Anti-Delay Distributed Optimization Protocols for Multiagent Systems With Coupled Constraints

Yao Zou, *Member, IEEE*, Wei Wang, *Member, IEEE*, Bomin Huang, Hui Wang, Ziyang Meng, *Senior Member, IEEE*, and Keum-Shik Hong, *Fellow, IEEE*

***Abstract*— This article focuses on the constrained optimization problem for second-order multiagent systems that experience heterogeneous communication delays. Specifically, the involved agents work together to find the optimal solution of a global payoff function, which is summed by multiple strongly convex local payoff functions, with each function being exclusively owned by an individual agent. However, the feasible solutions must satisfy a coupled equality constraint, formulated by individual parameters assigned to each agent. Initially, a basic anti-delay distributed protocol is developed, which leverages a scattering transformation to enhance the generation of received information. Using the Lyapunov framework, we demonstrate that the agents coordinated by this anti-delay distributed protocol can effectively reach a consensus on the expected optimal solution, despite the presence of communication delays. In addition, we present two results that extend the basic anti-delay distributed protocol. First, we consider the scenario of lacking velocity and develop a velocity-free anti-delay distributed protocol to achieve the concerned constrained optimization objective. Next, we augment the system order and develop an anti-delay distributed optimization protocol for integrator chain multiagent systems. Finally, we confirm the anti-delay performance of the developed distributed protocols through simulations.**

***Index Terms*— Communication delays, constrained optimization, distributed protocol, multiagent systems.**

I. INTRODUCTION

OPTIMIZATION problems are frequently encountered in engineering practice. As these problems become more complex, the efficiency to solve them using centralized solutions often decreases significantly. However, due to advancements in information technology, communication engineering, network science, and related fields, researchers have shifted their focus toward distributed approaches for solving optimization problems. Instead of calculating the optimal solution of a global payoff function using a centralized unit/agent, it is computed in parallel by multiple collaborative units/agents, which, respectively, correspond to local payoff functions disassembled from the global one [1]. This is important in the fields of cyber-physical estimation and dynamic programming [2], [3].

The distributed optimization solution is generally based on the concept of consensus, where all operating agents work together to converge on a global optimal solution through local interactions. This approach has led to the development of various distributed algorithms for online calculation of optimal solutions. For instance, several low-complexity gradient-consensus-based distributed protocols with all-time resource-demand feasibility were proposed in [4] and [5] for multiagent resource allocation, which is a simplified version of the distributed optimization regardless of state consensus. Next, for the nonconstrained distributed optimization, in terms of a basic zero-gradient-sum algorithm framework established in [6], multiple improved distributed algorithms were proposed in the cases of event-triggered communication [7] and fixed-time convergence [8]. In addition, a distributed projected subgradient algorithm using a differentiable projection operator was proposed in [9] such that the optimization problem with nonlinear set constraints was solved with an exponential convergence rate. Moreover, a primal–dual distributed algorithm was devised in [10] to solve the constrained convex optimization problem with finite-time convergence performance. To reinforce robust performance, a sliding mode-based robustification was implanted into the distributed optimization algorithm in [11], allowing it to reject disturbances and uncertainties. However, the distributed optimization algorithms in [6], [7], [8], and [11] do not consider any constraints on feasible solutions, while the constraints in [9] and [10] are decoupled for individual agents. In addition, the consensus-based distributed algorithms in [6], [7], [8], [9], [10], and [11] heavily depend on interagent communication. Due to the imperfectness in communication quality, delays are inevitable in the communication channels. Although several anti-delay distributed optimization algorithms have been developed in [12], [13], and [14], they become ineffec-

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Yao Zou is with the School of Advanced Agricultural Sciences, the School of Intelligence Science and Technology, and the Key Laboratory of Intelligent Bionic Unmanned Systems, Ministry of Education, University of Science and Technology Beijing, Beijing 100083, China (e-mail: zouyao@ustb.edu.cn).

Wei Wang is with the School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, China (e-mail: wangweiyh@bit.edu.cn).

Bomin Huang is with the College of Computer Engineering, Jimei University, Xiamen 361021, China (e-mail: huangbomin01@hotmail.com).

Hui Wang is with the Demonstration Center of Experimental Teaching in Comprehensive Engineering, Beijing Union University, Beijing 100101, China (e-mail: 838548613@qq.com).

Ziyang Meng is with the Department of Precision Instrument, Tsinghua University, Beijing 100084, China (e-mail: ziyangmeng@mail.tsinghua.edu.cn).

Keum-Shik Hong is with the Institute for Future, Qingdao University, Qingdao 266071, China, and also with the School of Mechanical Engineering, Pusan National University, Busan 46241, South Korea (e-mail: ksh@qdu.edu.cn).

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tive when coupled constraints on feasible solutions are imposed.

On the other hand, in practice, the executors of solving some practical (constrained) optimization problems are often physical objects with particular dynamics [15], [16]. Accordingly, it is important to build distributed optimization protocols for multiagent systems with physical dynamics. For a second-order multiagent system, anti-disturbance distributed optimization protocols were designed in [17] and [18] by leveraging appropriate disturbance estimators. For the same multiagent system, a distributed protocol was developed in [19] to tackle the constrained optimization problem in terms of event-based communication. In addition, for a second-order nonlinear system, a distributed fixed-time optimization protocol was developed in [20] via both state and output feedbacks. Another two velocity-free distributed protocols were designed for second-order multiagent systems in [21] and [22] such that the optimization problems with and without constraints were solved. Moreover, by introducing proper second-order auxiliary dynamics, two distributed protocols were established in [23] and [24] for Euler–Lagrange multiagent systems. In addition, several distributed optimization protocols were developed in [15], [25], and [26] for integrator chain multiagent systems using the zero-gradient-sum framework. However, these protocols do not accommodate any constraints on feasible solutions. A new class of distributed optimization protocols introduced in [27] addresses the optimization problem of uncertain nonlinear multiagent systems concerning inequality constraints. However, they still face limitations similar to those in [15], [17], [18], [19], [20], [21], [22], [23], [24], [25], and [26]. Specifically, these distributed protocols are only applicable in scenarios with no communication delays, meaning that they cannot guarantee convergence when communication delays are present.

In terms of the above discussions, we are mainly going to develop an anti-delay distributed protocol for second-order multiagent systems herein, such that the optimization problem with a coupled equality constraint is solved despite the presence of communication delays. Each local payoff function is only available to its corresponding agent, and in the meantime, the parameters associated with the coupled equality constraint are individually endowed to respective agents. In addition, the delays of different communication channels are heterogeneous. The anti-delay distributed protocol is developed by introducing the scattering transformation into the generation of received information. It is demonstrated in detail that the involved agents driven by the developed anti-delay distributed protocol reach a consensus on the expected optimal solution, while the consensus result is free from the heterogeneous communication delays. Upon this basis, we next develop an extended velocity-free anti-delay distributed optimization protocol such that the concerned constrained optimization objective of the second-order multiagent system is fulfilled in the absence of velocity information. Furthermore, we consider an integrator chain multiagent system and develop another extended anti-delay distributed protocol using a proper coordinate transformation for the fulfillment of the concerned constrained optimization objective. Compared with the current literature, three main contributions herein are enumerated as follows.

1) Unlike [6], [7], [8], [9], [10], [11], [12], [13], and [14], which focus solely on the online distributed calculation of the optimal solution without considering the physical dynamics of the nodes, we develop a distributed optimization protocol for a multiagent system with second-order dynamics. The fully distributed nature of the developed distributed protocol makes it accommodate to scenarios with arbitrary network variations. Moreover, the developed distributed protocol is ameliorated to the case with velocity unavailability. Furthermore, by applying a suitable coordinate transformation and introducing a new composite variable, we also ameliorate the distributed protocol to effectively accommodate integrator chain multiagent systems. Both the extended distributed protocols inherit the fully distributed nature of the basic one.
2) Rather than the delay-free distributed optimization protocols in [15], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], and [27] for multiagent systems with physical dynamics, which are probably not applicable in the presence of communication delays, the developed anti-delay distributed protocols in this article utilize the scattering transformation to enhance the received information. This process endows the developed distributed protocols with remarkable robustness against heterogeneous communication delays.
3) Unlike the optimization problems in [15], [17], [18], [19], [23], [25], and [26], which impose no constraints on feasible solutions, and those in [21], [24], and [27], which just permit decoupled constraints, our considered constrained optimization problem includes a coupled equality constraint on feasible solutions, wherein the involved parameters privately belong to respective agents. To comply with this coupled equality constraint, we incorporate appropriate auxiliary dynamics based on the scattering transformation to process the received information in the development of our distributed protocols.

The rest of this article is organized as follows. Section II presents fundamental theory. Section III discusses the constrained optimization problem under study. Section IV presents the main results, including the development of the anti-delay distributed optimization protocol and its convergence analysis. Building on this, Section V outlines two extension results. Section VI then validates the anti-delay performance of the developed distributed protocols through simulations. Finally, Section VII concludes this article.

Notations: In what follows, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$, respectively, stand for sets of real numbers, nonnegative real numbers, n-dimensional real vectors, and $m \times n$ dimensional real matrices. I_n represents an identity matrix of dimension $n \times n$, 1_n represents a vector of dimension n with all entries 1, and $\otimes$ stands for the Kronecker product. $\mathrm{col}(x_1, x_2, \ldots, x_n)$ stacks vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ into a column vector. $\mathrm{diag}\{A_1, A_2, \ldots, A_n\}$ is a diagonal block matrix with its entries $A_1, A_2, \ldots, A_n \in \mathbb{R}^{m\times n}$. In addition, we define two sets $\mathcal{L}_2 = \{g : \mathbb{R}^+ \to \mathbb{R}^n \mid g$ is locally integrable, $\int_0^\infty \|g(t)\|^2 dt < \infty\}$ and $\mathcal{L}_\infty = \{g : \mathbb{R}^+ \to \mathbb{R}^n | g$ is locally integrable, $\operatorname{ess\,sup}_{t\in\mathbb{R}^+} \|g(t)\| < \infty\}$. Furthermore, given a continuously differentiable function $g : \mathbb{R}^n \to \mathbb{R}$, its gradient is represented by ∇g.

