of an upper bound on ε_0 are usually extremely conservative (cf. Bellman *et al.*, 1985), therefore the value of ε_0 for a given nonlinear time lag system of the form (6) is best determined by a numerical simulation as described in the previous remark.

Remark 3. The choice of a fundamental matrix $\Phi(t)$ in (8) is immaterial and is dictated only by convenience. Indeed, since any fundamental matrix $\Phi(t)$ of (8) is related to any other fundamental matrix of (8), say $\Phi_1(t)$, via a constant nonsingular matrix (denote it as C) as $\Phi(t) = \Phi_1(t)C$, from (10) by direct substitution we have

$$R = C^{-1}R_1C, \qquad (13)$$

where

$$R_1 = \overline{\Phi_1^{-1}(t)A\Phi_1(t)} + \sum_{i=1}^{m} \overline{\Phi_1^{-1}(t)B_i\Phi_1(t-r_i)}.$$

Thus, R_1 is computed exactly as in (10) but with $\Phi(t)$ replaced by $\Phi_1(t)$, and due to (13) R and R_1 have identical spectra.

Remark 4. The assumption of almost periodicity of $\Phi(t)$ in t guarantees the existence of the averages (11) and (12). The elimination of this assumption results in admitting zero mean matrices $F(t)$ for which $\Phi(t)$ is unbounded. In this case (11) and (12) do not exist. One could potentially use Lyapunov's substitution to transform $\dot{x} = F(t)x$ into a time invariant system, find Floquet multipliers, and reject matrices $F(t)$ that give rise to unbounded $\Phi(t)$. However, currently there is no constructive method for finding closed form Lyapunov's substitutions for a general class of systems $\dot{x} = F(t)x$ with periodic zero mean $F(t)$.

Remark 5. Theorem 1 and Corollary 1 show that as in the case of nonlinear systems with no delays (cf. Bellman *et al.*, 1986a), the conditions of t-stabilizability of zero equilibrium of (6) and hence of (1) by linear multiplicative vibrations depend only on the properties of the linearization of (6) or of (1) at zero. However, for other types of vibrations, this is not true as will be shown further in the paper.

Remark 6. Theorem 1 also provides a clue to the robustness of t-stabilizability properties with respect to small delays. It is seen that if the delays are of the same order of magnitude as the period of oscillations, they cannot be neglected, except for the special system structures, since quantities $r_i = d_i/\varepsilon = O(1)$ significantly affect the spectrum of matrix R, as can be seen from its definition (10).

An example that demonstrates stabilization of a nonlinear time delay system by linear multiplicative vibrations is given next.

Example 1. Duffing equation with time lags. Consider a scalar Duffing equation with time lag $d = \text{const}$ in the states

$$\ddot{x}(t) + a_1\dot{x}(t) + a_2\dot{x}(t-d) - b_1x(t) - b_2x(t-d) + c_1x^3(t) + c_2x^3(t-d) = 0,$$
$$a_1, a_2, b_1, b_2, c_1, c_2 > 0, \qquad (14)$$

which in the state space form is given by

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -a_1x_2(t) - a_2x_2(t-d) + b_1x_1(t) + b_2x_1(t-d) - c_1x_1^3(t) - c_2x_1^3(t-d),\\ x_1 &\triangleq x, \quad x_2 \triangleq \dot{x}.\end{aligned} \qquad (15)$$

Equation (14) can serve as an approximate model of an inverted pendulum with a cavity filled with viscous liquid. The term with the delayed first derivative describes dissipation of the mechanical energy in the system due to viscous liquid. This term is a lumped approximation of a more complicated description of this dissipation by the retarded integrodifferential term given by Strizak (1982, p. 238). The linearization of (15) at $x_s = 0$ has the form

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= -a_1x_2(t) - a_2x_2(t-d) + b_1x_1(t) + b_2x_1(t-d).\end{aligned} \qquad (16)$$

Introducing vibrations $f(t) = (\alpha/\varepsilon)\cos(t/\varepsilon)$ into coefficient b_1,

$$b_1 \rightarrow b_1 + \frac{\alpha}{\varepsilon}\cos(t/\varepsilon), \qquad (17)$$

equation (8) is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ \alpha\cos t & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}, \qquad (18)$$

and its fundamental matrix is

$$\Phi(t) = \begin{bmatrix}1 & 0\\ \alpha\sin t & 1\end{bmatrix}. \qquad (19)$$

Replacing in (14) d by εr and computing R of (10) we obtain a corresponding averaged equation (9) given by

$$\ddot{z}(t) + r_1\dot{z}(t) + r_2z(t) = 0 \qquad (20)$$

where

$$r_1 = a_1 + a_2, \quad r_2 = -(b_1 + b_2) + \frac{\alpha^2}{2}.$$

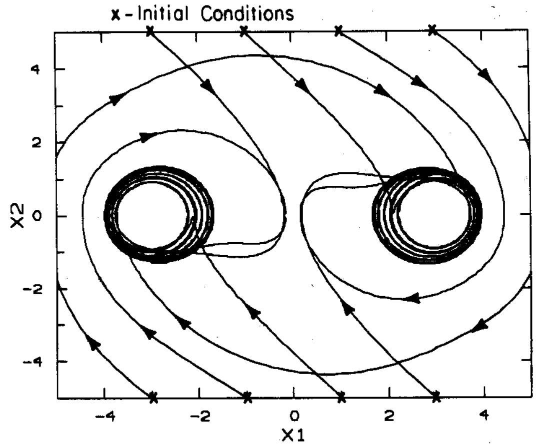

FIG. 1. Phase portrait of Duffing equation with a time lag $d = 0.4\pi$.

