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Adaptive Control of a Flexible Varying-length Beam with a Translating Base in the 3D Space

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Abstract: This paper investigates a control scheme for a variable-length beam attached to a translating base under an unknown boundary disturbance. The axial beam motion is assumed pre-defined. A hybrid system consisting of a gantry, a trolley, and an expandible cantilever beam attached to the trolley is considered. Two control forces are applied to the trolley and the gantry, respectively, to position them and suppress the vibration of the beam. According to Hamilton's principle, a nonlinear mathematical model is developed describing the dynamics of the transverse and lateral oscillations of the beam, trolley, and gantry. Based on this dynamic model, a robust adaptive control law is developed to handle the closed-loop stability of the axially moving system with unknown disturbances. Stability analysis using the Lyapunov method proves that the closed-loop system under the proposed control law is uniformly ultimately bounded. Finally, numerical simulations verify the proposed control laws' effectiveness.

Keywords: Adaptive control, axially moving system, boundary control, flexible cantilever beam, Lyapunov method, varying length.

1. INTRODUCTION

Structures consisting of a cantilever attached to a moving base are widely utilized in various applications such as Cartesian robots, industrial cranes [1], gantry robots (Fig. 1), flapping-wing robots [2], and refueling machines. A cantilever beam of elastic material is naturally flexible because one end is not hinged. A beam with low weight has advantages in terms of cost and mobility but becomes more flexible. With the advantages, elastic beams have received significant attention recently. However, in contrast to a stiff beam with negligible vibration, the vibration of a flexible beam becomes critical in high-speed operation. Specifically, in such systems consisting of an elastic beam affixed to a moving base, the motion of the base can cause vibrations along the beam. These vibrations are negative factors affecting the performance of the system. Therefore, suppressing the residual vibration after maneuvering is highly desirable. Additionally, in practice, disturbances such as frame vibration, wind, rail friction, or vibrations in the uncontrolled beam span (i.e., see the upper part in Fig. 2) can affect system dynamics. Therefore, this study targets the boundary control of the variable-length beam attached to a translating base in the presence of an unknown disturbance.

A flexible beam is a distributed parameter system. Therefore, its dynamics are described by partial differential equations (PDEs) [3-6]. The dynamic behavior of flexible cantilever beams is a classical problem researched for several decades. In such a configuration consisting of an elastic beam attached to a translating/rotating base, the beam's dynamics affect the base's motions and vice versa [4,5]. Early studies on this topic were published by Kane *et al*. [5] and Hanagud and Sarkar [7]. Park *et al*. [8] investigated the dynamic characteristics of a flexible beam mounted to a moving base, including natural frequencies and mode shapes. Later, Park and Youm [9] conducted an experimental study to investigate the vibrational behavior of the beam. The control problem of distributed parameter systems, whose dynamics are described by PDEs, has been investigated in the literature [10,11]. The boundary control technique, wherein the control input is exerted on the PDE through its boundary conditions [12-16], is an effective method for handling control spillover problems. The well-posedness issue of flexible cantilever beams was intriguingly discussed in [17-19]. A dynamic model derived from the extended Hamilton principle is well-posed. The closed-loop system is also well-posed if a feedback

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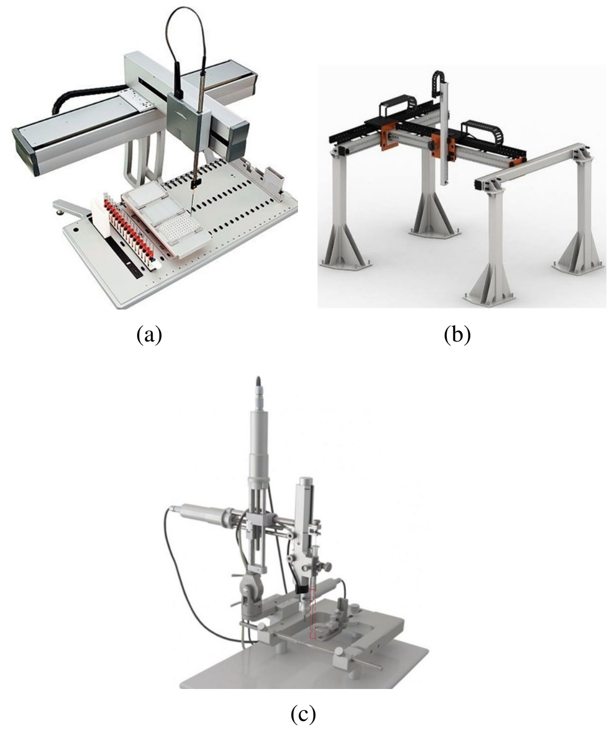


Fig. 1. Machines consisting of a cantilever beam attached to a moving base: (a) Liquid handling robot (www.medicalexpo.com/prod/tecan), (b) gantry manipulator (www.in-diamart.com/proddetail), and (c) drill and injection robot (https://animalab.eu/drill-and-injection-robot-with-stereotaxic-frame-1).