II. Preliminaries

A continuously differentiable function $g : \mathbb{R}^d \to \mathbb{R}$ is convex given that $(\nabla g(z_1) - \nabla g(z_2))^{\mathrm{T}}(z_1 - z_2) \geq 0, \ \ \forall z_1, z_2 \in \mathbb{R}^d$. It is further ω-strongly convex given that $(\nabla g(z_1) - \nabla g(z_2))^{\mathrm{T}}(z_1 - z_2) \geq \omega\|z_1 - z_2\|^2, \ \ \forall z_1, z_2 \in \mathbb{R}^d$.

For a multiagent system, graph theory is commonly used to build its internal network topology. Particularly, a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is constituted by a node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. For an undirected graph $\mathcal{G}$, $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$ means that node i and node j communicate reciprocally, and they are referred to as neighbors. Set $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ contains all the neighbors of node i. A path from one node to another is an ordered, unbroken node sequence connected by nonrepetitive edges. Moreover, an undirected graph is called connected if for each node, there is at least one path from it to any other node. In addition, for graph $\mathcal{G}$, its adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $a_{ij} >$ (or $=$) 0 given $(j, i) \in$ (or $\notin$) $\mathcal{E}$. Also, its Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $l_{ij} = -a_{ij}$ for $i \neq j$, and otherwise $l_{ii} = \sum_{j=1, j\neq i}^n a_{ij}$.

III. Problem Description

A. Constrained Optimization

Consider a multiagent system of n agents with second-order dynamics, labeled by $\mathcal{V} = \{1, 2, \ldots, n\}$, each of which is characterized by

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i \tag{1}$$

where $x_i, v_i \in \mathbb{R}^d$ denote its generalized position and velocity, respectively, and $u_i \in \mathbb{R}^d$ denotes its generalized input. In the studied constrained optimization problem, every agent is individually endowed with a local payoff function $g_i : \mathbb{R}^d \to \mathbb{R}$, while the involved agents are coordinated to compute the optimal solution of the global payoff function $g(z) = \sum_{i=1}^n g_i(z)$ subject to a coupled equality constraint equation

$$\sum_{i=1}^n A_i z = \sum_{i=1}^n b_i \tag{2}$$

where parameters $A_i \in \mathbb{R}^{l\times d}$ and $b_i \in \mathbb{R}^l$ are privately known to agent i. We assign a feasible set χ to incorporate all the feasible solutions that conform to this equality constraint. Formally, the definition of the constrained optimization objective is given below.

Definition 1: The multiagent system (1) is said to achieve the constrained optimization objective, if the involved agents reach a consensus at the optimal solution of the global payoff function g within the feasible set χ, i.e., $\lim_{t\to\infty} x_i(t) = z^*$ and $\lim_{t\to\infty} v_i(t) = 0$, $i \in \mathcal{V}$ with $z^* = \arg\min_{z\in\chi} g(z)$.

Remark 1: Note that each agent has just access to its own constraint parameters A_i and b_i. Thus, how to guarantee the coupled equality constraint (2) without using the constraint parameters of others becomes a difficulty.

In this article, we are primarily going to develop a distributed protocol for the multiagent system (1) to achieve the constrained optimization objective claimed in Definition 1. Particularly, we adopt an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ to represent the interagent interaction topology. In practice, a general way of information interaction is communication. However, suffering from imperfect communication quality, there inevitably exist delays in the communication channels. The delays can hardly be identified, and different channels can barely possess the same delay. Severe delays have harmful effects on the steady-state performance of the controlled system, or even destroy its stability. Thus, unlike the conventional distributed optimization protocols [6], [7], [8], [9], [10], [11], [15], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], the developed one herein is required to be tolerant against heterogeneous communication delays. Before moving on, several assumptions are made below.

Assumption 1: The feasible set χ is nonempty.

Assumption 2: Every local payoff function g_i is ω-strongly convex.

Assumption 3: The undirected graph $\mathcal{G}$ is connected.

Remark 2: Assumptions 1 and 2 guarantee the existence of a unique optimal solution of interest [28]. Moreover, Assumption 3 lays a sufficient condition for the state consensus [29].

B. Equivalent Problem

Note that, in the concerned constrained optimization problem, the global payoff function g is with respect to a common global variable. This causes difficulty in achieving the constrained optimization objective in the case where each agent has no global information. Inspired by [30], we next transform the concerned constrained optimization problem into an equivalent one with respect to a group of individual variables. Resorting to such a transformation, we can solve the concerned constrained optimization problem based on the multivariable consensus.

Lemma 1 [30]: Given Assumptions 1–3, the constrained optimization problem $\min_{z\in\chi} g(z)$ amounts to

$$\begin{aligned} &\min_{\mathbf{z}\in\mathbb{R}^{nd}} \bar{g}(\mathbf{z}) = \sum_{i=1}^n g_i(z_i) \\ &\text{s.t.} \ \ (L \otimes I_d)\mathbf{z} = 0, \quad \sum_{i=1}^n A_i z_i = \sum_{i=1}^n b_i \end{aligned} \tag{3}$$

where $\mathbf{z} = \operatorname{col}(z_1, z_2, \ldots, z_n)$.

In the equivalent constrained optimization problem (3), each variable z_i exclusively belongs to agent i. Accordingly, we next concentrate on the equivalent constrained optimization problem (3) and develop an anti-delay distributed protocol in Section IV to solve it. Before doing so, a useful lemma that characterizes the optimal solution of the equivalent constrained optimization problem (3) is presented below.

Lemma 2 [31]: Given Assumptions 1–3, $\boldsymbol{z}^* = \mathrm{col}(z_1^*, z_2^*, \ldots, z_n^*)$ is the optimal solution of the equivalent constrained optimization problem (3) if there are two multipliers $\boldsymbol{\lambda}^* = \mathrm{col}(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$ and $\boldsymbol{\beta}^* = \mathrm{col}(\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$ such that

$$\nabla \bar{g}\left(\boldsymbol{z}^*\right) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}^* + (L \otimes I_d)\,\boldsymbol{\beta}^* = 0 \tag{4a}$$

$$(L \otimes I_d)\,\boldsymbol{z}^* = 0, \quad (L \otimes I_l)\,\boldsymbol{\lambda}^* = 0 \sum_{i=1}^{n} A_i z_i^* \quad = \sum_{i=1}^{n} b_i \tag{4b}$$

where $\boldsymbol{A} = \mathrm{diag}\{A_1, A_2, \ldots, A_n\}$.

IV. Main Results

In what follows, we consider heterogeneous delays in the interagent communication channels. Particularly, the communication delay from agent j to agent i is denoted by T_{ij}.

A. Anti-Delay Distributed Optimization Protocol Development

To calculate the optimal solution of interest in the presence of the communication delays, we develop an anti-delay distributed protocol in the following form:

$$u_i = -kv_i - \nabla g_i(s_i) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) - A_i^{\mathrm{T}}\lambda_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i - r_{ij}^\beta\right) \tag{5a}$$

$$\dot{\lambda}_i = A_i s_i - b_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^\lambda\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\mu_i - r_{ij}^\mu\right) \tag{5b}$$

$$\dot{\beta}_i = \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) \tag{5c}$$

$$\dot{\mu}_i = \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^\lambda\right) \tag{5d}$$

where k is a positive parameter, $s_i = x_i + v_i$ is a newly introduced composite variable, λ_i, β_i, and μ_i are three auxiliary variables, r_{ij}^s, r_{ij}^β, r_{ij}^λ, and r_{ij}^μ are four variables calculated by the received information from its neighbor j, and a_{ij} is the (i, j)th entry of the adjacency matrix A regarding graph $\mathcal{G}$. In the developed anti-delay distributed protocol (5), the auxiliary dynamics (5c) account for the consensus of s_i, and the auxiliary dynamics (5b) and (5d) play a role in guaranteeing the coupled equality constraint (2). Substituting (5) into (1) gives rise to the following closed-loop system:

$$\begin{bmatrix} \dot{x}_i \\ \dot{v}_i \\ \dot{\lambda}_i \\ \dot{\beta}_i \\ \dot{\mu}_i \end{bmatrix} = \begin{bmatrix} v_i \\ -kv_i - \nabla g_i(s_i) - A_i^{\mathrm{T}}\lambda_i \\ A_i s_i - b_i \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma_i \end{bmatrix} \tag{6}$$

where $\gamma_i = \sum_{j\in\mathcal{N}_i} \gamma_{ij}$ with

$$\gamma_{ij} = E_{ij}\begin{bmatrix} s_i - r_{ij}^s \\ \lambda_i - r_{ij}^\lambda \\ \beta_i - r_{ij}^\beta \\ \mu_i - r_{ij}^\mu \end{bmatrix} \tag{7}$$

and

$$E_{ij} = \begin{bmatrix} -a_{ij} & 0 & -a_{ij} & 0 \\ 0 & -a_{ij} & 0 & -a_{ij} \\ a_{ij} & 0 & 0 & 0 \\ 0 & a_{ij} & 0 & 0 \end{bmatrix} \otimes I_d. \tag{8}$$

When the communication delays are considered, the conventional distributed protocol (5) that uses states as interaction items is executed as $r_{ij}^s(t) = x_j(t - T_{ij})$, $r_{ij}^\lambda(t) = \lambda_j(t - T_{ij})$, $r_{ij}^\beta(t) = \beta_j(t - T_{ij})$, and $r_{ij}^\mu(t) = \mu_j(t - T_{ij})$. In such a case, the delays may destabilize the closed-loop system, as shown in the simulation section. To this end, we need to modify r_{ij}^s, r_{ij}^λ, r_{ij}^β, and r_{ij}^μ in order for the states to uniformly converge to the optimal solution in spite of the communication delays.

