Vibrational Stabilization of Nonlinear Parabolic Systems with Neumann Boundary Conditions

Joseph Bentsman and Keum S. Hong

Abstract—**This note derives the conditions for the existence of the stabilizing vibrations for a class of distributed parameter systems governed by parabolic partial differential equations with Neumann boundary conditions and gives the guidelines for the choice of the vibration parameters that ensure stabilization. Examples of vibrational stabilization of unstable systems by linear multiplicative and vector additive vibrations are given to support the theory.**

I. Introduction

The concept of vibrational control proposed in [1] is especially attractive when it is applied to distributed parameter systems (DPS), generally known as not readily amenable to sensing and actuation. Indeed, being an open-loop strategy that can ensure desired system behavior via zero-mean parametric excitations, vibrational control requires no on-line sensing and it can stabilize all system modes simultaneously. The experimental and applied theoretical results on the vibrational control of DPS include stabilization of plasma pinches

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The authors are with the Department of Mechanical Industrial Engineering, University of Illinois, Urbana-Champaign, Urbana, IL 61801.

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[2], powerful continuous CO_2 lasers [3], Bernard instabilities in liquids [4], and Euler instabilities in elastic systems [4].

Along with the finite-dimensional vibrational control theory developed in [1], [5]–[7], and tested experimentally as reported in [8]–[10], vibrational stabilizability of infinite-dimensional systems described by hyperbolic partial differential equations (PDE's) has been addressed in [4] using finite-dimensional approximations, and those described by parabolic PDE's with Dirichlet boundary conditions (DBC)—in [11], [12] using operator theory and imposing rather restrictive assumptions on the solutions of the PDE's considered.

There is, however, a large number of practically important processes described by parabolic PDE's with Neumann boundary conditions (NBC). Physically, these boundary conditions define the gradient at the boundary, e.g., temperature gradient. In the present note, the case of zero gradient is considered which physically means that the boundary is insulated. This situation arises for example in combustion processes and chemical reactors when heat transfer through the walls of a combustion chamber or a reactor tank is insignificant and can be neglected. It is, therefore, of interest to investigate the effectiveness of vibrational control as applied to parabolic DPS with NBC. Intuitively, the effectiveness of vibrational control in this case is not obvious since NBC provide substantially more freedom for a system to evolve than DBC which prescribe the evolution along the boundary beforehand.

The purpose of this note is to show that vibrational control can ensure stabilization of unstable parabolic DPS with NBC and present the conditions of vibrational stabilizability and a procedure for the choice of vibration parameters for two classes of vibrations.

A class of DPS considered is described by a nonlinear parabolic PDE

$$u_t = Au_{xx} + Bu_x + C(u, \lambda), \quad u_x(0, t) = u_x(1, t) = 0,$$
$$t \geq 0, \quad u(x, 0) = u_0(x) \tag{1}$$

where $u = u(x, t)$: $R(0, 1) \times R_+ \to R^n$; $x \in (0, 1)$; $A, B \in R^{n \times n}$ are constant matrices; $\lambda \in R^m$ is a vibratile parameter; C: $R^n \times R^m \to R^n$ is a nonlinear vector function such that $C(0, \lambda) = 0$; subscripts of u denote corresponding partial derivatives with respect to t and x; the Neumann boundary conditions are given by $u_x(0, t) = u_x(1, t) = 0$, $t \geq 0$, and initial condition by $u(x, 0) = u_o(x)$.

Vibrational stabilizability properties of (1) depend on the mechanism through which excitations affect the spectrum of the linearized averaged system. The efficacy of linear multiplicative vibrations (cf. Section II) depends on the spectrum of the linearization of (1), and the efficacy of the other vibration types is, in addition, strongly influenced by the nonlinear terms of (1).

In this note, we use the terminology of [5]–[7]. The note has the following structure. Section II discusses vibrational stabilization by two vibration types. In this section, a stability criterion similar to [12, Lemma] is derived for parabolic PDE's with NBC. This condition then permits the application of the results of [13] on averaging without imposing any restrictive assumptions on the properties of the solutions of the averaged system. Numerical examples that demonstrate vibrational stabilization of unstable DPS governed by parabolic PDE's with NBC are presented as well. Section III gives the conclusions. Proofs of all formal statements are collected in the Appendix.