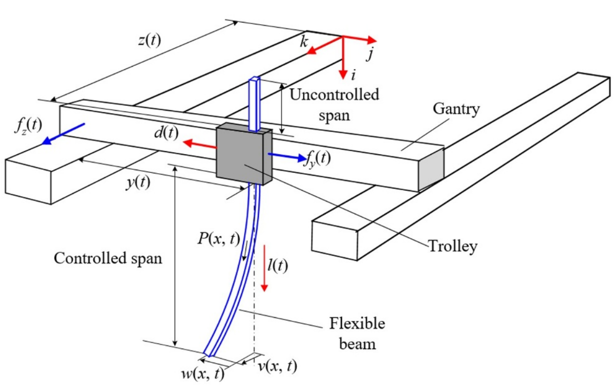


Fig. 2. Scheme of a flexible beam attached to a translating base operating in the 3D space.

control law is derived using a Lyapunov function. Due to space limitations, the well-posedness issue is skipped in this paper.

The control problem of an elastic beam attached to a moving base has also been considered. For beams attached to a rotating base, Liu *et al*. [20] addressed the adaptive neural network control of the beam attached to a rotating hub. Liu *et al*. [21] developed a boundary control law for a flexible robotic manipulator system, modeled as a varying-length beam attached to a rotating hub. Later, this system's asymmetric input–output constraint control was investigated in [22]. The vibration control of the rotating beam under external disturbance was also considered in [23]. For the beam attached to a translating base, Park *et al*. [12] proposed an input preshaping method to suppress the single-mode vibration of the beam, whereas Shah and Hong [24] designed an input shaping control scheme for an underwater elastic beam mounted to a moving trolley. Pham *et al*. [25] presented an experimental investigation on the performance of various types of input shaping control for vibration suppression of a non-uniform beam with a moving hub. Lin and Chao [26] applied an intelligent control strategy called adaptive neuro-fuzzy control to suppress the vibration of a beam-hub system. These studies assumed that the base moves in one direction, resulting in beam vibrations restricted to the 1-D plane. Zhang *et al*. [27] designed a control law for a three-dimensional cable hung from a helicopter under output constraints, and an input backlash scheme was designed. Shah and Hong [13] investigated the control problem of a flexible beam in the 3D space. The authors considered a flexible beam system in the presence of a hydrodynamic force, wherein the base moved along a plane. According to the linear dynamic model of the system, a robust adaptive control scheme was developed using the Lyapunov design method to suppress both the transverse and lateral vibrations of the beam. Pham *et al*. [28] extended the control technique in [13] to the nonlinear system and further considered the longitudinal displacement. Additionally, systems consisting of a flexible string hung from a moving base, similar to a beam attached to a moving hub, were also considered.

The mentioned studies on beams attached to a translating base assumed that the length of the beam/string is constant. In gantry manipulators, robotic arms, which can be treated as flexible beams, can extend or retract during operation. This results in changes in beam length. When the beam length changes over time, the beam can be treated as an axially moving beam with a time-varying length. Axially moving systems are characterized by gyroscopic and distributed parameter properties [29,30]. One of the most critical studies on the dynamics of axially moving strings/beams of variable length is that by Zhu and Ni [31]. The authors developed mathematical models of axially moving strings/beams with a time-varying length and discussed the energy of a system during extension and retraction. They highlighted that the vibration energy of axially moving systems with variable lengths decreases during extension and increases during retraction. Several studies have explored the control of axially moving systems [32]. Fung *et al*. [33] investigated sliding-mode control for a flexible cable with time-varying length, In contrast, Kim and Chung [34] introduced a boundary control law for an elastic beam deployed by a moving base. Zhu *et al*. [35]