To obtain r_{ij}^s, r_{ij}^λ, r_{ij}^β, and r_{ij}^μ in the anti-delay distributed protocol (5), we use the following scattering transformation:

$$\phi_{ij}^{\rightarrow} = \frac{1}{\sqrt{2}}\left(-\gamma_{ij} + \mu_{ij}\right), \quad \phi_{ij}^{\leftarrow} = \frac{1}{\sqrt{2}}\left(\gamma_{ij} + \mu_{ij}\right) \tag{9a}$$

$$\phi_{ji}^{\leftarrow} = \frac{1}{\sqrt{2}}\left(\gamma_{ji} + \mu_{ji}\right), \quad \phi_{ji}^{\rightarrow} = \frac{1}{\sqrt{2}}\left(-\gamma_{ji} + \mu_{ji}\right) \tag{9b}$$

where $\mu_{ij} = \mathrm{col}(r_{ij}^s, r_{ij}^\lambda, r_{ij}^\beta, r_{ij}^\mu)$. In particular, $\phi_{ij}^{\rightarrow}$ denotes the information to agent j delivered by agent i, and $\phi_{ji}^{\leftarrow}$ denotes the information from agent i received by agent j. Due to the communication delays, we trivially have that

$$\phi_{ji}^{\leftarrow}(t) = \phi_{ij}^{\rightarrow}\left(t - T_{ji}\right), \quad \phi_{ij}^{\leftarrow}(t) = \phi_{ji}^{\rightarrow}\left(t - T_{ij}\right). \tag{10}$$

These mean that the information received by agent i (or j) from agent j (or i) at time t is actually that which was delivered by agent j (or i) at $t - T_{ij}$ (or $t - T_{ji}$). In the distributed protocol execution, once agent i receives $\phi_{ij}^{\leftarrow}$ from agent j, it immediately calculates μ_{ij}. To be specific, based on (7) and (9), it follows that:

$$\mu_{ij} = \left(I_{4d} - E_{ij}\right)^{-1}\left(\sqrt{2}\phi_{ij}^{\leftarrow} - E_{ij}\begin{bmatrix} s_i \\ \lambda_i \\ \beta_i \\ \mu_i \end{bmatrix}\right). \tag{11}$$

The following theorem indicates the achievement of the concerned constrained optimization objective using the developed anti-delay distributed protocol (5).

Theorem 1: Consider the multiagent system (1) with Assumptions 1–3. The developed anti-delay distributed protocol (5) with $k > 1$ guarantees that the constrained optimization objective is achieved in the sense of Definition 1.

Proof: For each $i \in \mathcal{V}$, we define tracking errors $\tilde{x}_i = x_i - z_i^*$, $\tilde{v}_i = v_i$, $\tilde{\lambda}_i = \lambda_i - \lambda_i^*$, $\tilde{\beta}_i = \beta_i - 2\beta_i^*$, and $\tilde{\mu}_i = \mu_i - 2\mu_i^*$, where z_i^* is the ith component of the optimal solution of the equivalent constrained optimization problem (3), λ_i^* and β_i^* are the ith components of two multipliers given in Lemma 2, and μ_i^* is the ith component of $\boldsymbol{\mu^*} = \mathrm{col}(\mu_1^*, \mu_2^*, \ldots, \mu_n^*)$ that satisfies $A_i z_i^* - b_i + \sum_{i\in\mathcal{N}_i} a_{ij}(\mu_i^* - \mu_j^*)$.

Next, we assign a Lyapunov function, $Y = V + W$, where

$$V = \sum_{i=1}^{n}\left(\frac{k}{2}\|\tilde{x}_i\|^2 + \tilde{x}_i^{\mathrm{T}}\tilde{v}_i + \frac{1}{2}\|\tilde{v}_i\|^2 + \frac{1}{2}\left\|\tilde{\lambda}_i\right\|^2 + \frac{1}{2}\left\|\tilde{\beta}_i\right\| + \frac{1}{2}\|\tilde{\mu}_i\|^2\right) \quad (12)$$

and $W = \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} W_{ij}$ with

$$W_{ij} = \frac{1}{2}\left(\int_0^t \left\|\phi_{ij}^{\rightarrow}(\iota) - \rho_{ij}^{\circ}\right\|^2 - \left\|\phi_{ji}^{\leftarrow}(\iota) - \rho_{ij}^{\circ}\right\|^2 + \left\|\phi_{ji}^{\rightarrow}(\iota) - \varrho_{ij}^{\circ}\right\|^2 - \left\|\phi_{ij}^{\leftarrow}(\iota) - \varrho_{ij}^{\circ}\right\|^2\right) d\iota + \frac{T_{ij}}{2}\left(\rho_{ij}^{\circ}\right)^2 + \frac{T_{ji}}{2}\left(\varrho_{ij}^{\circ}\right)^2 \quad (13)$$

$\rho_{ij}^{\circ} = (-\gamma_{ij}^{\circ} + \mu_{ij}^{\circ})/\sqrt{2}$, $\varrho_{ij}^{\circ} = (\gamma_{ij}^{\circ} + \mu_{ij}^{\circ})/\sqrt{2}$, $\mu_{ij}^{\circ} = \mathrm{col}(z_j^*, \lambda_j^*, \beta_i^* + \beta_j^*, \mu_i^* + \mu_j^*)$, and $\gamma_{ij}^{\circ} = \mathrm{col}(-a_{ij}(\beta_i^* - \beta_j^*), -a_{ij}(\mu_i^* - \mu_j^*), 0, 0)$. In terms of (10) and the setup that $\phi_{ij}^{\rightarrow}(t) = \phi_{ji}^{\rightarrow}(t) = 0$, $\forall t \geq 0$, W_{ij} satisfies $W_{ij} = (\int_{t-T_{ij}}^{t}\|\phi_{ij}^{\rightarrow}(\iota) - \rho_{ij}^{\circ}\|^2 d\iota + \int_{t-T_{ji}}^{t}\|\phi_{ji}^{\rightarrow}(\iota) - \varrho_{ij}^{\circ}\|^2 d\iota)/2 \geq 0$. Thus, given $k > 1$, the Lyapunov function Y is positive definite.

To analyze the evolution of the Lyapunov function Y, we first consider V in (12). Its derivative along (6) satisfies

$$\begin{aligned}\dot{V} \leq &-(k-1)\sum_{i=1}^{n}\|\tilde{v}_i\|^2 + \sum_{i=1}^{n}\tilde{s}_i^{\mathrm{T}}\Bigg[-\nabla g_i(s_i) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^{s}\right) - A_i^{\mathrm{T}} s_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i - r_{ij}^{\beta}\right)\Bigg]\\ &+ \sum_{i=1}^{n}\tilde{\lambda}_i^{\mathrm{T}}\Bigg[A_i s_i - b_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^{\lambda}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\mu_i - r_{ij}^{\mu}\right)\Bigg] + \sum_{i=1}^{n}\tilde{\beta}_i^{\mathrm{T}}\\ &\cdot\sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^{s}\right) + \sum_{i=1}^{n}\tilde{\mu}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^{\lambda}\right). \quad (14)\end{aligned}$$

With Lemma 2, we have that $\nabla g_i(z_i^*) + A_i^{\mathrm{T}}\lambda_i^* + \sum_{j\in\mathcal{N}_i} a_{ij}(\beta_i^* - \beta_j^*) = 0$ and $A_i z_i^* - b_i - \sum_{j\in\mathcal{N}_i} a_{ij}(\mu_i^* - \mu_j^*) = 0$, $i \in \mathcal{V}$. Accordingly, $\dot{V}$ satisfies

$$\begin{aligned}\dot{V} = &-(k-1)\sum_{i=1}^{n}\|\tilde{v}_i\|^2\\ &+ \sum_{i=1}^{n}\tilde{s}_i^{\mathrm{T}}\Bigg[-\nabla g_i(s_i) + \nabla g_i\left(z_i^*\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^{s}\right) - A_i^{\mathrm{T}}\tilde{\lambda}_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i - r_{ij}^{\beta}\right) + \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i^* - \beta_j^*\right)\Bigg]\\ &+ \sum_{i=1}^{n}\tilde{\lambda}_i^{\mathrm{T}}\Bigg[A_i\tilde{s}_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^{\lambda}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\mu_i - r_{ij}^{\mu}\right) + \sum_{j\in\mathcal{N}_i} a_{ij}\left(\mu_i^* - \mu_j^*\right)\Bigg]\\ &+ \sum_{i=1}^{n}\tilde{\beta}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^{s}\right) + \sum_{i=1}^{n}\tilde{\mu}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^{\lambda}\right)\\ = &-(k-1)\sum_{i=1}^{n}\|\tilde{v}_i\|^2 - \sum_{i=1}^{n}\tilde{s}_i^{\mathrm{T}}\left(\nabla g_i(s_i) - \nabla g_i\left(z_i^*\right)\right)\\ &+ \sum_{i=1}^{n}\tilde{s}_i^{\mathrm{T}}\Bigg[-\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{s}_i - \tilde{r}_{ij}^{s}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\beta}_i - \tilde{r}_{ij}^{\beta}\right)\Bigg]\\ &+ \sum_{i=1}^{n}\tilde{\lambda}_i^{\mathrm{T}}\Bigg[-\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\mu}_i - \tilde{r}_{ij}^{\mu}\right)\Bigg]\\ &+ \sum_{i=1}^{n}\tilde{\beta}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{s}_i - \tilde{r}_{ij}^{s}\right) + \sum_{i=1}^{n}\tilde{\mu}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right)\end{aligned}$$