II. Vibrational Stabilization

A. Definitions and Problem Formulation

Assuming λ fixed, introduce in (1) parametric vibrations as

$$\lambda \to \lambda + f(t) \tag{2}$$

where $f(t)$ is a periodic vector function with average value equal to zero (PAZ vector). As a result, (1) becomes

$$u_t = Au_{xx} + Bu_x + C(u, \lambda + f(t)),$$
$$u_x(0, t) = u_x(1, t) = 0, \quad t \geq 0, \quad u(x, 0) = u_o(x). \tag{3}$$

Throughout this note, it will be assumed that (3) has the form

$$u_t = Au_{xx} + Bu_x + C(u) + C_1(f(t), u),$$
$$u_x(0, t) = u_x(1, t) = 0, \quad t \geq 0, \quad u(x, 0) = u_o(x)$$
$$C_1: R^m \times R^n \to R^n, \quad C(u) \equiv C(u, \lambda) \tag{4}$$

where $C_1(\cdot, \cdot)$ is a vector function linear with respect to its first argument. If $C_1(f(t), u) = l(t)$, where $l(t)$ is a PAZ vector, the introduced vibrations are referred to as *vector additive*, if $C_1(f(t), u) = D(t)u$, where $D(t)$ is an $n \times n$ PAZ matrix, the vibrations are called *linear multiplicative*, and if $C_1(f(t), u) = D(t)X(u)$, where X: $R^n \to R^n$ is a nonlinear map, the vibrations are termed *nonlinear multiplicative*.

Throughout the note, it is also assumed that for given initial and boundary conditions systems (1) and (3) are well posed in the Sobolev space $H^{n1}(0, 1)$ of vector functions $v(x) = [v_1(x), \cdots, v_n(x)]^T$ with components $v_i(x)$ in $L_2(0, 1)$ which have the first distributional derivatives. Norm on $H^{n1}(0, 1)$ is defined as

$$\|v\|_1 \triangleq \left(\int_0^1 v_x^T(x)v_x(x) + v^T(x)v(x)\, dx\right)^{1/2}. \tag{5}$$

Superscripts "n" and "1" in $H^{n1}(0, 1)$ indicate the dimension of the vector $v(x)$ and the order of the highest derivative with respect to x in the definition of norm (5).

Definition 1: The null solution of (1) is said to be *vibrationally stabilizable* (v-stabilizable) if for any $\delta > 0$ there exists a PAZ vector $f(t)$ such that system (3) has an asymptotically stable almost periodic solution $u^s(x, t) \in H^{n1}(0, 1)$ characterized by

$$\overline{\|u^s(x, t)\|_1} < \delta, \quad \overline{u^s(x, t)} \triangleq \lim_{T \to \infty} \frac{1}{T} \int_0^T u^s(x, t)\, dt. \tag{6}$$

Definition 2: The null solution of (1) is said to be totally vibrationally stabilizable (t-stabilizable) if there exists a PAZ vector $f(t)$ such that system (3) has the null solution asymptotically stable in $\|\cdot\|_1$ norm.

Problem Formulation: The problem of vibrational stabilization consists in finding 1) verifiable conditions for the *existence* of stabilizing vibrations for the system (1) (v- and t-stabilizability), and 2) the *parameters* of vibrations (amplitudes and frequencies) which ensure stabilization.

System (1) with Dirichlet boundary conditions $u(0, t) = u(1, t) = 0$ has been shown in [12] to be t-stabilizable by linear multiplicative vibrations. In this section, we extend the results of [12] to Neumann boundary conditions $u_x(0, t) = u_x(1, t) = 0$, remove the assumptions that are difficult to check *a priori* (see [12, expression (6)], formulate conditions of v-stabilizability of (1) by another vibration class—vector additive vibrations, and describe a procedure for the choice of the parameters of stabilizing vibrations.

B. Preliminary Lemmas

Lemma 1: The null solution of the linear system

$$u_t = Au_{xx} + Bu_x + C'u, \quad A, B, C' \in R^{n \times n},$$
$$u \in H^{n2}(0, 1), \quad u_x(0, t) = u_x(1, t) = 0, \quad t \geq 0 \tag{7}$$

is asymptotically stable with respect to the norm (5) if there exists a positive-definite matrix $M \in R^{n \times n}$ such that 1) $A^TM + MA$ is a positive-definite matrix; 2) $B^TM = MB$; 3) $C'^TM + MC'$ is negative definite, and 4) C' is a Hurwitz matrix.

Remark 1: Condition 4) and positive definiteness of M in Lemma 1 do not imply condition 3) in general (see [14, p. 176]).

Assume that the ordinary differential equation

$$\frac{d\xi}{dt} = C_1(\phi(t), \xi) \tag{8}$$

where $C_1(\cdot, \cdot)$ is defined in (4) and $\phi(\cdot)$ is a PAZ vector, has a unique solution defined by every initial condition $\xi_o \in \Omega \subset R^n$, $\forall t \geq 0$. Denote the general solution of (8) as $\xi(t) = h(t, q)$, h:

$R \times R^n \to R^n$, where $q \in R^n$ is a constant uniquely defined for every pair of initial conditions (ξ_o, t_o).