designed a pointwise control law for the variable-length beam/string systems considered in [31]. These studies developed control laws based on the order-reduced models described by ordinary differential equations (ODEs). However, control design using the ODE model can result in spillover problems [29]. Many researchers have conducted studies on control design using a PDE model to overcome these spillover problems. Kim and Hong [36] proposed boundary control for an overhead crane with a varying-length string attached to a translating trolley. Later, Ngo *et al*. [37] extended this control scheme to address unknown boundary disturbances. In these studies, the uniform stability of a closed-loop system was proven using the Lyapunov method. By using this method, the control problem of a varying-length Timoshenko beam mounted to translating support was addressed by Pham *et al*. [38]. For the vibration suppression of flexible strings with a time-varying length in the 3D space, Xing *et al*. [39] developed a boundary control law for eliminating the transverse, lateral, and longitudinal vibrations of a string under an input constraint. Subsequently, Xing and Liu [40] considered a 3D flexible cable hung from a moving trolley. A boundary control law was proposed for controlling the system's position and vibration suppression under input amplitude and rate constraints. Additionally, the Lyapunov design method was used to determine the control law and verify the stability of the system.

The robotic arms of Cartesian robots can be modeled as flexible beams of variable lengths affixed to a translating base. Additionally, in the case of large-amplitude vibration, dynamic tension cannot be ignored, resulting in beam vibrations described by nonlinear PDEs. Furthermore, unknown boundaries caused by the influences of frame vibration, rail friction, or vibrations of the uncontrolled beam span (i.e., see Fig. 2) may appear and affect the base dynamics. Therefore, this paper proposes an adaptive boundary control law for a variable-length beam attached to a translating base subjected to an unknown boundary disturbance. A nonlinear dynamic model of the system is established using the extended Hamilton principle. Accordingly, an adaptive control law with an adaptation law is designed to handle unknown disturbances. Based on the Lyapunov method, the ultimate uniform stability of the system under the proposed adaptive control law is proved. Finally, numerical simulations are conducted to verify the designed control laws in two cases, with and without disturbance.

This paper overcomes the limitations of the existing control strategies for varying-length beams/strings that require the implementation of control inputs at the free end of the beam/string [40]. These control strategies may be applicable in typical scenarios. However, as seen in Figs. 1 and 2, there is no way to apply control forces at the tip position of the end-effectors of gantry manipulators without hindering the system's operation. This paper proposes a control strategy that directly uses the control forces applied to the base to suppress the vibrations of the beam. The main contributions of this study are as follows:

1) Develop a novel nonlinear dynamic model of a variable-length beam attached to a translating base in the presence of unknown disturbance, where the coupling dynamics of the transverse and lateral vibrations and the base are considered.
2) Design an adaptive control law for handling an unknown boundary disturbance. Additionally, the uniform stability of the closed-loop system is proved, and numerical simulations are conducted.

The remainder of this paper is organized as follows: The dynamic model of the system is presented in Section 2, and Section 3 develops an adaptive control law. In Section 4, numerical results are provided. Finally, our conclusions are summarized in Section 5.

2. DYNAMIC MODEL

Fig. 2 presents a flexible cantilever beam of time-varying length $l(t)$ attached to a trolley of mass M, where the trolley translationally moves along a gantry of mass M_g. The cantilever beam is treated as an Euler-Bernoulli beam with mass density ρ, cross-sectional area A, Young's modulus E, and area moments of inertia I_y and I_z. The trolley and gantry are controlled by two forces, f_y and f_z, respectively. The trolley separates the vertical beam into two spans: The upper span and the lower span, see Fig. 2. In most practical systems, only the vibration of the beam's lower span is considered. Therefore, this span is referred to as the controlled span. The influence of the vibration of the upper span (i.e., the uncontrolled span) on the system is manifested by the disturbance $d(t)$ at the trolley. This disturbance is assumed to be bounded by an unknown positive constant d_b (i.e., $|d(t)| < d_b$). Additionally, we assume that the beam length $l(t)$ is a predefined time function. Therefore, the flexible beam can be treated as an axially moving beam with a time-varying length.

Let $y(t)$ and $z(t)$ denote the position of the trolley and the gantry, respectively. The vibrations of the beam in the j-axis and k-axis are defined as the transverse vibration $w(x,t)$ and the lateral vibration $v(x,t)$, respectively. In this paper, $\dot{y}$, $\dot{z}$, and $\dot{l}$ denote the total derivatives of $y(t), z(t)$, and $l(t)$ with respect to t, respectively; the subscripts in $(\cdot)_x$ and $(\cdot)_t$ are the partial derivatives of the spatiotemporal functions with respect to x and t, respectively; and $D(\cdot)/Dt = (\cdot)_t + \dot{l}(\cdot)_x$ denotes the material derivative.