Therefore equation (15) with $d \rightarrow \varepsilon r$, $r = O(1)$, and vibrations introduced into the coefficient b_1 as in (17) has a stable zero equilibrium point when $a_1 + a_2 > 0$ and $\alpha > [2(b_1 + b_2)]^{1/2}$ for any positive ε smaller than some ε_0, the existence of which for fixed r is guaranteed by Theorem 1. Consider the case when $a_1 = 0.6$, $a_2 = 0.4$, $b_1 = b_2 = 0.5$, and $c_1 = c_2 = 0.05$. Then for $a > \sqrt{2}$, $r = \text{const.} = O(1)$, and $0 < \varepsilon \leq \varepsilon_0$, vibrations (17) must stabilize the originally unstable zero equilibrium point of (15) with any $d \leq \varepsilon_0 r$. Such stabilization is indeed demonstrated in Figs 1 and 2. Figure 1 shows the phase portrait of Duffing equation with time lags without parametric excitation. It is seen from this diagram that $x_s = 0$ is an unstable equilibrium point. Two other equilibrium points are given on the graph by $x_s = \pm\sqrt{10}$. Figure 2 shows a trajectory of Duffing equation with oscillations with $x_1(t) = 3$ and $x_2(t) = 5$ for $t \geq 0$, $\alpha = 2$, $d = 0.4\pi$, and $\varepsilon_1 = 0.05$. Since the trajectory in Fig. 2 converges to zero, we can

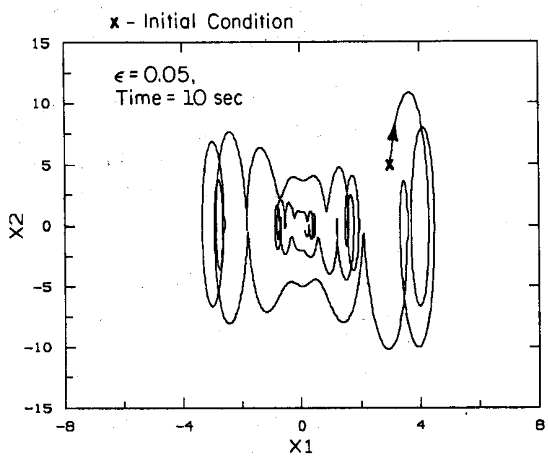

FIG. 2. A trajectory of Duffing equation with a time lag $d = 0.4\pi$ and oscillations.

observe the qualitative changes in system behavior, and specifically, a t-stabilization of an unstable zero equilibrium.

B. *Vector additive vibrations*

Consider the Taylor expansion of $P_i(\xi, \eta, \lambda) \equiv P_i(\xi, \eta)$ around an equilibrium point $x_s = \xi = \eta = 0$

$$P_i(\xi, \eta) = \sum_{j=1}^{\infty} \frac{1}{j!} v_i^{j\otimes} + \sum_{k=1}^{\infty} \frac{1}{k!} w_i^{k\otimes} \quad (21)$$

where

$$v_i^{r\otimes} = \{[v_{i1}, \ldots, v_{in}]^T\}^{r\otimes} \triangleq [v_{i1}^r, \ldots, v_{in}^r]^T,$$

$$v_{il}^r \triangleq \left(\xi_1 \frac{\partial}{\partial \xi_1} + \cdots + \xi_n \frac{\partial}{\partial \xi_n}\right)^r p_{il}(\xi, \eta)|_{\xi=\eta=0},$$

$$w_i^{r\otimes} = \{[w_{i1}, \ldots, w_{in}]^T\}^{r\otimes} \triangleq [w_{i1}^r, \ldots, w_{in}^r]^T,$$

$$w_{il}^r \triangleq \left(\eta_1 \frac{\partial}{\partial \eta_1} + \cdots + \eta_n \frac{\partial}{\partial \eta_n}\right)^r p_{il}(\xi, \eta)|_{\xi=\eta=0},$$

$$l = 1, \ldots, n, \quad (22)$$

where subscript $\xi = \eta = 0$ means that derivatives of a term $p_{il}(\cdot, \cdot)$ are evaluated at zero values of its arguments. $P_i(\xi, \eta)$ will be referred to as an *odd* r_{i1}, r_{i2}*-algebraic function in the vicinity of* 0 if

(1) expansion (21) may have nonzero terms only for $j \in [1, r_{i1}]$, $k \in [1, r_{i2}]$, $r_{i1}, r_{i2} < \infty$ with the last nonzero terms at $j = r_{i1}$, $k = r_{i2}$,

(2) expansion (21) has no terms with $j = 2n$, $k = 2n$, $n = 1, 2, 3, \ldots$.

In (21) a term $(1/j!)v_i^{j\otimes}$ with $\xi = y + u$ can be represented as

$$\frac{1}{j!} v_i^{j\otimes}|_{\xi=y+u} = \frac{1}{j!}\beta_j + \frac{1}{(j-1)!} S_j^i y + \mathrm{HOT}(y) \quad (23)$$

where the elements of vector β_j are algebraic forms of order j with respect to the components of vector u, the elements of matrix S_j^i are algebraic forms of order $j-1$ with respect to u, and HOT(y) denotes higher order terms in y. For example, the element s_{lm}^i of matrix S_3^i is

$$s_{lm}^i = d_{lm}^i[(d_{l1}^i u_1 + \cdots + d_{ln}^i u_n)^2 - \sum_{r=1}^{n} \sum_{k=1}^{n} d_{lr}^i u_r d_{lk}^i u_k \quad (24)$$

$$l, m = 1, \ldots, n; \quad r \neq k, \quad r \neq m, \quad k \neq m$$

where

$$d_{lm}^i = d_{lm}^i\{p_i(\xi, \eta)\}|_{\xi=\eta=0} = \frac{\partial p_{il}(\xi, \eta)}{\partial \xi_m}\bigg|_{\xi=\eta=0},$$

and

$$d_{lm}^i d_{lk}^i = d_{lm}^i d_{lk}^i\{p_i(\xi, \eta)\}|_{\xi=\eta=0} = \frac{\partial^2 p_{il}(\xi, \eta)}{\partial \xi_m \, \partial \xi_k}\bigg|_{\xi=\eta=0}. \quad (25)$$