Introduce into (4) a substitution of the form

$$u(x,t) = h(t, v(x,t)), \quad v: R(0,1) \times R_+ \to R^n. \tag{9}$$

Assuming that $C_1(\cdot, u)$ is differentiable with respect to u, (4) takes the form

$$\begin{aligned} v_t &= [\partial h/\partial v]^{-1}[Ah(t,v)_{xx} + Bh(t,v)_x + C(h(t,v))] \\ &= F_1(t,v) + F_2(t,v) + F_3(t,v) \end{aligned} \tag{10}$$

where $[\partial h/\partial v]^{-1}$ always exists (cf. [5, Section III]) and

$$\begin{aligned} F_1(t,v) &\triangleq [\partial h/\partial v]^{-1} Ah(t,v)_{xx}, \\ F_2(t,v) &\triangleq [\partial h/\partial v]^{-1} Bh(t,v)_x, \\ F_3(t,v) &\triangleq [\partial h/\partial v]^{-1} C(h(t,v)). \end{aligned} \tag{11}$$

Introduce the equation

$$\begin{aligned} w_t = P_1(w) + P_2(w) + P_3(w), \quad & w_x(0,t) = w_x(1,t) = 0, \\ t \geq 0, \quad & w(x,0) = w_o(x) \end{aligned} \tag{12}$$

where $w: R(0,1) \times R_+ \to R^n$

$$P_1(v) \triangleq \overline{F_1(t,v)} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_o^T F_1(t,v)\, dt,$$

$$P_2(v) \triangleq \overline{F_2(t,v)}, \quad \text{and} \quad P_3(v) \triangleq \overline{F_3(t,v)}.$$

Lemma 2: Assume that (1) with vibrations $\lambda \to \lambda + \phi(t)$, where $\phi(t)$ is a PAZ vector with the period T^*, has the form (4), $u \in H^{n2}(0,1)$, and both $C(u)$ and $C_1(\cdot, u)$ are continuously differentiable with respect to u. Assume also that 1) $h(t,q)$ is periodic in t for any q and is linear or affine in q; 2) equations (10) and (12) are parabolic; 3) equation (12) has the null solution $w = 0$.

Then the null solution of (1) is 1) v-stabilizable if there exists $\phi(t)$ such that the null solution of

$$\begin{aligned} w_t = P_1(w) + P_2(w) + P_{30}w, \quad & P_{30} \triangleq \partial P_3(0)/\partial w, \\ w_x(0,t) = w_x(1,t) &= 0 \end{aligned} \tag{13}$$

is asymptotically stable and $\overline{h(t,0)} = 0$; 2) t-stabilizable if it is v-stabilizable, $C_1(\cdot, 0) = 0$, and $h(t,0) = 0$.

Lemma 3: Let all assumptions of Lemma 2 hold. Then there exists positive ϵ = const such that the trivial solution of (1) is 1) v-stabilizable by vibrations

$$\lambda \to \lambda + (1/\epsilon)\phi(t/\epsilon), \qquad 0 < \epsilon \leq \epsilon_o \tag{14}$$

if all the conditions of Lemma 2, assertion 1) hold; 2) t-stabilizable by vibrations (14) if all the conditions of assertion 2) of Lemma 2 hold.

Remark 2: Lemma 2 gives conditions for v- and t-stabilizability in terms of the existence of a PAZ vector $\phi(t)$. Once $\phi(t)$ that satisfies conditions of Lemma 2 is found, according to Lemma 3, the actual stabilizing vibrations are obtained via one-parameter (ϵ) numerical search by rescaling $\phi(t)$ as $(1/\epsilon)\phi(t/\epsilon)$ and simulating system (4) with $f(t) = (1/\epsilon)\phi(t/\epsilon)$ until ϵ_o is determined. Such $\epsilon_o > 0$ must exist due to Lemma 3.

C. Linear Multiplicative Vibrations

The main contribution of this section is in presenting the results similar to [12, the Theorem and Corollary] for parabolic DPS with Neumann boundary conditions, and most importantly, in removing the assumption [12, expression (6)] on the identical stability properties of the solutions in $L_2(0,1)$ and in Sobolev space ($H_o^{n1}(0,1)$ in the case of [12]), for all the equations considered. Besides being overly restrictive, this assumption is usually impossible to check, and its removal makes the vibrational stabilizability conditions for parabolic DPS with NBC easily applicable and constructive.

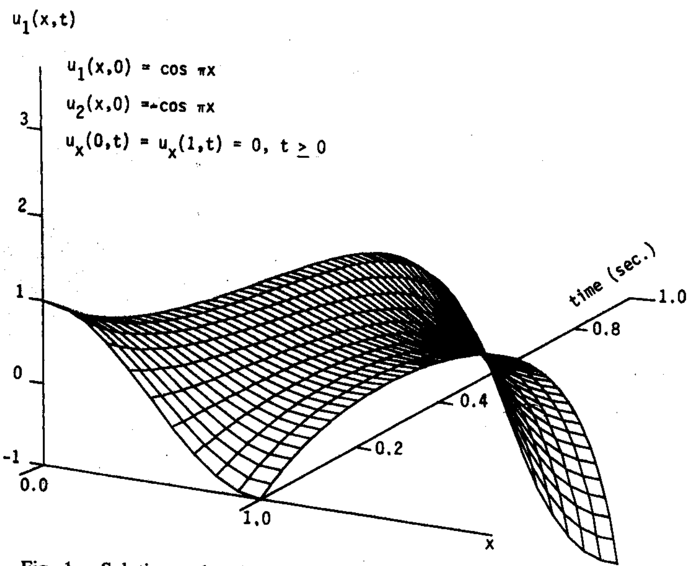

Fig. 1. Solution $u_1(x,t)$ of system (19) without vibrations, which grows without bound as $t \to \infty$.