According to the Euler-Bernoulli beam theory, the kinetic energy K and the potential energy U are derived as follows:

$$K = \frac{1}{2}\rho A \int_0^l \left[\dot{l}^2 + (\dot{y} + Dw/Dt)^2 + (\dot{z} + Dv/Dt)^2\right]$$

$$+\frac{1}{2}m\dot{y}^2+\frac{1}{2}M\dot{z}^2, \tag{1}$$

$$U=\frac{1}{2}\int_0^l P(x,t)\left(w_x^2+v_x^2\right)dx + \frac{1}{8}\int_0^l EA\left(w_x^2+v_x^2\right)^2dx+\frac{1}{2}EI_y\int_0^l w_{xx}^2dx + \frac{1}{2}EI_z\int_0^l v_{xx}^2dx, \tag{2}$$

where $m=M+M_{\mathrm{g}}$ and $P(x,t)=\rho A(l-x)(g-\ddot{l})$ is the axial force [13]. The work done due to the control forces, disturbance, and structural damping is derived as follows:

$$\delta W=(f_y+d)\,\delta y+f_z\delta y-c_w\int_0^l w_t\delta w dx - c_v\int_0^l v_t\delta v dx, \tag{3}$$

where c_w and c_v are structural damping coefficients. According to (1)-(3), a dynamic model of the system can be derived using the extended Hamilton principle as follows:

$$m\ddot{y}+\rho A\dot{l}\dot{y}-c_w\int_0^l w_t dx+EI_y w_{xxx}(0,t)=f_y+d, \tag{4}$$

$$M\ddot{z}+\rho A\dot{l}\dot{z}-c_v\int_0^l v_t dx+EI_z v_{xxx}(0,t)=f_z, \tag{5}$$

$$\rho A\left(\ddot{y}+D^2w/Dt^2\right)+c_w w_t-(Pw_x)_x - EA\left[w_x\left(w_x^2+v_x^2\right)\right]_x/2+EI_y w_{xxxx}=0, \tag{6}$$

$$w(0,t)=w_x(0,t)=0, \tag{7}$$

$$EAw_x(l,t)\left[w_x^2(l,t)+v_x^2(l,t)\right]/2-EI_y w_{xxx}(l,t)=0, \quad w_{xx}(l,t)=0, \tag{8}$$

$$\rho A\left(\ddot{z}+D^2v/Dt^2\right)+c_v v_t-(Pv_x)_x - EA\left[v_x\left(w_x^2+v_x^2\right)\right]_x/2+EI_z v_{xxxx}=0, \tag{9}$$

$$v(0,t)=v_x(0,t)=0, \tag{10}$$

$$EAv_x(l,t)\left[w_x^2(l,t)+v_x^2(l,t)\right]/2-EI_z v_{xxx}(l,t)=0, \quad v_{xx}(l,t)=0. \tag{11}$$

The nonlinear PDE-ODE model in (4)-(11) describes all the dynamics of the considered system, where the trolley and gantry's motions are represented by two ODEs in (4) and (5), respectively, and the two PDEs and boundary conditions in (6)-(11) describe the transverse and lateral vibrations of the flexible beam.

3. CONTROL DESIGN

This section describes the development of boundary control laws for feedback control using the Lyapunov design method. The control objectives are to position the gantry and trolley and simultaneously suppress the vibration energy of the beam's controlled span (the lower part). Accordingly, the two control forces f_y and f_z applied to the trolley and gantry are designed to guarantee closed-loop stability.

The length of the lower cantilever beam (the controlled span) in this paper is time-varying, but its length changes in a pre-described manner. The mechanical energy of the vibrating beam consists of the energies due to the transverse motion, lateral motion, and longitudinal motion of the beam. But, the energy due to the longitudinal motion is finite due to its prescribed motion and can be omitted from the stability analysis of the closed-loop system [36]. The following assumption and lemmas are presented for analyzing system stability.