where $\tilde{s}_i = s_i - z_i^*$, $\tilde{r}_{ij}^{s} = r_{ij}^{s} - z_j^*$, $\tilde{r}_{ij}^{\lambda} = r_{ij}^{\lambda} - \lambda_j^*$, $\tilde{r}_{ij}^{\beta} = r_{ij}^{\beta} - \beta_i^* - \beta_j^*$, and $\tilde{r}_{ij}^{\mu} = r_{ij}^{\mu} - \mu_i^* - \mu_j^*$. It next follows that:

$$\begin{aligned}&\tilde{s}_i^{\mathrm{T}}\Bigg[-\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{s}_i - \tilde{r}_{ij}^{s}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\beta}_i - \tilde{r}_{ij}^{\beta}\right)\Bigg]\\ &+ \tilde{\lambda}_i^{\mathrm{T}}\Bigg[-\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\mu}_i - \tilde{r}_{ij}^{\mu}\right)\Bigg]\\ &+ \tilde{\beta}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{s}_i - \tilde{r}_{ij}^{s}\right) + \tilde{\mu}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right)\\ = &-\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{r}_{ij}^{s}\right)^{\mathrm{T}}\left(\tilde{s}_i - \tilde{r}_{ij}^{s}\right) - \tilde{s}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\beta}_i - \tilde{r}_{ij}^{\beta}\right)\\ &-\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{r}_{ij}^{\lambda}\right)^{\mathrm{T}}\left(\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right) - \tilde{\lambda}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\mu}_i - \tilde{r}_{ij}^{\mu}\right)\\ &+ \tilde{\beta}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{s}_i - \tilde{r}_{ij}^{s}\right) + \tilde{\mu}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right)\\ &-\sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{s}_i - \tilde{r}_{ij}^{s}\right\|^2 - \sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right\|^2\\ = &-\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{r}_{ij}^{s}\right)^{\mathrm{T}}\left(\tilde{s}_i - \tilde{r}_{ij}^{s}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{r}_{ij}^{s}\right)^{\mathrm{T}}\left(\tilde{\beta}_i - \tilde{r}_{ij}^{\beta}\right)\\ &-\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{r}_{ij}^{\lambda}\right)^{\mathrm{T}}\left(\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{r}_{ij}^{\lambda}\right)^{\mathrm{T}}\left(\tilde{\mu}_i - \tilde{r}_{ij}^{\mu}\right)\\ &+\sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{r}_{ij}^{\beta}\right)^{\mathrm{T}}\left(\tilde{s}_i - \tilde{r}_{ij}^{s}\right) + \sum_{j\in\mathcal{N}_i} a_{ij}\left(\tilde{r}_{ij}^{\lambda}\right)^{\mathrm{T}}\left(\tilde{\lambda}_i - \tilde{r}_{ij}^{\lambda}\right)\end{aligned}$$

$$- \sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{s}_i - \tilde{r}^s_{ij}\right\|^2 - \sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{\lambda}_i - \tilde{r}^\lambda_{ij}\right\|^2$$
$$= \sum_{j\in\mathcal{N}_i} \tilde{\mu}^{\mathrm{T}}_{ij}\tilde{\gamma}_{ij} - \sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{s}_i - \tilde{r}^s_{ij}\right\|^2 - \sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{\lambda}_i - \tilde{r}^\lambda_{ij}\right\|^2$$

where $\tilde{\mu}_{ij} = \mathrm{col}(\tilde{r}^s_{ij}, \tilde{r}^\lambda_{ij}, \tilde{r}^\beta_{ij}, \tilde{r}^\mu_{ij}) = \mu_{ij} - \mu^\circ_{ij}$, and

$$\tilde{\gamma}_{ij} = E_{ij}\begin{bmatrix}\tilde{s}_i - \tilde{r}^s_{ij}\\ \tilde{\lambda}_i - \tilde{r}^\lambda_{ij}\\ \tilde{\beta}_i - \tilde{r}^\beta_{ij}\\ \tilde{\mu}_i - \tilde{r}^\mu_{ij}\end{bmatrix} = \gamma_{ij} - \gamma^\circ_{ij}.$$

By substitution, $\dot{V}$ further satisfies

$$\begin{aligned}\dot{V} = &-(k-1)\sum_{i=1}^n \|\tilde{v}_i\|^2 - \sum_{i=1}^n \tilde{s}_i^{\mathrm{T}}\left(\nabla g_i(s_i) - \nabla g_i\left(z_i^*\right)\right)\\ &+ \sum_{i=1}^n\sum_{j\in\mathcal{N}_i}\tilde{\mu}^{\mathrm{T}}_{ij}\tilde{\gamma}_{ij} - \sum_{i=1}^n\sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{s}_i - \tilde{r}^s_{ij}\right\|^2\\ &- \sum_{i=1}^n\sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{\lambda}_i - \tilde{r}^\lambda_{ij}\right\|^2. \end{aligned} \quad (15)$$

Due to Assumption 2, we have that $\tilde{s}_i^{\mathrm{T}}(\nabla g_i(s_i) - \nabla g_i(z_i^*)) \geq \omega\|\tilde{s}_i\|^2$, $i \in \mathcal{V}$. Accordingly, $\dot{V}$ finally satisfies

$$\begin{aligned}\dot{V} \leq &-(k-1)\sum_{i=1}^n \|\tilde{v}_i\|^2 - \omega\sum_{i=1}^n\|\tilde{s}_i\|^2 + \sum_{i=1}^n\sum_{j\in\mathcal{N}_i}\tilde{\mu}^{\mathrm{T}}_{ij}\tilde{\gamma}_{ij}\\ &- \sum_{i=1}^n\sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{s}_i - \tilde{r}^s_{ij}\right\|^2 - \sum_{i=1}^n\sum_{j\in\mathcal{N}_i} a_{ij}\left\|\tilde{\lambda}_i - \tilde{r}^\lambda_{ij}\right\|^2\\ \leq &-(k-1)\sum_{i=1}^n \|\tilde{v}_i\|^2 - \omega\sum_{i=1}^n\|\tilde{s}_i\|^2 + \sum_{i=1}^n\sum_{j\in\mathcal{N}_i}\tilde{\mu}^{\mathrm{T}}_{ij}\tilde{\gamma}_{ij}. \end{aligned} \quad (16)$$

Next, we consider W_{ij} in (13). Due to the definitions of μ°_{ij} and γ°_{ij}, we have that $\mu^\circ_{ij} = \mu^\circ_{ji}$ and $\gamma^\circ_{ij} = -\gamma^\circ_{ji}$. Thus, using (9), the derivative of W_{ij} satisfies

$$\begin{aligned}\dot{W}_{ij} = &\frac{1}{2}\left(\left\|\phi^{\rightarrow}_{ij} - \rho^\circ_{ij}\right\|^2 - \left\|\phi^{\leftarrow}_{ji} - \rho^\circ_{ij}\right\|^2 + \left\|\phi^{\rightarrow}_{ji} - \varrho^\circ_{ij}\right\|^2 - \left\|\phi^{\leftarrow}_{ij} - \varrho^\circ_{ij}\right\|^2\right)\\ = &\frac{1}{4}\left(\left\|-\gamma_{ij} + \mu_{ij} + \gamma^\circ_{ij} - \mu^\circ_{ij}\right\|^2 - \left\|\gamma_{ji} + \mu_{ji} + \gamma^\circ_{ij} - \mu^\circ_{ij}\right\|^2\right.\\ &\left. + \left\|-\gamma_{ji} + \mu_{ji} - \gamma^\circ_{ij} - \mu^\circ_{ij}\right\|^2 - \left\|\gamma_{ij} + \mu_{ij} - \gamma^\circ_{ij} - \mu^\circ_{ij}\right\|^2\right)\\ = &\frac{1}{4}\left(\left\|-\gamma_{ij} + \mu_{ij} + \gamma^\circ_{ij} - \mu^\circ_{ij}\right\|^2 - \left\|\gamma_{ji} + \mu_{ji} - \gamma^\circ_{ji} - \mu^\circ_{ji}\right\|^2\right.\\ &\left. + \left\|-\gamma_{ji} + \mu_{ji} + \gamma^\circ_{ji} - \mu^\circ_{ji}\right\|^2 - \left\|\gamma_{ij} + \mu_{ij} - \gamma^\circ_{ij} - \mu^\circ_{ij}\right\|^2\right)\\ \times &\frac{1}{4}\left(\left\|-\tilde{\gamma}_{ij} + \tilde{\mu}_{ij}\right\|^2 - \left\|\tilde{\gamma}_{ji} + \tilde{\mu}_{ji}\right\|^2 + \left\|-\tilde{\gamma}_{ji} + \tilde{\mu}_{ji}\right\|^2 - \left\|\tilde{\gamma}_{ij} + \tilde{\mu}_{ij}\right\|^2\right)\\ = &-\tilde{\gamma}^{\mathrm{T}}_{ij}\tilde{\mu}_{ij} - \tilde{\gamma}^{\mathrm{T}}_{ji}\tilde{\mu}_{ji}. \end{aligned} \quad (17)$$

Under Assumption 3, this thus implies that $\dot{W}$ satisfies

$$\dot{W} = -\sum_{i=1}^n\sum_{j\in\mathcal{N}_i}\tilde{\gamma}^{\mathrm{T}}_{ij}\tilde{\mu}_{ij}. \quad (18)$$

By combining (16) and (18), the derivative of Y satisfies $\dot{Y} \leq -(k-1)\sum_{i=1}^n\|\tilde{v}_i\|^2 - \omega\sum_{i=1}^n\|\tilde{s}_i\|^2$. Using the comparison principle [32], we obtain that

$$Y(t) + (k-1)\int_0^\infty\sum_{i=1}^n\|\tilde{v}_i(\iota)\|^2 d\iota + \omega\int_0^\infty\sum_{i=1}^n\|\tilde{s}_i(\iota)\|^2 d\iota \leq Y(0) \quad \forall t \geq 0.$$