Similarly, in (21) a term $(1/j!)w_i^{j\otimes}$ with $\eta = y + u$ can be represented as

$$\frac{1}{j!} w_i^{j\otimes}|_{\eta=y+u} = \frac{1}{j!}\beta_j + \frac{1}{(j-1)!} E_j^i y + \mathrm{HOT}(y) \quad (26)$$

where the elements of matrix E_j^i are algebraic forms of order $j-1$ with respect to u, and, for example the element e_{lm}^i of matrix E_3^i is given by the right hand side of (24) and by (25) with differentiation with respect to η, i.e. the second argument of $P_i(\cdot, \cdot)$.

Let u in (23) and (26) be the zero average primitive of vector $m(t)$

$$u(t) \triangleq \int m(t)\, dt \quad \text{and} \quad \overline{u(t)} = 0. \quad (27)$$

Introduce a matrix

$$H_i \triangleq \frac{\partial P_i(\xi, \eta)}{\partial \xi}\bigg|_{\xi=\eta=0} + \frac{\partial P_i(\xi, \eta)}{\partial \eta}\bigg|_{\xi=\eta=0} + \frac{1}{2!}\overline{S_3^i(u(t))} + \frac{1}{4!}\overline{S_5^i(u(t))} + \cdots + \frac{1}{(r_{i1}-1)!}\overline{S_{r_{i1}}^i(u(t))} + \frac{1}{2!}\overline{E_3^i(u(t))} + \frac{1}{4!}\overline{E_5^i(u(t))} + \cdots + \frac{1}{(r_{i2}-1)!}\overline{E_{r_{i2}}^i(u(t))}. \quad (28)$$

Theorem 2. Assume that in (6) each $P_i(\cdot, \cdot)$ is an odd r_{i1}, r_{i2}-algebraic function in a sufficiently large neighborhood of 0. Then

(i) 0 of (6) is v-stabilizable if there exists a PAZ vector $m(t)$ such that with $u(t)$ defined in (27) $H \triangleq \sum_{i=1}^{m} H_i$ is a Hurwitz matrix;

(ii) there exists positive $\varepsilon_0 = $ const. such that $x_s = 0$ of (6) is v-stabilizable by vector additive vibrations $l(t) = (1/\varepsilon)m(t/\varepsilon)$, $0 < \varepsilon \leq \varepsilon_0$, where $m(t)$ satisfies all the conditions of assertion (i) above.

Corollary 2. Let an assumption of Theorem 2 hold for system (1) with fixed delays d_i, $i = 1, \ldots, m$. Suppose that there exists a set of constants r_i, $i = 1, \ldots, m$, such that $d_i/r_i = d_j/r_j$, $i, j = 1, \ldots, m$, and $x_x = 0$ of system (6) with these constants r_i, $i = 1, \ldots, m$, is v-stabilizable by vector additive vibrations $(1/\varepsilon)m(t/\varepsilon)$, $0 < \varepsilon \leq \varepsilon_0$.

Then the zero equilibrium of (1) is v-stabilizable by vector additive vibrations $(1/\varepsilon_1)m(t/\varepsilon_1)$, $\varepsilon_1 \triangleq d_i/r_i$, if $\varepsilon_1 \leq \varepsilon_0$.

Remark 7. Unlike linear multiplicative vibrations vector additive vibrations are incapable of stabilizing an unstable linear system. Indeed, in general, an unstable linear system has an unstable impulse response and is not bounded-input-bounded-state stable, therefore, any additive time-periodic input will give rise to an unbounded growth of system state. Consequently, nonlinearity is necessary for the v-stabilizability of an unstable equilibrium of a dynamical system by vector additive vibrations.

Example 2. Van der Pol equation with time delays. Consider equation

$$\ddot{x}(t) + \mu_1(x^2(t) - 1)\dot{x}(t) + b_1x(t) + \mu_2(x^2(t-d) - 1)\dot{x}(t-d) + b_2x(t-d) = 0 \quad (29)$$

or in state space form with vector additive vibrations

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} x_2(t) \\ -b_1x_1(t) + \mu_1x_2(t) - \mu_1x_1^2(t)x_2(t)\end{bmatrix} + \begin{bmatrix} 0 \\ -b_2x_1(t-d) + \mu_2x_2(t-d) - \mu_2x_1^2(t-d)x_2(t-d)\end{bmatrix} + \begin{bmatrix} l_1(t) \\ l_2(t)\end{bmatrix}. \quad (30)$$

Here the right hand side of (30) with $l_1(t) = l_2(t) = 0$ and d replaced by εr satisfies all assumptions of Theorem 2 and it is an odd 3,3-algebraic function around $x_{1s} = x_{2s} = 0$. Several trajectories of equation (30) without vibrations, i.e. with $l_1(t) = l_2(t) = 0$, shown in Fig. 3 for $b_1 = b_2 = \mu_1 = \mu_2 = 0.5$ demonstrate an instability of an equilibrium $x_s = 0$. Choosing $m_1(t) = \alpha\cos t$ and $m_2(t) = 0$, matrix H of Theorem 2 is given by

$$H = \begin{bmatrix} 0 & 1 \\ -(b_1 + b_2) & (\mu_1 + \mu_2)\left(1 - \dfrac{\alpha^2}{2}\right)\end{bmatrix}. \quad (31)$$