Assumption 1: The axial force $P(x,t)$ is bounded as follows:

$$0\le P(x,t)\le P_{\max}, \quad P_{\mathrm{D}\min}\le DP(x,t)/Dt\le P_{\mathrm{D}\max}. \tag{12}$$

Lemma 1 [41]**:** Let $\psi_1(x,t)\in\mathbb{R}$ and $\psi_2(x,t)\in\mathbb{R}$ be two functions defined on $x\in[0,\ l]$ and $t\in[0,\ \infty)$. Then, the following inequalities hold:

$$\psi_1(x,t)\psi_2(x,t)\le\psi_1^2(x,t)/\delta+\delta\psi_2^2(x,t),\ \forall\delta>0. \tag{13}$$

Lemma 2 [42]**:** Let $\psi(x,t)\in\mathbb{R}$ be a function defined on $x\in[0,\ l]$ and $t\in[0,\ \infty)$ that satisfies the boundary condition $\psi(0,t)=\psi_x(0,t)=0$, $\forall t\in[0,\ \infty)$. Then, the following inequalities hold $\forall x\in[0,\ l]$:

$$\int_0^l\psi^2(x,t)dx\le l^2\int_0^l\psi_x^2(x,t)dx\le l^4\int_0^l\psi_{xx}^2(x,t)dx, \tag{14}$$

$$\psi^2(x,t)\le l\int_0^l\psi_x^2(x,t)dx\le l^3\int_0^l\psi_{xx}^2(x,t)dx. \tag{15}$$

Lemma 3 [43]**:** If $\psi(x,t):[0,\ l]\times\mathbb{R}^+\to\mathbb{R}$ is uniformly bounded, $\{\psi(x,t)\}_{x\in[0,l]}$ is equicontinuous on t, and $\lim_{t\to\infty}\int_0^t\|\psi(x,\tau)\|^2d\tau$ exists and is finite, then $\lim_{t\to\infty}\|\psi(x,\tau)\|=0$, where $\{\psi(x,t)\}_{x\in[0,l]}$ denotes the function $\psi(x,t)$ with $x\in[0,l]$; $\|\cdot\|$ is used to denote the norm of an infinite dimensional vector, i.e., $\|\psi(x,t)\|=\left(\int_0^l\psi^2(x,t)dx\right)^{1/2}$.

The disturbance is assumed to be a periodic function bounded by an unknown value d_{b}. A robust adaptive boundary control law is developed to handle an unknown disturbance, wherein the adaptive law is designed to estimate the unknown bound d_{b}. The design procedure for the control law is illustrated in Fig. 3. One can see that the Lyapunov function is the summation of the system's mechanical energy and auxiliary functions.

$$V=V_1+V_2+V_3+V_4+V_5, \tag{16}$$

where

$$V_1=\rho A\Big[\int_0^l(\dot{y}+Dw/Dt)^2dx$$

$$
\begin{aligned}
&+\int_0^l (\dot{z}+Dv/Dt)^2\,dx\Big]/2\\
&+(k_1+1)\int_0^l \Big[P\left(w_x^2+v_x^2\right)/2\\
&+EA\left(w_x^2+v_x^2\right)^2/4\Big]dx\\
&+(k_1+1)EI_y\int_0^l w_{xx}^2dx/2\\
&+(k_1+1)EI_z\int_0^l v_{xx}^2dx/2,
\end{aligned}
\tag{17}
$$

$$
V_2 = m\dot{y}^2/2+k_2e_y^2/2+M\dot{z}^2/2+k_4e_z^2/2, \tag{18}
$$

$$
\begin{aligned}
V_3 = &\frac{1}{2}k_1\rho A\int_0^l\left(\frac{Dw}{Dt}\right)^2\mathrm{d}x\\
&+k_1\rho A\int_0^l \dot{y}\left(\dot{y}+\frac{Dw}{Dt}\right)dx+k_3\dot{y}e_y\\
&+\rho A a_1\int_0^l w\left(\dot{y}+\frac{Dw}{Dt}\right)dx\\
&+a_2\int_0^l e_y\left(\dot{y}+\frac{Dw}{Dt}\right)dx,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
V_4 = &\frac{1}{2}k_1\rho A\int_0^l\left(\frac{Dv}{Dt}\right)^2\mathrm{d}x\\
&+k_1\rho A\int_0^l \dot{z}\left(\dot{z}+\frac{Dv}{Dt}\right)dx+k_5\dot{z}e_z\\
&+\rho A b_1\int_0^l v\left(\dot{z}+\frac{Dv}{Dt}\right)dx\\
&+b_2\int_0^l e_z\left(\dot{z}+\frac{Dv}{Dt}\right)dx,
\end{aligned}
\tag{20}
$$