Thus, $Y \in \mathcal{L}_\infty$ and $\tilde{v}_i, \tilde{s}_i \in \mathcal{L}_2$, $i \in \mathcal{V}$. By the definition of Y, it then follows that $\tilde{x}_i, \tilde{v}_i, \tilde{\lambda}_i, \tilde{\beta}_i, \tilde{\mu}_i \in \mathcal{L}_\infty$, $i \in \mathcal{V}$, and further that $\tilde{s}_i, s_i, \lambda_i, \beta_i, \mu_i \in \mathcal{L}_\infty$, $i \in \mathcal{V}$. In addition, by virtue of (7), (9), and (11), it is trivial to obtain that

$$\mu_{ij}(t) = E'_{ij}\mu_{ij}\left(t - T_{ij} - T_{ji}\right) + \varepsilon_{ij}(t) \quad (19)$$
$$\mu_{ji}(t) = E'_{ij}\mu_{ji}\left(t - T_{ij} - T_{ji}\right) + \varepsilon_{ji}(t) \quad (20)$$

where $E'_{ij} = (I_{2d} - E_{ij})^{-2}(I_{2d} + E_{ij})^2$ and ε_{ij} and ε_{ji} are the linear combinations of $\mathrm{col}(s_i, \lambda_i, \beta_i, \mu_i)$ and $\mathrm{col}(s_j, \lambda_j, \beta_j, \mu_j)$ at moments t, $t - T_{ij}$, $t - T_{ji}$, and $t - T_{ij} - T_{ji}$. Since all s_i, λ_i, β_i, and μ_i have been shown bounded, and all the eigenvalues of E'_{ij} can be shown within the unit circle for any a_{ij}, (19) and (20) are two stable difference equations driven by bounded inputs. Hence, μ_{ij} (i.e., $r^s_{ij}, r^\lambda_{ij}, r^\beta_{ij}$, and r^μ_{ij}), μ_{ji} (i.e., $r^s_{ji}, r^\lambda_{ji}, r^\beta_{ji}$, and r^μ_{ji}) $\in \mathcal{L}_\infty$, $j \in \mathcal{N}_i$, $i \in \mathcal{V}$, and so is γ_{ij}, $j \in \mathcal{N}_i$, $i \in \mathcal{V}$ given in (7). This implies that $\dot{v}_i \in \mathcal{L}_\infty$, $i \in \mathcal{V}$ given (6), and so is $\dot{s}_i$, $i \in \mathcal{V}$. Using Barbalat lemma [32], it follows that $\lim_{t\to\infty}\tilde{v}_i(t) = 0$ and $\lim_{t\to\infty}\tilde{s}_i(t) = 0$, $i \in \mathcal{V}$. This implies that $\lim_{t\to\infty}s_i(t) = z^*$ and $\lim_{t\to\infty}v_i(t) = 0$, $i \in \mathcal{V}$, and further that $\lim_{t\to\infty}x_i(t) = z_i^*$, $i \in \mathcal{V}$. Thus, with Lemma 1, the constrained optimization objective in Definition 1 is achieved. □

Remark 3: From Theorem 1, it is worthwhile to note that, the developed anti-delay distributed protocol (5) enables the multiagent system (1) to converge to the expected optimal solution independently of the parameters associated with network, optimization functions, and communication delays. This indicates its fully distributed nature, thereby widening its application scope in practice.

V. Extensions

In this section, we extend the anti-delay results of the previous section into two cases. First, we develop a velocity-free anti-delay distributed protocol for the second-order multiagent system (1) to fulfill the constrained optimization objective in Definition 1 in the absence of generalized velocity. Next, we augment the system order and consider the integrator chain multiagent system. Resorting to a proper coordinate transformation and a newly introduced composite variable, we develop a viable anti-delay distributed protocol for the concerned constrained optimization objective.

A. Velocity Absence Case

We consider the case where the generalized velocity v_i in the multiagent system (1) is unknown, and develop a velocity-free anti-delay distributed protocol in the following form:

$$u_i = -c(x_i - p_i) - q_i \tag{21a}$$
$$\dot{p}_i = q_i \tag{21b}$$
$$\dot{q}_i = -kq_i - c(p_i - x_i) - \nabla g_i(s_i) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) - A_i^{\mathrm{T}}\lambda_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i - r_{ij}^{\beta}\right) \tag{21c}$$
$$\dot{\lambda}_i = A_i s_i - b_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^{\lambda}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\mu_i - r_{ij}^{\mu}\right) \tag{21d}$$
$$\dot{\beta}_i = \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) \tag{21e}$$
$$\dot{\mu}_i = \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^{\lambda}\right) \tag{21f}$$

where $s_i = p_i + q_i$, and k and c are positive parameters. Substituting (21) into (1) yields the following closed-loop system:

$$\begin{bmatrix}\dot{x}_i\\ \dot{v}_i\\ \dot{p}_i\\ \dot{q}_i\\ \dot{\lambda}_i\\ \dot{\beta}_i\\ \dot{\mu}_i\end{bmatrix} = \begin{bmatrix} v_i\\ -c(x_i - p_i) - q_i\\ q_i\\ -kq_i - c(p_i - x_i) - \nabla g_i(s_i) - A_i^{\mathrm{T}}\lambda_i\\ A_i s_i - b_i\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\ \gamma_i\end{bmatrix} \tag{22}$$

where γ_i is defined similar to that of (6). To obtain the received information μ_{ij} in this case, we still introduce the scattering transformation (9) and use (11).

The following theorem demonstrates that the developed velocity-free anti-delay distributed protocol (21) guarantees the fulfillment of the concerned constrained objective despite both the velocity unavailability and communication delays.

Theorem 2: Consider the multiagent system (1) with Assumptions 1–3. The developed velocity-free anti-delay distributed protocol (21) with $k > 1$ guarantees that the constrained optimization objective is achieved in the sense of Definition 1.

Proof: For each $i \in \mathcal{V}$, we define tracking errors $\tilde{x}_i = x_i - z_i^*$, $\tilde{v}_i = v_i$, $\tilde{p}_i = p_i - z_i^*$, $\tilde{q}_i = q_i$, $\tilde{\lambda}_i = \lambda_i - \lambda_i^*$, $\tilde{\beta}_i = \beta_i - 2\beta_i^*$, and $\tilde{\mu}_i = \mu_i - 2\mu_i^*$, where z_i^*, λ_i^*, β_i^*, and μ_i^* are defined identically with those in the proof of Theorem 1.

Assign a Lyapunov function $Y_1 = V_1 + W$, where

$$V_1 = \sum_{i=1}^{n}\left(\frac{c}{2}\|\tilde{x}_i - \tilde{p}_i\|^2 + \frac{1}{2}\|\tilde{v}_i + \tilde{p}_i\|^2 + \frac{k}{2}\|\tilde{p}_i\|^2 + \tilde{p}_i^{\mathrm{T}}\tilde{q}_i + \frac{1}{2}\|\tilde{q}_i\|^2 + \frac{1}{2}\left\|\tilde{\lambda}_i\right\|^2 + \frac{1}{2}\left\|\tilde{\beta}_i\right\| + \frac{1}{2}\|\tilde{\mu}_i\|^2\right) \tag{23}$$

and W is defined the same with that in (13). The derivative of V_1 can be derived to satisfy

$$\begin{aligned}\dot{V}_1 =& -(k-1)\sum_{i=1}^{n}\|\tilde{q}_i\|^2\\ &+ \sum_{i=1}^{n}\tilde{s}_i^{\mathrm{T}}\left[-\nabla g_i(s_i) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) - A_i^{\mathrm{T}}s_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i - r_{ij}^{\beta}\right)\right]\\ &+ \sum_{i=1}^{n}\tilde{\lambda}_i^{\mathrm{T}}\left[A_i s_i - b_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^{\lambda}\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\mu_i - r_{ij}^{\mu}\right)\right]\\ &+ \sum_{i=1}^{n}\tilde{\beta}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) + \sum_{i=1}^{n}\tilde{\mu}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^{\lambda}\right)\end{aligned}$$

which is in the same form with (14). Therefore, by following the arguments after (14), $\dot{V}_1$ finally satisfies $\dot{V}_1 \le -(k-1)\sum_{i=1}^{n}\|\tilde{q}_i\|^2 - \omega\sum_{i=1}^{n}\|\tilde{s}_i\|^2 + \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i}\tilde{\mu}_{ij}^{\mathrm{T}}\tilde{\gamma}_{ij}$. In addition, by combining (18), the derivative of Y_1 satisfies $\dot{Y}_1 \le -(k-1)\sum_{i=1}^{n}\|\tilde{q}_i\|^2 - \omega\sum_{i=1}^{n}\|\tilde{s}_i\|^2$. Using the comparison principle [32], this implies that

$$Y_1(t) + (k-1)\int_0^{\infty}\sum_{i=1}^{n}\|\tilde{q}_i(\iota)\|^2 d\iota + \omega\int_0^{\infty}\sum_{i=1}^{n}\|\tilde{s}_i(\iota)\|^2 d\iota \le Y_1(0) \quad \forall t \ge 0.$$