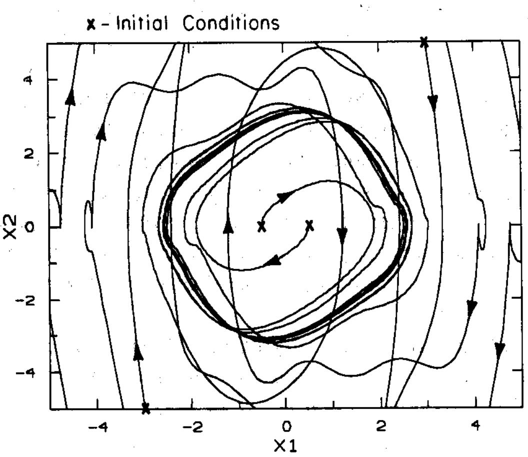

FIG. 3. Phase portrait of Van der Pol equation with a time lag $d = 0.4\pi$.

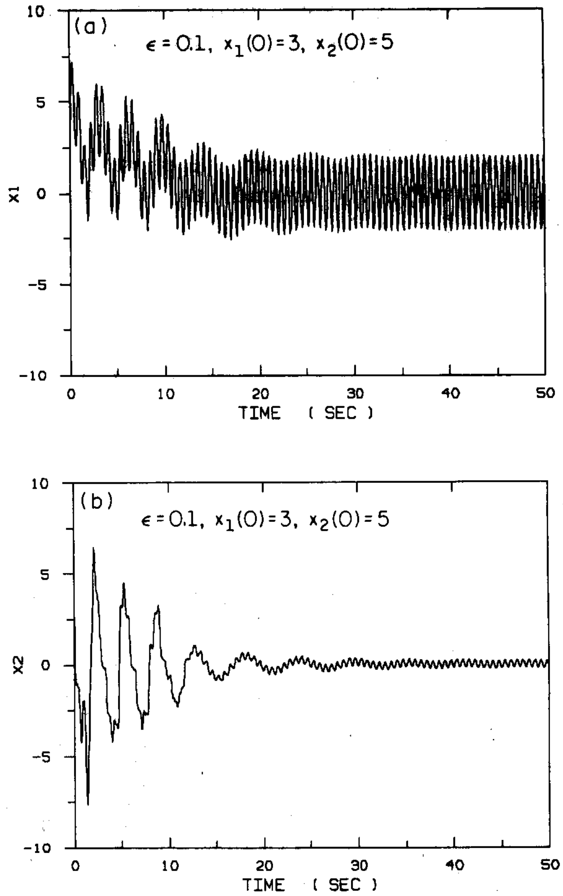

FIG. 4. Solutions $x_1(t)$ and $x_2(t)$ versus time of Van der Pol equation with oscillations and a time lag $d = 0.4\pi$.

Hence for $(b_1 + b_2) > 0$ and $(\mu_1 + \mu_2) > 0$ zero equilibrium of the Van der Pol equation with delays (29) is v-stabilizable by vector additive vibrations $l_1(t) = (\alpha/\varepsilon)\cos(t/\varepsilon)$, $l_2(t) = 0$, if $\alpha > \sqrt{2}$ and delay d is sufficiently small. Such stabilization is indeed shown in Fig. 4 for $d = 0.4\pi$. The last element in the lower row of matrix H in (31) demonstrates that nonlinearities play a decisive role in the vibrational stabilization of this class of systems by vector additive vibrations.

III. TRANSIENT BEHAVIOR ANALYSIS OF VIBRATIONALLY CONTROLLED NONLINEAR TIME LAG SYSTEMS

Vibrational stabilization considered in the previous sections addresses the local behavior of vibrationally controlled nonlinear systems around an equilibrium point with the main emphasis on attractivity, i.e. on behavior as $t \to \infty$. For control purposes it is also of interest to analyze the nonlocal system behavior at every time moment starting from $t = 0$, i.e. the transient behavior of a system. Analysis of trajectories of vibrationally controlled system (3) is a difficult task; however, if vibrations are of the form $f(t) = (1/\varepsilon)\phi(t/\varepsilon)$, $\phi(\cdot)$-PAZ function, $0 < \varepsilon \ll 1$, $\varepsilon = \text{const.}$, i.e. sufficiently fast, then trajectories of system (3) are usually composed of a fast oscillatory part with the period of $\phi(t/\varepsilon)$ superimposed on a slow part. A comparison of a slow part of a trajectory of the oscillatory system with a trajectory of the corresponding system without vibrations for the same initial conditions reveals the qualititive changes in system behavior caused by oscillations.

In order to analyze the transient behavior of a vibrationally controlled system, a solution of the initial value problem for every delay equation in this paper will be denoted as $x(t, x_0, 0)$ and interpreted in the sense of (Driver, 1977, p. 257) as a continuous function $x: [-r, \infty) \to R^n$ that reproduces the initial data, curve $x_0(s)$, $s \in [-r, 0]$, and satisfies the equation considered for $t \ge 0$ with $\dot{x}(0)$ being understood as the right-hand derivative.