Theorem 1: Assume that in (1) $C(u)$ is continuously differentiable with respect to u in the vicinity of $u = 0$, and $u \in H^{n2}(0,1)$.

Then the null solution $u = 0$ of (1) is t-stabilizable by linear multiplicative vibrations if there exist 1) a PAZ matrix $F(t)$ with a fundamental matrix $\Phi(t)$, $t \in (-\infty, \infty)$, of $\dot{y} = F(t)y$, $y: R \to R^n$, being periodic, and 2) a positive-definite matrix M such that

a) $\bar{A}^T M + M\bar{A}$ is positive definite,

$$\text{where } \bar{A} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi^{-1}(t) A \Phi(t)\, dt, \tag{15}$$

$$\text{b) } M\bar{B} = \bar{B}^T M, \quad \bar{B} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi^{-1}(t) B \Phi(t)\, dt, \tag{16}$$

c) $\bar{C}_o \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T \Phi^{-1}(t) C_o \Phi(t)\, dt$ is a Hurwitz matrix,

$$\text{where } C_o \triangleq \partial C(0)/\partial u, \tag{17}$$

$$\text{d) } \bar{C}_o^T M + M\bar{C}_o \text{ is negative definite} \tag{18}$$

Corollary 1: Let all assumptions of Theorem 1 hold. Assume also that matrices A and B in (1) are, respectively, given by aI and bI, $a > 0$, a, b = const, and matrix C_o is nonderogatory. Then the null solution of (1) is v-stabilizable by linear multiplicative vibrations if tr $C_o < 0$.

Theorem 2: Let all assumptions of Theorem 1 hold. Then there exists positive ϵ_o = const such that $u = 0$ of (1) is t-stabilizable by linear multiplicative vibrations $D(t)x(t) = (1/\epsilon)F(t/\epsilon)x(t)$, $0 < \epsilon \leq \epsilon_o$, where $F(t)$ satisfies all the conditions of Theorem 1.

Example 1: Consider (1) with

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad C(u) = C_o u + R(u),$$

$$C_o = \begin{bmatrix} 2 & 7 \\ 3 & -3 \end{bmatrix}, \quad R(u) = \begin{bmatrix} 0.1 & u_1^3 \\ 0.1 & u_2^3 \end{bmatrix}, \tag{19}$$

and initial conditions $u_1(x,0) = \cos \pi x$, $u_2(x,0) = -\cos \pi x$, further referred to as system (19). The null solution of (19) is unstable (cf. Fig. 1). Noting that by Corollary 1 system (19) is vibrationally stabilizable by linear multiplicative vibrations, let us choose matrix $F(t)$ of Theorem 1 as

$$F(t) = \begin{bmatrix} 0 & 0 \\ \alpha \cos t & 0 \end{bmatrix}.$$

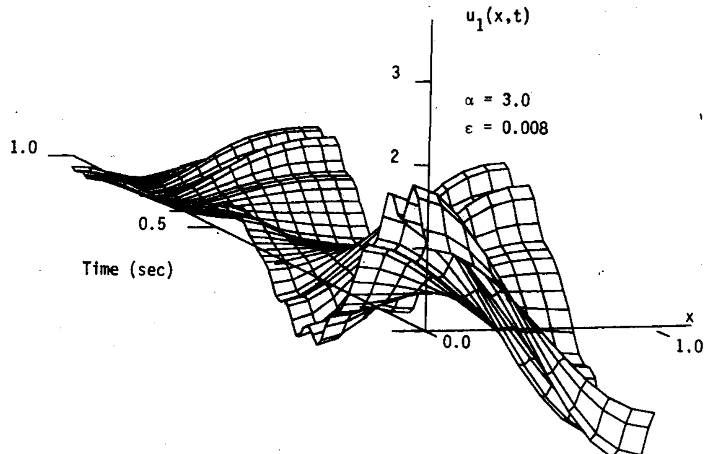

Fig. 2. Solution $u_1(x, t)$ of system (19) with linear multiplicative vibrations.

Then matrix $\Phi(t)$ of Theorem 1 can be taken as

$$\Phi(t) = \exp\left\{\alpha\begin{bmatrix} 0 & 0 \\ \sin t & 0\end{bmatrix}\right\},$$

and matrix $\overline{C}_o$ is given by

$$\overline{C}_o = \begin{bmatrix} 2 & 7 \\ 3 - \frac{7}{2}\alpha^2 & -3 \end{bmatrix}.$$

Since for $\alpha > 1.050$ $\overline{C}_o$ is a Hurwitz matrix, by Theorem 2 linear multiplicative vibrations with

$$D(t) = \begin{bmatrix} 0 & 0 \\ \frac{\alpha}{\epsilon}\sin\frac{t}{\epsilon} & 0 \end{bmatrix}$$

will ensure t-stabilization of (19) for $0 < \epsilon \le \epsilon_o$. Such stabilization is indeed demonstrated in Fig. 2 for $\alpha = 3$ and $\epsilon = 0.01$.