Thus, $Y_1 \in \mathcal{L}_\infty$ and $\tilde{q}_i, \tilde{s}_i \in \mathcal{L}_2$, $i \in \mathcal{V}$. By the definition of Y_1, we next have that $\tilde{x}_i - \tilde{p}_i, \tilde{v}_i + \tilde{p}_i, \tilde{p}_i, \tilde{q}_i, \tilde{\lambda}_i, \tilde{\beta}_i, \tilde{\mu}_i \in \mathcal{L}_\infty$, $i \in \mathcal{V}$, and further that $\tilde{x}_i, \tilde{v}_i, \tilde{s}_i \in \mathcal{L}_\infty$, $i \in \mathcal{V}$. Hence, $x_i, v_i, p_i, q_i, s_i, \lambda_i, \beta_i, \mu_i \in \mathcal{L}_\infty$, $i \in \mathcal{V}$. In addition, by following the arguments in the proof of Theorem 1, we can also obtain that $\mu_{ij}, \gamma_{ij} \in \mathcal{L}_\infty$ and $j \in \mathcal{N}_i$, $i \in \mathcal{V}$. It thus follows from (22) that $\dot{v}_i, \dot{q}_i \in \mathcal{L}_\infty$, $i \in \mathcal{V}$, and so is $\dot{s}_i, \dot{\tilde{q}}_i, \dot{\tilde{s}}_i$, $i \in \mathcal{V}$. Based on Barbalat lemma [32], it follows that $\lim_{t\to\infty}\tilde{v}_i(t) = 0$, $\lim_{t\to\infty}\tilde{q}_i(t) = 0$, and $\lim_{t\to\infty}\tilde{s}_i(t) = 0$, $i \in \mathcal{V}$, and further that $\lim_{t\to\infty}\tilde{p}_i(t) = 0$, $i \in \mathcal{V}$. This thus implies that $\lim_{t\to\infty}v_i(t) = 0$, $\lim_{t\to\infty}q_i(t) = 0$, and $\lim_{t\to\infty}p_i(t) = z_i^*$, $i \in \mathcal{V}$. Next, differentiating $\dot{v}_i$ along (22) trivially yields $\ddot{v}_i \in \mathcal{L}_\infty$, $i \in \mathcal{V}$. Using Barbalat lemma [32], it follows that $\lim_{t\to\infty}\dot{v}_i(t) = 0$, $i \in \mathcal{V}$. This, from (22), implies that $\lim_{t\to\infty}(x_i(t) - p_i(t)) = 0$, $i \in \mathcal{V}$, and further that $\lim_{t\to\infty}x_i(t) = z_i^*$, $i \in \mathcal{V}$. Thus, by Lemma 1, the constrained optimization objective in Definition 1 is achieved. □

Remark 4: In practice, in order to save cost or lighten payload for a low-cost autonomous agent, it is feasible not to equip it with a velocity measurement unit. In addition, to keep the long-duration running of an autonomous agent, it is necessary to make it tolerant against sensor malfunction. Under such a background, the developed velocity-free anti-delay distributed protocol (21) is of significant engineering interest for the scenario with velocity information unavailability.

B. Integrator Chain Systems

Consider the following integrator chain multiagent system:

$$x_i^{(m_i+1)} = u_i \tag{24}$$

where $x_i, \dot{x}_i, \ldots, x_i^{(m_i)} \in \mathbb{R}^d$ denote the generalized states, $u_i \in \mathbb{R}^d$ denotes the generalized input, and $m_i \geq 1$ denotes the system order. Different agents are allowed to have distinct system orders. Due to the system order augmentation, the constrained optimization objective in Definition 1 turns into

$$\begin{cases} \lim_{t\to\infty} x_i(t) = z^* \\ \lim_{t\to\infty} \dot{x}_i(t) = 0 \\ \vdots \\ \lim_{t\to\infty} x_i^{(m_i)}(t) = 0 \end{cases} \tag{25}$$

where z^* is the optimal solution given in Definition 1.

For the purpose of the concerned constrained optimization, we first introduce the following coordinate transformation:

$$\begin{cases} p_{i,m_i} = x_i^{(m_i)} \\ p_{i,m_i-\bar{\kappa}} = \displaystyle\sum_{\kappa=0}^{\bar{\kappa}-1} \frac{(\bar{\kappa}-1)!}{\kappa!\,(\bar{\kappa}-1-\kappa)!} x_i^{(m_i-\kappa-1)}, \\ \quad 1 \leq \bar{\kappa} \leq m_i - 1. \end{cases} \tag{26}$$

With (24), the dynamics of the new states can be derived as

$$\begin{cases} \dot{p}_{i,1} = p_{i,2} + p_{i,3} + \cdots + p_{i,m_i} \\ \dot{p}_{i,2} = p_{i,3} + \cdots + p_{i,m_i} \\ \vdots \\ \dot{p}_{i,m_i-1} = p_{i,m_i} \\ \dot{p}_{i,m_i} = u_i. \end{cases} \tag{27}$$

We next introduce a composite variable for each agent i as

$$s_i = p_{i,1} + p_{i,2} + \cdots + p_{i,m_i}. \tag{28}$$

Using s_i, an anti-delay distributed protocol for the constrained optimization objective is developed as follows:

$$u_i = -m_i s_i + \sum_{\kappa=1}^{m_i-1} (m_i - \kappa)\, p_{i,\kappa} + p_{i,1} - \nabla g_i(s_i) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) - A_i^{\mathrm{T}}\lambda_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i - r_{ij}^\beta\right) \tag{29a}$$

$$\dot{\lambda}_i = A_i s_i - b_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^\lambda\right) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\mu_i - r_{ij}^\mu\right) \tag{29b}$$

$$\dot{\beta}_i = \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) \tag{29c}$$

$$\dot{\mu}_i = \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^\lambda\right). \tag{29d}$$

In terms of (27) and (29) and after some simple calculations, the following closed-loop system is available:

$$\begin{bmatrix} \dot{p}_{i,1} \\ \dot{s}_i \\ \dot{\lambda}_i \\ \dot{\beta}_i \\ \dot{\mu}_i \end{bmatrix} = \begin{bmatrix} s_i - p_{i,1} \\ -s_i + p_{i,1} - \nabla g_i(s_i) - A_i^{\mathrm{T}}\lambda_i \\ A_i s_i - b_i \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \gamma_i \end{bmatrix} \tag{30}$$

where γ_i is defined similar to that of (6). Still, we implement the scattering transformation (9) and adopt (11) to obtain the received information μ_{ij}.

The following theorem indicates that the developed anti-delay distributed protocol (29) enables the integrator chain multiagent system (24) to achieve the concerned constrained objective despite the communication delays.

Theorem 3: Consider the integrator chain multiagent system (24) with Assumptions 1–3. The developed anti-delay distributed protocol (29) guarantees that the constrained optimization objective (25) is achieved.

Proof: For each $i \in \mathcal{V}$, we define tracking errors $\tilde{p}_{i,1} = p_{i,1} - z_i^*$, $\tilde{s}_i = s_i - z_i^*$, $\tilde{\lambda}_i = \lambda_i - \lambda_i^*$, $\tilde{\beta}_i = \beta_i - 2\beta_i^*$, and $\tilde{\mu}_i = \mu_i - 2\mu_i^*$, where z_i^*, λ_i^*, β_i^*, and μ_i^* are defined identically with those in the proof of Theorem 1.

Assign a Lyapunov function $Y_2 = V_2 + W$, where

$$V_2 = \frac{1}{2}\sum_{i=1}^{n}\left(\left\|\tilde{p}_{i,1}\right\|^2 + \|\tilde{s}_i\|^2 + \left\|\tilde{\lambda}_i\right\|^2 + \left\|\tilde{\beta}_i\right\| + \|\tilde{\mu}_i\|^2\right) \tag{31}$$

and W is defined identically with that in (13). It can be derived that the derivative of V_2 satisfies

$$\begin{aligned} \dot{V}_2 = &-\sum_{i=1}^{n}\left\|\tilde{s}_i - \tilde{p}_{i,1}\right\|^2 \\ &+ \sum_{i=1}^{n}\tilde{s}_i^{\mathrm{T}}\Bigg[-\nabla g_i(s_i) - \sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) \\ &\qquad - A_i^{\mathrm{T}} s_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\beta_i - r_{ij}^\beta\right)\Bigg] \\ &+ \sum_{i=1}^{n}\tilde{\lambda}_i^{\mathrm{T}}\Bigg[A_i s_i - b_i - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\lambda_i - r_{ij}^\lambda\right) \\ &\qquad - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\mu_i - r_{ij}^\mu\right)\Bigg] \\ &+ \sum_{i=1}^{n}\tilde{\beta}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i} a_{ij}\left(s_i - r_{ij}^s\right) \end{aligned}$$

$$+\sum_{i=1}^{n}\tilde{\mu}_i^{\mathrm{T}}\sum_{j\in\mathcal{N}_i}a_{ij}\left(\lambda_i-r_{ij}^{\lambda}\right)$$

which is in the same form with (14). Next, by following the arguments after (14), $\dot{V}_2$ finally satisfies $\dot{V}_2 \leq -\sum_{i=1}^{n}\|\tilde{s}_i - \tilde{p}_{i,1}\|^2 - \omega\sum_{i=1}^{n}\|\tilde{s}_i\|^2 + \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i}\tilde{\mu}_{ij}^{\mathrm{T}}\tilde{\gamma}_{ij}$. In addition, by combining (18), the derivative of Y_2 satisfies $\dot{Y}_2 \leq -\sum_{i=1}^{n}\|\tilde{s}_i - \tilde{p}_{i,1}\|^2 - \omega\sum_{i=1}^{n}\|\tilde{s}_i\|^2$. Using the comparison principle [32], this implies that

$$Y_2(t)+\int_0^{\infty}\sum_{i=1}^{n}\left\|\tilde{s}_i(\iota)-\tilde{p}_{i,1}(\iota)\right\|^2 d\iota+\omega\int_0^{\infty}\sum_{i=1}^{n}\|\tilde{s}_i(\iota)\|^2 d\iota \leq Y_2(0)\quad \forall t\geq 0.$$