In this section we consider the transient behavior analysis of the system (6) with linear multiplicative and vector additive vibrations on a finite time interval. Then we relate it to system (4) with fixed delays, d_i, $i = 1, \ldots, m$, as in the previous section. Thus we consider system

$$\dot{x}(t) = \sum_{i=1}^{m} P_i(x(t), x(t - \varepsilon r_i), \lambda_0) + Q((1/\varepsilon)\phi(t/\varepsilon), x(t)) \tag{32}$$

with the right-hand side defined in (1) and (4) and r_i, $i = 1, \ldots, m$, being positive constants of the order $O(1)$.

Let $x(t, x_0, 0)$, where $x(0, x_0, 0) = x_0$ be a trajectory of equation (32). In order to strip $x(t, x_0, 0)$ of its fast oscillating component introduce a *moving average along a trajectory* $x(t, x_0, 0)$ as

$$\bar{x}(t) \triangleq \frac{1}{T} \int_t^{t+T} x(s, x_0, 0)\, ds, \quad 0 \le t < \infty, \tag{33}$$

where T is a period of $\phi(t/\varepsilon)$.

If the quantity $\bar{x}(t)$ can be closely approximated by the trajectory of a time-invariant system then the transient behavior analysis of system (32) for various magnitudes and frequencies of the oscillations can be greatly simplified, resulting in the constructive procedure for the design of the parametric excitations that induce the desired qualitative changes in the system behavior.

A. *Linear multiplicative vibrations*

Let linear multiplicative vibrations in (32) be of the form $(1/\varepsilon)F(t/\varepsilon)x(t)$. Assume that equation

$$\dot{x}(t) = F(t)x(t) \tag{34}$$

has a periodic in t fundamental matrix $\Phi(t)$.

Along with (33) introduce

$$\bar{x}_M(z(t)) \triangleq \bar{\Phi}(t)z(t, z_0, 0), \tag{35}$$

where $\bar{\Phi}(t) = (1/T^*)\int_0^{T^*} \Phi(t)\, dt$, $T^* \triangleq T/\varepsilon$, and $z(t, z_0, 0)$ with $z(0, z_0, 0) = z_0 = \text{const.}$ is a solution of the equation

$$\dot{z}(t) = \bar{Q}(z(t)), \quad \bar{Q}(y) \triangleq \lim_{T\to\infty} (1/T) \int_0^T Q(t, y)\, dt \tag{36}$$

where

$$Q(t, y) \triangleq \Phi^{-1}(t) \sum_{i=1}^{m} P_i(\Phi(t)y, \Phi(t - r_i)y, \lambda_0). \tag{37}$$

If $\bar{x}_M(z(t))$ stays close to $\bar{x}(t)$ on a time interval of interest, then $\bar{x}_M(z(t))$ can be viewed as an *approximate moving average along a trajectory* $x(t, x_0, 0)$ of (32) with linear multiplicative vibrations on this time interval. Theorem 3 below gives the conditions under which $\bar{x}_M(z(t))$ and $\bar{x}(t)$ can be made arbitrarily close on the arbitrarily large finite time inverval.

Theorem 3. Assume that (a) functions $P_i(\zeta, \eta, \lambda_0)$, $i = 1, \ldots, m$, of system (32) are continuously differentiable with respect to $\zeta, \eta \in \Omega_1 \subset R^n$ and (b) fundamental matrix $\Phi(t)$ of (34) is T^*-periodic where T^* is a period of $F(\cdot)$ in (34).

Then for any δ as small as desired and κ as large as desired there exists $\varepsilon_0 = \varepsilon_0(\delta, \kappa)$ such that for $0 < \varepsilon \le \varepsilon_0$ the following holds

$$\|\bar{x}(t) - \bar{x}_M(z(t))\| < \delta, \quad t \in [0, \kappa], \tag{38}$$

where $z_0 = \Phi(0)x_0(0)$.

Corollary 3. Let assumptions (a) and (b) of Theorem 3 hold for system (4) with fixed delays d_i, $i = 1, \ldots, m$, and linear multiplicative vibrations $D(t)x(t)$. Suppose that there exists a set of constants r_i, $i = 1, \ldots, m$, such that $d_i/r_i = d_j/r_j$, $i, j = 1, \ldots, m$, and solutions of system (32) with these constants r_i satisfy inequality (38) for given δ and κ for $0 < \varepsilon < \varepsilon_0$. Then solutions of system (4) with vibrations $D(t)x(t) = (1/\varepsilon_1)F(t/\varepsilon_1)x(t)$, $\varepsilon_1 \triangleq d_i/r_i$, satisfy inequality (38) for the same δ and κ if $\varepsilon_1 \le \varepsilon_0$.

Example 3. Duffing equation (15) with time lags and linear multiplicative vibrations

$$\frac{1}{\varepsilon} F\left(\frac{t}{\varepsilon}\right) x = \begin{bmatrix} 0 & 0 \\ \frac{\alpha}{\varepsilon}\cos\frac{t}{\varepsilon} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{39}$$

Since $\Phi(t)$ in this case is given by (19), from (35) we have

$$\bar{x}_{M1}(z(t)) = z_1(t) \quad \text{and} \quad \bar{x}_{M2}(z(t)) = z_2(t) \quad (40)$$

where $z(t) \equiv z(t, z_0, 0)$ is a solution of equation (36) which for this specific case takes the form

$$\begin{aligned} \dot{z}_1(t) &= z_2(t), \\ \dot{z}_2(t) &= -(a_1 + a_2)z_2(t) \\ &\quad + \left(b_1 + b_2 - \frac{\alpha^2}{2}\right)z_1(t) - (c_1 + c_2)z_1^3(t). \end{aligned} \quad (41)$$

Figure 5 shows that $\bar{x}_{M1}(z(t))$ and $\bar{x}_{M2}(z(t))$ given by dashed curves indeed represent approximate moving averages along $x_1(t)$ and $x_2(t)$, respectively, given by solid curves for delay $d = 0.1\pi$, $\varepsilon = 0.05$, $a_1 = 0.6$, $a_2 = 0.4$, $b_1 = b_2 = 0.5$, and $c_1 = c_2 = 0.05$.