$$
V_5 = \xi_2\tilde{d}_{\mathrm{b}}^2/2. \tag{21}
$$

In (18)-(21), $e_y = y-y_{\mathrm{d}}$ and $e_z = z-z_{\mathrm{d}}$ denote the position errors of the trolley and gantry, respectively. $\tilde{d}_{\mathrm{b}} = d_{\mathrm{b}}-\hat{d}_{\mathrm{b}}$ is the estimation error of the disturbance. k_1, k_2, k_3, k_4, k_5, a_1, a_2, b_1, b_2, and ξ_2 are positive coefficients. Note that V_1 is the mechanical energy of the vibrations of the axially moving beam, V_2 represents the mechanical energy of the trolley and gantry, V_3 and V_4 are the auxiliary functions consisting of the motions in the j-axis and the k-axis, respectively, and V_5 corresponds to the disturbance. As indicated above, the longitudinal energy of the beam was omitted in (17) for simplicity. Otherwise, the total energy after suppression of the transverse and lateral vibrations will converge to this energy.

A robust adaptive boundary control law is proposed as follows:

$$
\begin{aligned}
f_y = &\left(m/(m+k_1\rho Al)\right)\left[-k_1\left(1-\rho Al/m\right)EI_yw_{xxx}(0,t)\right.\\
&-\left(k_2+k_3k_6/(m+k_1\rho Al)-\rho A\dot{l}k_3/m\right)e_y\\
&\left.-k_6\dot{y}+(k_1+1)\dot{l}w_{xx}^2(0,t)/(g(t)+\vartheta_y)\right]\\
&-\operatorname{sgn}\left(a_2e_y/(m+k_1\rho Al)+\dot{y}\right)\hat{d}_{\mathrm{b}},
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
f_z = &\left(M/(M+k_1\rho Al)\right)\left[-k_1\left(1-\rho Al/M\right)EI_zv_{xxx}(0,t)\right.\\
&-\left(k_4+k_5k_7/(M+k_1\rho Al)-\rho A\dot{l}k_5/M\right)e_y\\
&\left.-k_7\dot{y}+(k_1+1)\dot{l}v_{xx}^2(0,t)/\left(h(t)+\vartheta_z\right)\right],
\end{aligned}
\tag{23}
$$

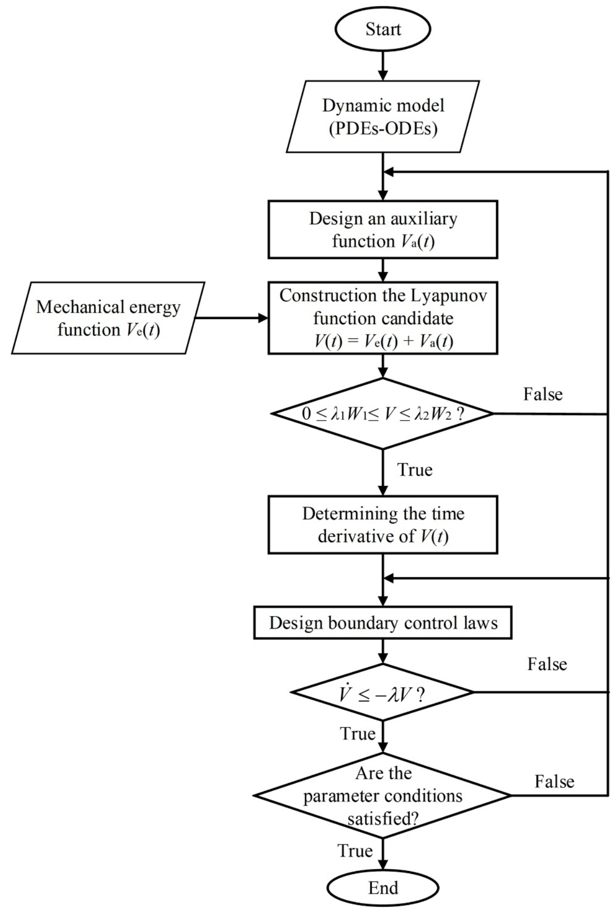


Fig. 3. Design procedure for the boundary control law.