Hence, $Y_2 \in \mathcal{L}_\infty$ and $\tilde{s}_i-\tilde{p}_{i,1}, \tilde{s}_i \in \mathcal{L}_2, i \in \mathcal{V}$. By the definition of Y_2, it follows that $\tilde{p}_{i,1}, \tilde{s}_i, \tilde{\lambda}_i, \tilde{\beta}_i, \tilde{\mu}_i \in \mathcal{L}_\infty, i \in \mathcal{V}$, and further that $\tilde{s}_i - \tilde{p}_{i,1} \in \mathcal{L}_\infty, i \in \mathcal{V}$. Moreover, by following the arguments in the proof of Theorem 1, we can also obtain that $\mu_{ij}, \gamma_{ij} \in \mathcal{L}_\infty, j \in \mathcal{N}_i, i \in \mathcal{V}$. It thus follows that $\dot{\tilde{p}}_{i,1}, \dot{\tilde{s}}_i \in \mathcal{L}_\infty, i \in \mathcal{V}$, and further that $\dot{\tilde{s}}_i - \dot{\tilde{p}}_{i,1} \in \mathcal{L}_\infty, i \in \mathcal{V}$. With Barbalat lemma [32], we thus have that $\lim_{t\to\infty}(\tilde{s}_i(t) - \tilde{p}_{i,1}(t)) = 0$ and $\lim_{t\to\infty}\tilde{s}_i(t) = 0, i \in \mathcal{V}$, and further that $\lim_{t\to\infty}\tilde{p}_{i,1}(t) = 0, i \in \mathcal{V}$. This implies that $\lim_{t\to\infty}s_i(t) = z_i^*$ and $\lim_{t\to\infty}p_{i,1}(t) = z_i^*, i \in \mathcal{V}$. Next, due to (27), we have that $\dot{p}_{i,2} = -p_{i,2} + s_i - p_{i,1}$. Using the input-to-state stability theory [32], this implies that $\lim_{t\to\infty}p_{i,2}(t) = 0, i \in \mathcal{V}$. Iterating this analysis yields $\lim_{t\to\infty}p_{i,3}(t) = 0, \ldots, \lim_{t\to\infty}p_{i,m_i}(t) = 0, i \in \mathcal{V}$. Finally, by conducting an inverse transformation on (26), it follows that $\lim_{t\to\infty}x_i(t) = z_i^*$, $\lim_{t\to\infty}\dot{x}_i(t) = 0, \ldots, \lim_{t\to\infty}x_i^{(m_i)}(t) = 0, i \in \mathcal{V}$. Thus, with Lemma 1, the constrained optimization objective (25) is achieved. □

Remark 5: Note from Theorem 3 that, using the composite variable s_i in (28) regarding the new states p_{i_1}, $p_{i,2}, \ldots, p_{i,m_i}$ from the coordinate transformation (26), the developed anti-delay distributed protocol (29) still achieves the constrained optimization objective, regardless of the system dimension augmentation. Also, note from (30) that the closed-loop system for each agent is in a uniform form, which is just related to two system-related states $p_{i,1}$ and s_i, despite the heterogeneity in agent system orders. Accordingly, the introduction of the composite variable s_i in (28) from the coordinate transformation (26) plays a role in solving the issues caused by dimension augmentation and order heterogeneity.

VI. Simulation Verification

In this section, a practical source localization example [24] is given to confirm the optimization performance of the developed anti-delay distributed protocols. Specifically, given a multiagent system of five agents with $d = 3$, they cooperate to search for an acoustic source $x_0 \in \mathbb{R}^3$ guided by five anchors $\alpha_j \in \mathbb{R}^3$, $j \in \{1, 2, \ldots, 5\}$ around the source. However, each anchor is only available to some agents. Accordingly, an anchor availability matrix $\Delta = [\delta_{ij}] \in \mathbb{R}^{5\times 5}$ is introduced to build the availability relationship between the anchors and agents. Particularly, $\delta_{ij} = 1$ (or 0) means that agent i has (or no) access to anchor j. Each agent chooses its local payoff function as $g_i(x) = \sum_{j=1}^{5}\delta_{ij}\|x - r_j\|^2$, which means that

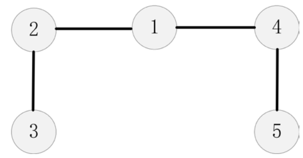

Fig. 1. Communication topology.

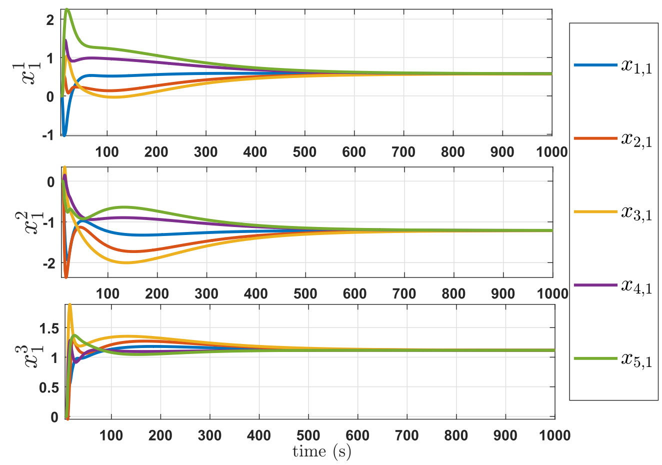

Fig. 2. Trajectories of generalized positions driven by distributed protocol (5).

it would like to move closer to its available anchors. The optimization objective is to regulate all the agents to rendezvous on the optimal solution to the global payoff function $g(x) = \sum_{i=1}^{5}g_i(x)$, which is the optimal localization of the unknown source.

To be specific, the source is posited at $x_0 = [0, 2, 0]^{\mathrm{T}}$, and five anchors are posited at $\alpha_1 = [3, 4, 2]^{\mathrm{T}}$, $\alpha_2 = [1, 1, 3]^{\mathrm{T}}$, $\alpha_3 = [-2, 2, -1]^{\mathrm{T}}$, $\alpha_4 = [-3, -6, 2]^{\mathrm{T}}$, and $\alpha_5 = [2, 5, 0]^{\mathrm{T}}$, respectively. The anchor availability matrix $\Delta = [0, 0, 1, 1, 0; 0, 0, 0, 0, 1; 0, 1, 0, 0, 0; 1, 1, 0, 0, 1; 1, 0, 0, 0, 1; 1, 0, 1, 0, 0]$ is chosen. In addition, five agents suffer from a coupled equality constraint (2) parameterized by $A_1 = [1, 1, 0]$, $b_1 = 0$, $A_2 = [1, -1, 1]$, $b_2 = 1$, $A_3 = [0, 2, -1]$, $b_3 = -1$, $A_4 = [1, -3, 2]$, $b_4 = 2$, and $A_5 = [1, 3, -1]$, $b_5 = -1$. Moreover, the interagent communication topology is depicted in Fig. 1. It can be confirmed that Assumptions 1–3 are met.

A. Example 1: Anti-Delay Performance Verification

Consider the multiagent system (1). To highlight the anti-delay performance of the developed distributed protocol (5) that uses the scattering transformation (9) for the generation of the received information [i.e., (11)], we compare it with the one proposed in [24] that directly uses the delayed states. The delays of different communication channels are distinct, which are specified as

$$\mathbb{T} = [T_{ij}] = \begin{bmatrix} 0 & 0.2 & 0 & 0.4 & 0 \\ 0.1 & 0 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 & 0 \end{bmatrix}.$$

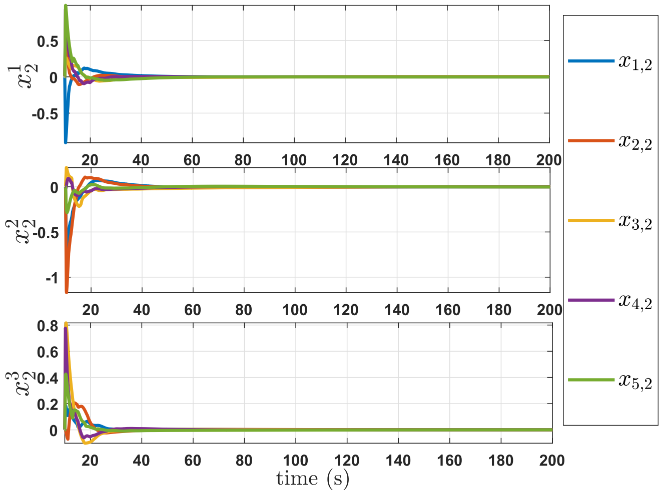

Fig. 3. Trajectories of generalized velocities driven by distributed protocol (5).

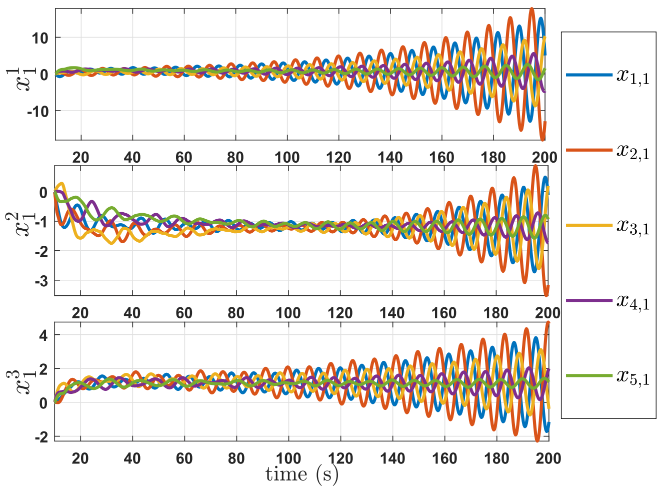

Fig. 4. Trajectories of generalized positions driven by distributed protocol in [24].

The nondiagonal zeros come from the nonexistence of communication channels. In addition, the parameter $k = 4$ is set. The simulation results in this example are given in Figs. 2–5.