B. *Vector additive vibrations*

Let vector additive vibrations in (32) be of the form $(1/\varepsilon)m(t/\varepsilon)$.

Theorem 4. Let assumption (a) of Theorem 3 hold and $u(t)$ be the T^*-periodic zero mean primitive of $m(t)$.

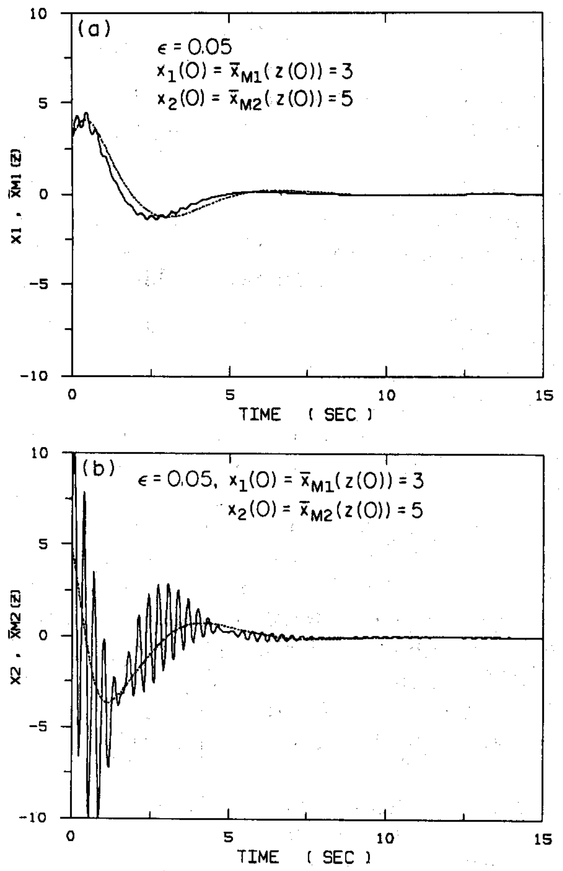

FIG. 5. Solutions $x_1(t)$ and $x_2(t)$ versus time of Duffing equation with oscillations and a time lag $d = 0.1\pi$ and their approximate moving averages $\bar{x}_{M1}(z(t))$ and $\bar{x}_{M2}(z(t))$ versus time.

Then for any δ as small as desired and κ as large as desired there exists $\varepsilon_0 = \varepsilon_0(\delta, \kappa)$ such that for $0 < \varepsilon \le \varepsilon_0$ the following holds

$$\|\bar{x}(t) - z(t)\| < \delta, \quad t \in [0, \kappa] \quad (42)$$

where $z_0 = u(0) + x_0(0)$, and $z(t, z_0, 0)$ is a solution of an ordinary differential equation

$$\dot{z}(t) = \frac{1}{T^*}\int_0^{T^*} \sum_{i=1}^{m} P_i[z(t) + u(s), z(t) + u(s - r_i), \lambda_0]\, ds. \quad (43)$$

Corollary 4. Let all assumptions of Theorem 4 hold for system (4) with fixed delays d_i, $i = 1, \ldots, m$, and vector additive vibrations $l(t)$. Suppose that there exists a set of constants r_i, $i = 1, \ldots, m$, such that $d_i/r_i = d_j/r_j$, $i, j = 1, \ldots, m$ and solutions of system (32) with these constants r_i satisfy inequality (42) for given δ and κ for $0 < \varepsilon \le \varepsilon_0$. Then solutions of system (4) with vibrations $l(t) = (1/\varepsilon_1)m(t/\varepsilon_1)$, $\varepsilon_1 \triangleq d_i/r_i$, satisfy inequality (42) for the same δ and κ if $\varepsilon_1 \le \varepsilon_0$.

Example 4. Van der Pol equation (30) with time lags and vector additive vibrations

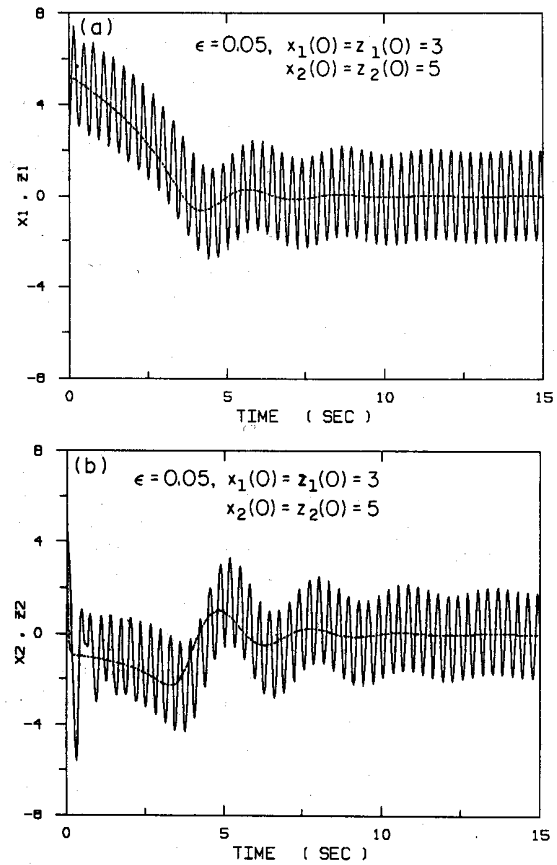

FIG. 6. Solutions $x_1(t)$ and $x_2(t)$ versus time of Van der Pol equation with oscillations and a time lag $d = 0.2$ and their approximate moving average, $z_1(t)$ and $z_2(t)$ versus time.