where k_i, i = 1, 2, ..., 7 are control parameters, $\hat{d}_{\mathrm{b}}$ is the estimated value of d_b, and

$$
g(t) = \dot{y}+\frac{k_3e_y}{m+k_1\rho Al},\; h(t) = \dot{z}+\frac{k_5e_z}{M+k_1\rho Al}, \tag{24}
$$

$$
\begin{aligned}
\vartheta_y &= \begin{cases}\operatorname{sgn}(\dot{l}g)\,|g|/2, & \text{if } g\neq 0,\\ \text{constant}, & \text{if } g=0,\end{cases}\\
\vartheta_z &= \begin{cases}\operatorname{sgn}(\dot{l}h)\,|h|/2, & \text{if } h\neq 0,\\ \text{constant}, & \text{if } h=0,\end{cases}
\end{aligned}
\tag{25}
$$

The adaptive law is designed as follows:

$$
\dot{\hat{d}}_{\mathrm{b}} = -\xi_1\hat{d}_{\mathrm{b}}+\frac{1}{\xi_2}\left(1+\frac{k_1\rho Al}{m}\right)\left|\frac{a_2}{m+k_1\rho Al}e_y+\dot{y}\right|, \tag{26}
$$

where ξ_1 and ξ_2 define the adaptive gains.

Under the proposed control law, the following lemmas and theorem are made.

Lemma 4: The Lyapunov function candidate in (16) is upper and lower bounded as follows:

$$
0 \le \lambda_1W_1 \le V \le \lambda_2W_2, \tag{27}
$$

where λ_1 and λ_2 are positive parameters and

$$W_1 = \int_0^l (Dw/Dt)^2 dx + \int_0^l (Dv/Dt)^2 dx + \int_0^l w_{xx}^2 dx + \int_0^l v_{xx}^2 dx + e_y^2 + e_z^2 + \dot{y}^2 + \dot{z}^2, \tag{28}$$

$$W_2 = \tilde{d}_b^2 + \int_0^l (Dw/Dt)^2 dx + \int_0^l (Dv/Dt)^2 dx + \int_0^l P\left(w_x^2 + v_x^2\right) dx/2 + \int_0^l \left(w_x^2 + v_x^2\right)^2 dx/4 + \int_0^l w_{xx}^2 dx + \int_0^l v_{xx}^2 dx + e_y^2 + e_z^2 + \dot{y}^2 + \dot{z}^2. \tag{29}$$

Proof: See Appendix A. □

Lemma 5: Under the boundary control laws in (22)-(23) and adaptive law in (26), the time derivative of the Lyapunov function candidate in (16) is upper bounded as follows:

$$\dot{V} \leq -\lambda V + \varepsilon, \tag{30}$$

where $\varepsilon = \xi_1 \xi_2 d_b^2/2$ and λ is a positive parameter.

Proof: See Appendix B. □

According to Lemma 5, the time derivative of V_d can be evaluated as

$$\dot{V} \leq e^{-\lambda t} V(0) + \varepsilon(1 - e^{-\lambda t})/\lambda_3. \tag{31}$$

The following inequalities are obtained based on Lemma 2 and the inequality in (31):

$$EI_y(k_1+1)w(x,t)^2/2 \leq l^3 EI_y(k_1+1)\int_0^l w_{xx}^2 dx/2 \leq l^3 V(t) \leq l^3 V(0)e^{-t\lambda} + l^3\varepsilon(1-e^{-t\lambda})/\lambda, \tag{32}$$

$$EI_z(k_1+1)v(x,t)^2/2 \leq l^3 EI_z(k_1+1)\int_0^l v_{xx}^2 dx/2 \leq l^3 V(t) \leq l^3 V(0)e^{-t\lambda} + l^3\varepsilon(1-e^{-t\lambda})/\lambda, \tag{33}$$

$$k_2 e_y^2(t)/2 \leq V(t) \leq V(0)e^{-t\lambda} + \varepsilon(1-e^{-t\lambda})/\lambda_3, \tag{34}$$

$$k_4 e_z^2(t)/2 \leq V(t) \leq V(0)e^{-t\lambda} + \varepsilon(1-e^{-t\lambda})/\lambda_3, \tag{35}$$

$$\xi_2 \tilde{d}_b^2(t)/2 \leq V(t) \leq V(0)e^{-t\lambda} + \varepsilon(1-e^{-t\lambda})/\lambda. \tag{36}$$