D. Vector Additive Vibrations

This section demonstrates that under certain conditions vector additive vibrations introduced into a parabolic DPS with NBC are capable of inducing an asymptotically stable periodic solution with the average value in the vicinity of an unstable equilibrium of this DPS.

Using the notation of [5], consider the Taylor expansion of $C(u) = [c_1(u),\cdots, c_n(u)]T$ of (1) around an equilibrium point $u_s = 0$

$$C(u) = \sum_{i=1}^{\infty} \frac{1}{i!} v^{i\otimes} \tag{20}$$

where

$$v^{r\otimes} = \{[v_1,\cdots,v_n]^T\}^{r\otimes} \triangleq [v_1^r,\cdots,v_r^r]^T,$$

$$v_l^r \triangleq \left[u_1\frac{\partial}{\partial u_1} + \cdots + u_n\frac{\partial}{\partial u_n}\right]^r c_l(u)\,|_{u=0}, \quad l = 1,\cdots,n.$$

$C(u)$ will be referred to as an *odd r-algebraic function in the vicinity of u_s* if 1) expansion (20) may have nonzero terms only for $i \in [1, r]$, with the last nonzero term at $i = r$, and r being a bounded number; 2) expansion (20) has no terms with $i = 2k$, $k = 1, 2, 3, \cdots$.

Represent the ith term of (20) with $u = y + \psi$ as

$$\frac{1}{i!}v^{i\otimes}|_{u=y+\psi} = \frac{1}{i!}\xi_i + \frac{1}{(i-1)!}S_i y + \text{HOT}(y) \tag{21}$$

where the elements of vector ξ_i are algebraic forms of order i with respect to the components of vector ψ, the elements of matrix S_i are algebraic forms of order $i - 1$ with respect to ψ, and HOT denotes the higher order terms in y. The form of an element s_{lm} of matrix S_3 is given in [5].

Define ψ in (21) as

$$\psi(t) \triangleq \int m(t)\,dt, \qquad \overline{\psi}(t) = 0 \tag{22}$$

where $m(t) = [m_1(t),\cdots, m_n(t)]^T$ is a PAZ vector and introduce a matrix

$$H = C_o + \frac{1}{2!}\overline{S_3(\psi(t))} + \frac{1}{4!}\overline{S_5(\psi(t))} + \cdots + \frac{1}{(r-1)!}\overline{S_r(\psi(t))}. \tag{23}$$

Theorem 3: Assume that in (1) $u \in H^{n2}(0, 1)$, $C(u)$ is an odd r-algebraic function in a sufficiently large neighborhood of 0. Assume further that $A = aI$, $a = \text{const} > 0$, $B = bI$, $b = \text{const}$. Then the null solution $u = 0$ of (1) is v-stabilizable by vector additive vibrations if there exists a PAZ vector $m(t)$ such that H is a Hurwitz matrix.

Theorem 4: Let all assumptions of Theorem 3 hold. Then there exists positive $\epsilon_o = \text{const}$ such that $u = 0$ of (1) is v-stabilizable by vector additive vibrations $l(t) = (1/\epsilon)m(t/\epsilon)$, $0 < \epsilon \le \epsilon_o$, where $m(t)$ satisfies all the conditions of Theorem 3.

Example 2: Consider (1), (2) with A, B, C_o as in (19), $C(u) = C_o u + R(u)$, and

$$R(u) = \begin{bmatrix} u_2 \\ -u_1 + \mu u_2 - \mu u_1^2 u_2 \end{bmatrix}, \qquad \mu = \text{const} \tag{24}$$

further referred to as system (24). The solution $u(x, t)$ of this system grows without bound (cf. Fig. 3). Noting that $C(u)$ is an odd 3-algebraic function around $u = 0$, matrix H of (23) takes the form

$$H = \begin{bmatrix} 2 & 8 \\ 2 - 2\mu\overline{\psi_1^2(t)} & -3 + \mu - \mu\overline{\psi_1^2(t)} \end{bmatrix}.$$

Setting $\mu = 1.0$, H is Hurwitz for $\overline{\psi_1^2(t)} > 1.430$ or for $\alpha > 1.690$ if $m_1(t) = m_2(t) = \alpha \sin t$. Hence, by Theorem 4, vector additive vibrations $l_1(t) = (\alpha/\epsilon)\sin(t/\epsilon)$, $l_2(t) = (\alpha/\epsilon)\sin(t/\epsilon)$, will guarantee v-stabilization of (24) for sufficiently small ϵ, which is shown in Fig. 4 for $\alpha = 3$ and $\epsilon = 0.008$.