$[(\alpha/\varepsilon)\sin(t/\varepsilon),\ (\alpha/\varepsilon)\sin(t/\varepsilon)]^T$. Equation (43) in this case has the form

$$\begin{aligned}\dot z_1(t) &= z_2(t),\\ \dot z_2(t) &= [(b_1+b_2)+(\mu_1+\mu_2)\alpha^2]z_1(t)\\ &\quad+(\mu_1+\mu_2)\left(1-\frac{\alpha^2}{2}\right)z_2(t)\\ &\quad-(\mu_1+\mu_2)z_1^2(t)z_2(t).\end{aligned} \tag{44}$$

Approximate moving averages $z_1(t)$ and $z_2(t)$ (dashed curves) along with the corresponding solutions $x_1(t)$ and $x_2(t)$ (solid curves), respectively, are shown in Fig. 6 for $b_1=b_2=\mu_1=\mu_2=0.5$, $d=0.2$ and $\varepsilon=0.05$.

IV. CONCLUSIONS

This paper demonstrates that under certain conditions, parametric vibrations approximately introduced into a nonlinear time lag system of the form (1) are capable of converting an unstable equilibrium of a system into an asymptotically stable one or creating an asymptotically stable oscillatory regime with the average located at an unstable equilibrium point. The criteria presented enable one to investigate the existence of the stabilizing vibrations and give procedures for the choice of their parameters. While the theory presented is restricted to the delays of the order $O(\varepsilon)$, $0<\varepsilon\ll 1$, Examples 1 and 2 demonstrate that vibrations are capable of stabilizing systems with delays of the order $O(1)$. The method for the transient behavior analysis of a vibrationally controlled system is also given and is supported by numerical examples. Thus, vibrational control is shown to be a possible alternative for control of nonlinear systems with time delays in the situations where conventional methods are expensive, difficult, or impossible to apply due to restrictions on sensing and actuation.

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APPENDIX: PROOFS OF THE FORMAL STATEMENTS

Proof of Theorem 1. With vibrations

$$Q(f(t), x(t)) = D(t)x(t) = (1/\varepsilon)F(t/\varepsilon)x(t), \tag{1a}$$

$\tau \triangleq t/\varepsilon$, and $d_i \to \varepsilon r_i$, $i=1,\ldots,m$, equation (4) takes the form

$$\frac{dx(\tau)}{d\tau} = \varepsilon\sum_{i=1}^{m} P_i(x(\tau), x(\tau-r_i), \lambda_0) + F(\tau)x(\tau). \tag{2a}$$

Introducing into equation (2a) coordinate transformation

$$x(\tau)=\Phi(\tau)y(\tau), \tag{3a}$$

which for the delayed states takes the form

$$x(\tau-r_i)=\Phi(\tau-r_i)y(\tau-r_i),\quad i=1,\ldots,m, \tag{4a}$$

(2a) reduces to a standard form

$$\frac{dy(\tau)}{d\tau}=\varepsilon\Phi^{-1}(\tau)\sum_{i=1}^{m} \times P_i(\Phi(\tau)y(\tau), \Phi(\tau-r_i)y(\tau-r_i), \lambda_0). \tag{5a}$$

Since $F(\tau)$ is a periodic bounded zero mean function of τ, by Abel's formula (cf. Brockett, 1970, Theorem 3.3) $\Phi^{-1}(\tau)$ is bounded for $\tau\in(-\infty,\infty)$ and the right-hand side of (5a) is well defined for all τ.

Averaging the right hand side of (5a) with respect to τ, linearizing it at $y(\tau)=y(t-r_i)=0$ and dropping the delays in the states we obtain an ordinary differential equation

$$\frac{dv(\tau)}{d\tau}=\varepsilon\left[\overline{\Phi^{-1}(\tau)A\Phi(\tau)}+\sum_{i=1}^{m}\overline{\Phi^{-1}(\tau)B_i\Phi(\tau-r_i)}\right]v(\tau) \tag{6a}$$

that for sufficiently small ε governs the stability properties of the trivial solution of (5a). Finally, noting that the averaged equation corresponding to (5a) is given in time t by $\dot z(t)=Rz(t)$, where matrix R is defined in (10), and that if $\Phi(\tau)$ is almost periodic then (3a) and (4a) are stability preserving, the assertions of the theorem directly follow from Theorem 3.3 of Hale (1966). Q.E.D.

Proof of Corollary 1. Under the assumptions of Corollary 1, conditions of assertion (i) of Theorem 1 are satisfied for systems (1). Therefore, for $\varepsilon_1 < \varepsilon_0$, the proof of Corollary 1 directly follows from assertion (ii) of Theorem 1. Q.E.D.