Figs. 2 and 3 show the trajectories of the generalized positions and velocities driven by the developed anti-delay distributed protocol (5). It can be observed that, despite the heterogeneous communication delays, all the generalized velocities converge to zero asymptotically, and the generalized positions reach a consensus on the unique optimal solution $[0.582, -1.208, 1.116]^{\mathrm{T}}$. This indicates the fulfillment of the constrained optimization objective claimed in Definition 1. For comparison, Figs. 4 and 5 give the trajectories of the generalized positions and velocities driven by the distributed protocol proposed in [24] that directly uses the delayed states. It can be seen that all the trajectories diverge as time slides due to the delay effect. By comparison, the developed anti-delay distributed protocol (5) reveals the favorable robust performance against communication delays.

B. Example 2: Velocity Absence

We consider the absence of the generalized velocity for the multiagent system (1) and verify the anti-delay performance

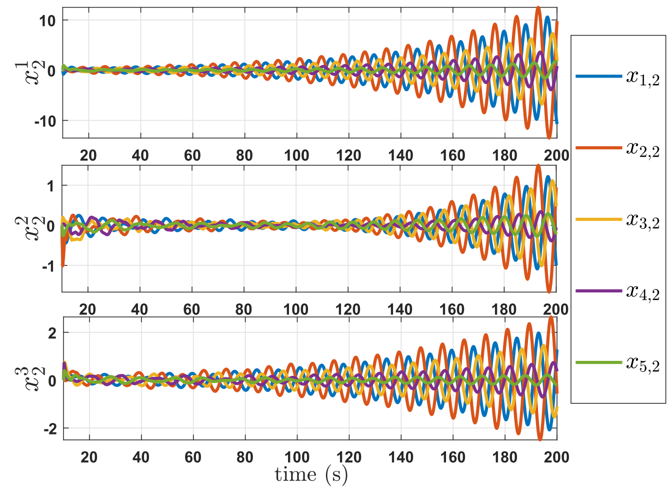

Fig. 5. Trajectories of generalized velocities driven by distributed protocol in [24].

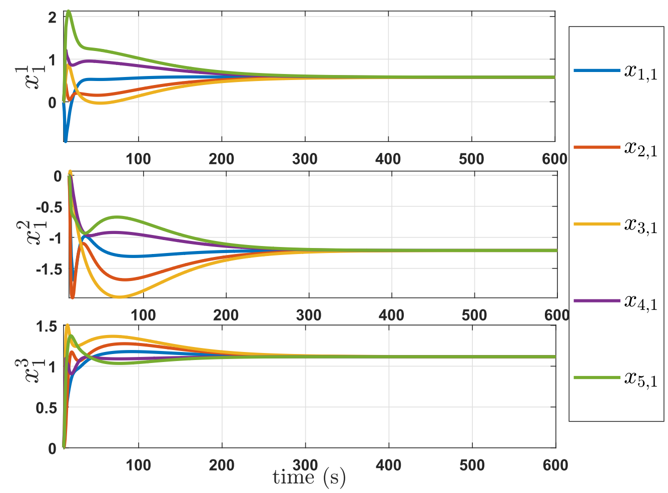

Fig. 6. Trajectories of generalized positions driven by distributed protocol (21).

of the developed velocity-free distributed protocol (21). The parameters $k = 4$ and $c = 4$ are specified. The communication delays are in the same form as those in the previous example. The trajectories of the generalized positions and velocities are given in Figs. 6 and 7. Note therein that, despite the communication delays and velocity unavailability, the developed velocity-free anti-delay distributed protocol (21) still guarantees that the generalized positions reach a consensus on the optimal solution of interest, and the generalized velocities asymptotically converge to 0. This verifies the robustness of the developed velocity-free anti-delay distributed protocol (21) against both communication delays and velocity unavailability.

C. Example 3: Integrator Chain System

Next, we consider the integrator chain multiagent system (24) with $l_i = 3$ and validate the anti-delay performance of the improved distributed protocol (29). Also, the communication delays are identical with those in Example 1. The trajectories of three order states are, respectively, illustrated in Figs. 8–10. Note therein that the first-order states reach a

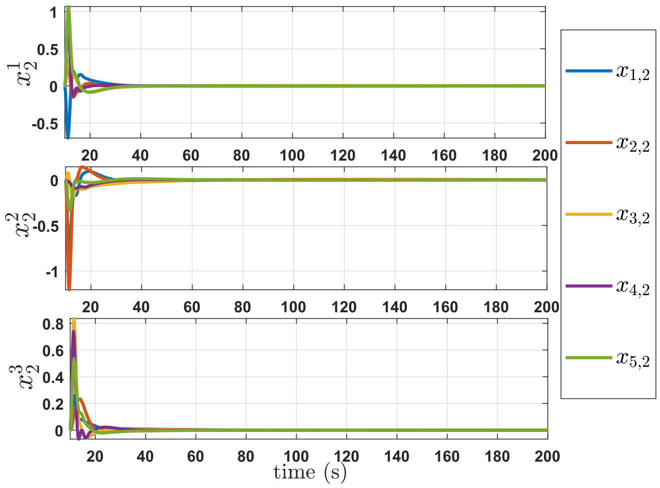

Fig. 7. Trajectories of generalized velocities driven by distributed protocol (21).

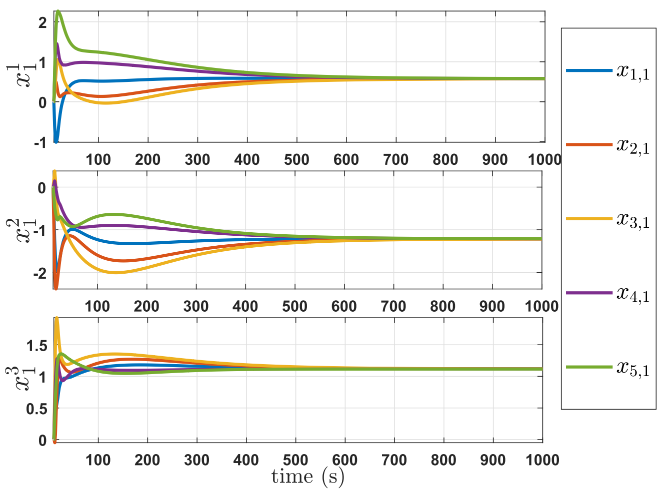

Fig. 8. Trajectories of first-order states driven by distributed protocol (29).

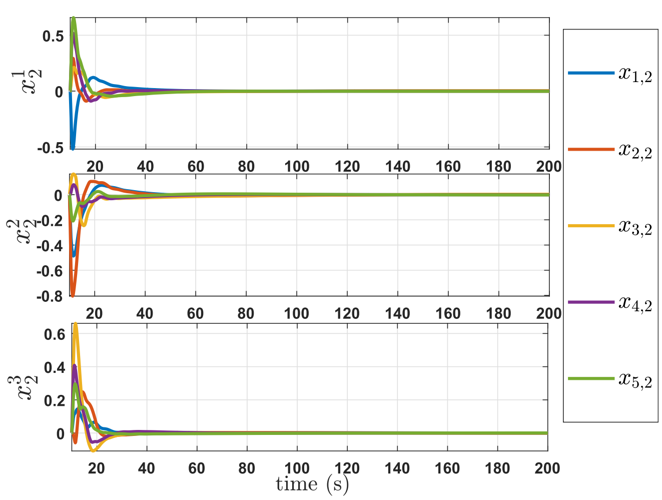

Fig. 9. Trajectories of second-order states driven by distributed protocol (29).

consensus on the expected optimal solution, and the second- and third-order states both converge to zero asymptotically. This substantiates that the improved distributed protocol (29) for the integrator chain multiagent system is also well-tolerant against communication delays.

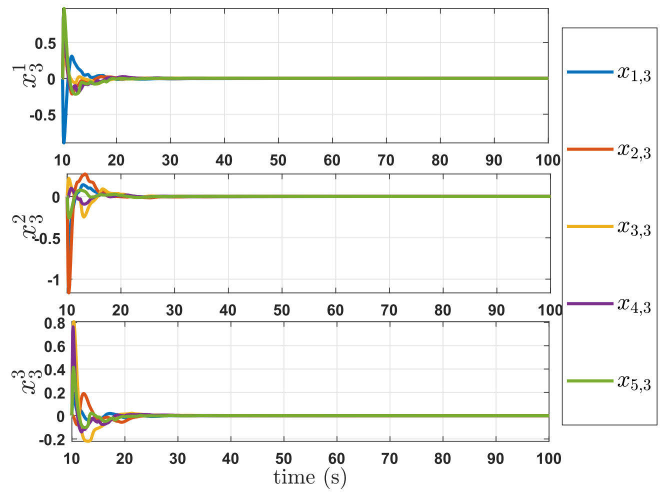

Fig. 10. Trajectories of third-order states driven by distributed protocol (29).

VII. Conclusion

We develop an anti-delay distributed protocol for a multiagent system with second-order dynamics to achieve the constrained optimization objective in spite of heterogeneous communication delays. In particular, the optimal solution of a global payoff function accumulated by strongly convex local ones is calculated in parallel within a feasible set involving a coupled equality constraint, while each local payoff function and each group of constraint parameters exclusively belong to a respective agent. By introducing proper scattering transformation and internal auxiliary dynamics, it is proven that the developed anti-delay distributed protocol enables the multiagent system to reach a consensus on the optimal solution of interest. Upon this basis, we consider the case of no velocity availability and develop a velocity-free anti-delay distributed protocol for the sake of the concerned constrained optimization objective. Besides this, we augment the system order and develop an anti-delay distributed optimization protocol, resorting to a proper coordinate transformation for the integrator chain multiagent system. In the future, we will work on the solution to the studied constrained optimization problem for nonlinear systems in the presence of time-varying communication delays over a directed topology. Also, to adapt to practical applications, the prescribed output performance constraint will be considered [33], [34].

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