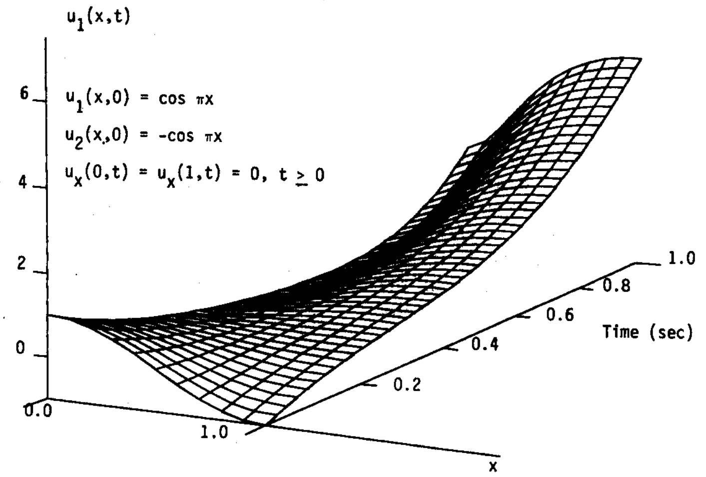

Fig. 3. Solution $u_1(x, t)$ of system (24) without vibrations, which grows without bound as $t \to \infty$.

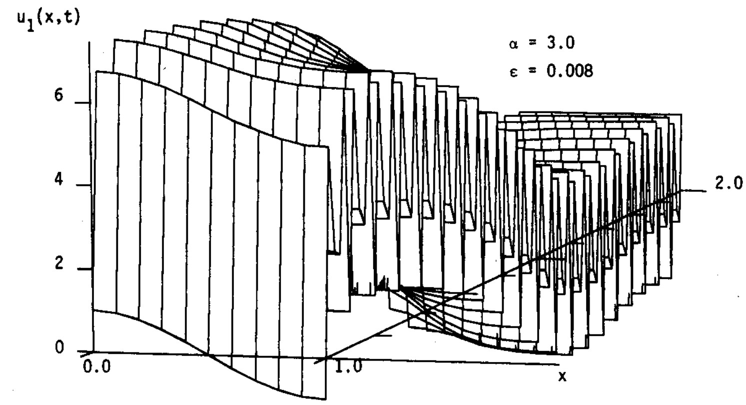

Fig. 4. Solution $u_1(x, t)$ of system (24) with vector additive vibrations.

IV. Conclusions

This note presents the vibrational control theory for a class of nonlinear parabolic DPS with Neumann boundary conditions. The theorems presented enable one to determine the existence of the stabilizing vibrations for a class of systems and to organize a simple one parameter (ϵ) numerical search for their actual magnitudes and frequencies. Since vibrational control strategy requires no on-line measurements, vibrational stabilization can be an attractive alternative in the situation when feedback and/or feedforward are difficult or impossible to apply due to the restrictions on sensing and actuation. To facilitate the application of the technique, however, it is important to investigate the transient behavior of a vibrationally controlled DPS. Parabolic systems with DBC have been partially investigated in [18]. The transient behavior analysis of vibrationally controlled DPS with NBC considered in this note needs to be addressed in the future.

Appendix

Proof of Lemma 1: Let

$$\tilde{u}(x,t) \triangleq u(x,t) - p(t), \quad p(t) = [p_1(t), \cdots, p_2(t)]^T,$$

$$p(t) \triangleq \int_0^1 u(x,t)\,dx. \tag{A.1}$$

Consider a functional with a positive-definite matrix M such that

$$V(t) = \int_0^1 \tilde{u}_x^T M \tilde{u}_x(x,t)\,dx = \int_0^1 u_x^T M u_x\,dx. \tag{A.2}$$

Differentiating (A.2) with respect to t, integrating by parts, and making use of the given Neumann boundary conditions yields

$$\frac{dV}{dt} = -\int_0^1 u_{xx}^T(A^TM + MA)u_{xx}\,dx + \int_0^1 u_{xx}^T(B^TM - MB)u_x\,dx + \int_0^1 u_x^T(C'^TM + MC')u_x\,dx. \tag{A.3}$$

For any continuously differentiable function $\phi(x)$ with $\phi(0) = \phi(1) = 0$, the following Wirtinger's inequality holds (cf. [15])

$$\int_0^1 \left(\frac{d\phi}{dx}\right)^2 dx \geq \left(\frac{\pi}{2}\right)^2 \int_0^1 \phi^2\,dx. \tag{A.4}$$

If we consider $d\phi/dx$ to be a generalized derivative of $\phi(\cdot)$, then inequality (A.4) also holds for $\phi \in H_0^1(0,1)$ and consequently, when the Lemma 1, condition 1) is satisfied, with $\phi \to u_x$ it follows from (A.4) and NBC in (1) that

$$\int_0^1 u_{xx}^T(A^TM + MA)u_{xx}\,dx \geq \left(\frac{\pi}{2}\right)^2 \int_0^1 u_x^T(A^TM + MA)u_x\,dx. \tag{A.5}$$