Proof of Theorem 2. System (6) with vector additive vibrations $l(t) = (1/\varepsilon)m(t/\varepsilon)$ in time $\tau \triangleq t/\varepsilon$ is given by

$$\frac{dx(t)}{d\tau} = \varepsilon \sum_{i=1}^{m} P_i(x(\tau), x(\tau - r_i), \lambda_0) + m(\tau). \tag{7a}$$

With $u(\tau)$ defined by (27), substitutions

$$x(\tau) = y(\tau) + u(\tau) \tag{8a}$$

and

$$x(\tau - r_i) = y(\tau - r_i) + u(\tau - r_i) \tag{9a}$$

reduce (7a) to a standard form

$$\frac{dy(\tau)}{d\tau} = \varepsilon \sum_{i=1}^{m} P_i[y(\tau) + u(\tau), y(\tau - r_i) + u(\tau - r_i), \lambda_0]. \tag{10a}$$

Expanding every $P_i(\xi, \eta, \lambda_0) \equiv P_i(\xi, \eta)$ around $\xi = \eta = 0$, under the assumption of Theorem 2, we obtain

$$\frac{dy(\tau)}{d\tau} = \varepsilon \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n_i} \frac{1}{j!} v_i^{j\otimes}|_{\xi = y(\tau)+u(\tau)} + \sum_{k=1}^{n_i} \frac{1}{k!} w_i^{k\otimes}|_{\eta = y(\tau - r_i)+u(\tau - r_i)} \right\}. \tag{11a}$$

Averaging the right hand side of (11a) with respect to τ and dropping the delays in the states, we have

$$\frac{dz(\tau)}{d\tau} = \left(\sum_{i=1}^{m} H_i\right) z(\tau) + \mathrm{HOT}(z) \tag{12a}$$

where matrices H_i, $i = 1, \ldots, m$, are defined in (28).

Assume now that $\sum_{i=1}^{m} H_i$ is a Hurwitz matrix. Then, noting that (12a) has a zero equilibrium, by Theorem 3.3 of Hale (1966) for every $\delta > 0$ there exists $\varepsilon_0(\delta)$ such that (10a) with $0 < \varepsilon \le \varepsilon_0$ has a unique asymptotically stable almost periodic solution $y^s(\tau)$ characterized by

$$\|y^s(\tau)\| < \delta. \tag{13a}$$

Now, the proof Theorem 2 follows from (13a) upon noting that due to (8a)

$$\bar{x}^s(\tau) = \bar{y}^s(\tau) + \bar{u}(\tau) = \bar{y}^s(\tau).$$

Q.E.D.

Proof of Corollary 2. Directly follows from Theorem 2.

Proof of Theorem 3. Follows that of Theorem 1 until and including the sentence after equation (5a). Dropping the delays in state variable $y(\cdot)$ and averaging the right hand side of (5a) we obtain

$$\frac{dz(\tau)}{d\tau} = \varepsilon \bar{Q}(z(t)) \tag{14a}$$

where $\bar{Q}(\cdot)$ is given in (36) and (37).

Now by Theorem 4.32 of Halanay (1966, p. 460) for $y(0) = y_0(0) = z(0) = z_0$ and every $\kappa > 0$ and $\delta > 0$ there exists $\varepsilon_2 > 0$ such that for $0 < \varepsilon \le \varepsilon_2$ we have

$$\|y(\tau, y_0, 0) - z(\tau, z_0, 0)\| < \eta, \quad \forall \tau \in [0, \kappa/\varepsilon]. \tag{15a}$$

Since

$$x(t, x_0, 0) = \Phi(t/\varepsilon) y(t, x_0, 0), \tag{16a}$$

$T = O(\varepsilon)$, and $\dot{x}(t)$ and $\dot{y}(t)$ are of the order $O(1)$ in time $t = \tau/\varepsilon$, we have

$$\begin{aligned} \bar{x}(t) &= \frac{1}{T} \int_t^{t+T} \Phi(t/\varepsilon) y(t)\, dt \\ &= \left[\frac{1}{T} \int_t^{t+T} \Phi\left(\frac{s}{\varepsilon}\right) ds\right] y(t) + K_1(\varepsilon) \\ &= \left[\frac{1}{T^*} \int_0^{T^*} \Phi(s)\, ds\right] y(t) + K_1(\varepsilon) \\ &= \overline{\Phi(t)} y(t) + K_1(\varepsilon), \end{aligned} \tag{17a}$$

where $y(t) \equiv y(t, x_0, 0)$, $K_1(\varepsilon) = O(\varepsilon)$ and $\|K_1(\varepsilon)\|$ uniformly approaches 0 as $\varepsilon \to 0$.

Denoting $z(t) \equiv z(t, z_0, 0)$, for the left hand side of inequality (38) from (15a) and (17a) for any given η and κ we have in time t

$$\begin{aligned} \|\bar{x}(t) - \bar{x}_M(z(t))\| &= \|\bar{\Phi}(t) y(t) + K_1(\varepsilon) - \bar{\Phi}(t) z(t)\| \\ &\le \|\bar{\Phi}(t)[y(t) - z(t)]\| + \|K_1(\varepsilon)\| \\ &\le N\, \|y(t) - z(t)\| + \|K_1(\varepsilon)\| \\ &\le N\eta + \|K_1(\varepsilon)\|, \quad 0 < \varepsilon \le \varepsilon_2, \end{aligned} \tag{18a}$$

where N is some positive constant which exists since $\bar{\Phi}(t)$ is a constant matrix with bounded elements.

Finally, since we can always choose $\varepsilon_0 < \varepsilon_2$ and η such that

$$N\eta + \|K_1(\varepsilon_0)\| < \delta, \quad \forall t \in [0, \kappa],$$

Theorem 3 is proven. Q.E.D.

Proof of Corollary 3. Directly follows from Theorem 3.

Proof of Theorem 4. Follows that of Theorem 2 until and including equation (10a). Dropping the delays in state variable $y(\cdot)$ and averaging the right hand side of (10a) we obtain in time t equation (43). Now the proof follows that of Theorem 3 after equation (14a), with $\Phi(t/\varepsilon) y(t, x_0, 0)$ replaced by $u(t/\varepsilon) + y(t, x_0, 0)$ and $\bar{x}_M(z(t))$ replaced by $z(t)$. Q.E.D.

Proof of Corollary 4. Directly follows from Theorem 4.