Further, if Lemma 1, conditions 2) and 3) hold, from (A.3) and

(A.5) it follows that

$$\frac{dV}{dt} \leq \int_0^1 u_x^T\left[C'^TM + MC' - \left(\frac{\pi}{2}\right)^2 (A^TM + MA)\right]u_x\,dx < 0. \tag{A.6}$$

Therefore, $u_x(x, t)$ and hence $\tilde{u}_x(x, t)$ converges to 0 almost everywhere, which in turn implies that $\tilde{u}(x, t)$ converges to 0 pointwise. Further, since $p_x(t) = p_{xx}(t) \equiv 0$, convergence properties of $p(t)$ are defined by the spectrum of matrix C, therefore, if Lemma 1, condition 4) is satisfied, $p(t)$ converges to 0, and consequently $u(x, t)$ converges to zero pointwise. Thus, if all the conditions of Lemma 1 hold, the null solution of (7) is globally asymptotically stable with respect to the norm (5). Q.E.D.

Proof of Lemmas 2 and 3: Consider system (4) with $f(t) = (1/\epsilon)\phi(t/\epsilon)$, i.e., system of the form

$$u_t = Au_{xx} + Bu_x + C(u) + C_1\left(\frac{1}{\epsilon}\phi\left(\frac{t}{\epsilon}\right), u\right). \tag{A.7}$$

Since $C_1(\cdot\,,\cdot)$ is linear with respect to its first argument, (A.6) can be rewritten in time $\tau = t/\epsilon$ as

$$u_\tau = \epsilon[Au_{xx} + Bu_x + C(u)] + C_1(\phi(\tau), u). \tag{A.8}$$

Equation (8) with t replaced by τ yields a substitution

$$u(x, \tau) = h(\tau, v(x, \tau)) \tag{A.9}$$

where $h(\cdot\,,\cdot)$ is defined in of Section II-B. Introducing into (A.8), substitution (A.9) yields the equation

$$v_\tau = \epsilon[F_1(\tau, v) + F_2(\tau, v) + F_3(\tau, v)] \tag{A.10}$$

where $F_i(\cdot\,,\cdot)$, $i = 1, 2, 3$, are defined in (11). Averaging the right-hand side of (A.10) with respect to τ yields

$$w_\tau = \epsilon[P_1(w) + P_2(w) + P_3(w)] \tag{A.11}$$

where $P_i(\cdot)$, $i = 1, 2, 3$, are defined in (12).

Since by assumption $h(\cdot\,,\cdot)$ is linear or affine with respect to the second argument, $P_1(w)$ and $P_2(w)$ can be represented as $P_1(w) = P_1'w_{xx}$ and $P_2(w) = P_2'w_x$, where P_1' and P_2' are constant $n \times n$ matrices. Therefore, defining

$$P\psi(x) \triangleq [P_1'(\partial^2/\partial x^2) + P_2'(\partial/\partial x) + P_{30}]\psi(x),$$

where $\psi(x) \in H^{n1}(0, 1)$, and noting that (12) is assumed to be parabolic, operator P: $H^{n1}(0, 1) \to L_2(0, 1)$ is m-sectorial in the sense of Kato (cf. [16, p. 280]) or sectorial in the sense of Henry (cf. [13, p. 18]). Now, representing $P_3(w)$ in terms of linear part P_{30} at $w = 0$ and high-order terms as $P_3(w) = P_{30}w + P_{3h}(w)$, (A.10) can be rewritten in time t as

$$v_t = P_1'v_{xx} + P_2'v_x + P_{30}v + P_4\left(\frac{t}{\epsilon}, v\right) \tag{A.12}$$

where $P_4(t/\epsilon, v) = \sum_{i=1}^3 F_i(t/\epsilon, v) - \sum_{i=1}^3 P_i(v) + P_{3h}(v)$. Representing (A.12) as an evolution equation in Sobolev space $H^{n1}(1, 0)$, [13, p. 222, assertion 2)] is applicable to (A.12). Indeed, if conditions of Lemma 2, assertion 1) hold, spectrum of the linear operator P in the evolution equation lies in the open left-half plane. Identifying $P_4(t/\epsilon, v)$ in (A.12) with $f(t, x)$ on [13, p. 222] and following the notation of [13], we have $\partial f_o(0)/\partial x = 0$ where $f_o(x) = (1/T)\int_0^T f(t, x)\,dt$. Consequently, defining $\epsilon \triangleq 1/\omega$, due to [13, p. 222, assertions (i) and (ii)], for any given $\eta > 0$ there exists ω_o and, hence $\epsilon_o = 1/\omega_o = \epsilon_o(\eta)$ such that for any $0 < \epsilon < \epsilon_o$ or $\omega > \omega_o$ (A.12) has a unique T-periodic asymptotically stable solution $v^s(t, x)$ for which

$$\sup\|v^s(x, t)\|_1 < \eta. \tag{A.13}$$

Now, since $h(t, q)$ is a solution of (8) defined for all $t \in [0, \infty)$, there exists a constant μ such that

$$\left\|h\left(\frac{t}{\epsilon}, v^s(x, t)\right) - h\left(\frac{t}{\epsilon}, 0\right)\right\|_1 \leq \mu\|v^s(x, t)\|_1 \quad \forall t \geq 0. \tag{A.14}$$

Consequently, choosing η in (A.13) as δ/μ and taking the time average of (A.14), we obtain

$$\left\|\overline{h\left(\frac{t}{\epsilon}, v^s(x, t)\right)} - \overline{h\left(\frac{t}{\epsilon}, 0\right)}\right\|_1 \leq \mu\overline{\|v^s(x, t)\|_1} < \delta. \tag{A.15}$$

If $\overline{h(t, 0)} = 0$ holds, then from (A.15)

$$\left\|\overline{h\left(\frac{t}{\epsilon}, v^s(x, t)\right)}\right\|_1 < \delta. \tag{A.16}$$

Due to (A.9), inequality (A.16) coincides with (6) for $f(t) = (1/\epsilon)\phi(t/\epsilon)$. Finally, asymptotic stability of $u^s(x, t)$ follows from asymptotic stability of $v^s(x, t)$ via mapping $h((t/\epsilon), \cdot)$, i.e., $f(t) = (1/\epsilon)\phi(t/\epsilon)$ indeed are stabilizing vibrations in the sense of Definition 1. This proves Lemmas 2 and 3, assertions 1).

If conditions of Lemma 2, assertion 2) hold, then (1) with vibrations $f(t) = (1/\epsilon)\phi(t/\epsilon)$ must have the null solution and so does (A.10). Now, of Lemmas 2 and 3, assertions 2) follow from the uniqueness of the asymptotically stable solution of (A.12) in the vicinity of 0 which must be the null solution itself, since it is already known to exist. Q.E.D.

Proof of Theorem 1: Under the assumptions and conditions of Theorem 1, function $h(t, v(x, t))$ in (9) is given by $\Phi(t)v(x, t)$, therefore (13) takes the form

$$w_t = \bar{A}w_{xx} + \bar{B}w_x + \bar{C}_o w. \tag{A.17}$$

If all the conditions of Theorem 1 hold, then all the conditions of Lemma 1 are satisfied as well and, consequently, the null solution of (A.17) is asymptotically stable. Now, noting that in this case $h(t, 0) = 0$, the assertion of Theorem 1 directly follows from Lemma 2. Q.E.D.

Proof of Corollary 1: Since $A = aI$, $a > 0$, and $B = bI$, Theorem 1, conditions a), b) hold for any positive-definite matrix M. Further, if matrix C_o is nonderogatory, there exists a nonsingular matrix K such that matrix $K^{-1}C_oK$ is in the companion form. Assume now that tr $C_o < 0$. Then

$$\operatorname{tr} C_o = \operatorname{tr} K^{-1}C_oK < 0. \tag{A.18}$$

In this case from the Theorem of [17] it follows that there exists a periodic matrix $D_1(t)$ such that a fundamental matrix $\Phi_1(t)$ of $\dot{y} = D_1(t)y$ is periodic and matrix

$$Q_1 \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T \Phi_1^{-1}(t)K^{-1}C_oK\Phi_1(t)\,dt$$

is Hurwitz. Setting $F(t) = KD_1(t)K^{-1}$ in Theorem 1, condition, 1) a fundamental matrix $\Phi(t) = K\Phi_1(t)$ of $\dot{y} = F(t)y$ guarantees that matrix $\bar{C}_o$ defined in (17) is Hurwitz and therefore, there exists positive-definite matrix M such that matrix $\bar{C}_o^TM + M\bar{C}_o$ is negative definite. Now, the assertion of Corollary 1 follows from Theorem 1. Q.E.D.

Proof of Theorem 2: It directly follows from Lemma 3, assertion 2). Q.E.D.

Proof of Theorem 3: Under the assumptions and conditions of Theorem 3 function $h(t, v(x, t))$ in (9) is given by $\psi(t) + v(x, t)$, with $\psi(t)$ defined in (22), therefore (13) takes the form

$$w_t = aIw_{xx} + bIw_x + Hw \tag{A.19}$$

where H is given in (23). If H is a Hurwitz matrix, due to $A = aI$ and $B = bI$, all the conditions of Lemma 1 are satisfied, consequently the null solution of (A.19) is asymptotically stable. Finally, since in this case $h(t, q) = \psi(t) + q$ and therefore according to (22) $\overline{h(t, 0)} = 0$, the assertion of Theorem 3 directly follows from assertion 1) of Lemma 2. Q.E.D.

Proof of Theorem 4: It directly follows from Lemma 3, assertion 1). Q.E.D.

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