The inequalities in (32)-(36) indicate that the transverse and lateral vibrations of the beam, the position errors of the trolley and gantry, and estimated error are uniformly bounded. If the initial conditions are bounded, by using (32)-(36), we can obtain

$$\lim_{t\to\infty}|w(x,t)| \leq \sqrt{2l^3\varepsilon/(\lambda EI_y(k_1+1))}, \tag{37}$$

$$\lim_{t\to\infty}|v(x,t)| \leq \sqrt{2l^3\varepsilon/(\lambda EI_z(k_1+1))}, \tag{38}$$

$$\lim_{t\to\infty}|e_y(t)| \leq \sqrt{2\varepsilon/\lambda k_2}, \tag{39}$$

$$\lim_{t\to\infty}|e_z(t)| \leq \sqrt{2\varepsilon/\lambda k_4}. \tag{40}$$

As shown in (37)-(40), due to the presence of the disturbance, the vibrations of the beam and position errors cannot converge to zero. However, the solutions of the closed-loop system, namely $w(x,t)$, $v(x,t)$, $e_y(t)$, and $e_z(t)$, are uniformly ultimately bounded. If the design parameters are selected such that λ, k_1, k_2, and k_4 are large, and ε is small, then the uniform ultimate boundedness region can be arbitrarily made small near zero. Additionally, the control law is bounded. Furthermore, because the estimation error is bounded, the boundedness of the adaptive law is ensured. All the above results are summarized in the following theorem.

Theorem 1: Consider a hybrid system described by (4)-(11) under the boundary control laws in (22) and (23) with the adaptive law in (26). The solutions of the closed-loop system, namely $w(x,t)$, $v(x,t)$, $e_y(t)$, and $e_z(t)$, are uniformly ultimately bounded.

Remark 1: If the disturbance is ignored, we further conclude that the closed-loop system is exponentially stable in the sense that the transverse vibration $w(x,t)$, lateral vibration $v(x,t)$, and position errors of the trolley and gantry exponentially converge to zero.

The convergence of the vibration and position errors can be proven by using Lemma 3. If the disturbance is zero ($d = 0$ and $d_b = 0$) and the initial value of the Lyapunov function candidate, $V(0)$, is bounded, the Lyapunov function can be evaluated as follows:

$$V(t) \leq e^{-\lambda t} V(0) < \infty. \tag{41}$$

Using Lemmas 1 and 4, the following results are obtained.

$$w^2(x,t) \leq l^3 \int_0^l w_{xx}^2(x,t)dx \leq l^3 W_1(t) \leq l^3 V(t)/\lambda_1 < \infty, \tag{42}$$

$$\|w(x,t)\|^2 \leq l^4 W_1(t) \leq l^4 V(t)/\lambda_1 \leq -l^4 \dot{V}/\lambda\lambda_1. \tag{43}$$

Inequality (42) implies that the $w(x,t)$ is uniformly bounded, whereas inequality (43) leads to the following result.

$$\lim_{t\to\infty}\int_0^t \|w(x,t)\|^2 d\tau \leq -l^4 \lim_{t\to\infty}(V(t) - V(0))/\lambda\lambda_1 < \infty. \tag{44}$$

Furthermore, we also have

$$d\|w(x,t)\|^2/dt = 2\int_0^l (w(x,t)Dw(x,t)/Dt)\,dx \leq \|w(x,t)\|^2 + \|Dw(x,t)/Dt\|^2 < \infty. \tag{45}$$

This inequality implies that $w(x,t)$ is equicontinuous in t. According to Lemma 3, we can conclude that $\|w(x,t)\| \to$

0 as $t \to \infty$. Using the same approach, we can prove that $\|v(x,t)\| \to 0$ as $t \to \infty$. Additionally, the convergence of the position errors is also proven based on Barbalat's lemma: $\|e_y(t)\|$, $\|e_z(t)\| \to 0$ as $t \to \infty$.

4. SIMULATION RESULTS

This section verifies the effectiveness of the proposed control law through numerical simulations. Consider the system described by (4)-(11), where the system parameters are given as follows: $M = 8$ kg, $M_g = 15$ kg, $\rho = 2,700$ kg/m^3, $A = 3 \times 10^{-3}$ m^2, $E = 69$ GPa, $I_y = 10^{-8}$ m^2, $I_z = 5.63 \times 10^{-9}$ m^2, $c_w = c_v = 0.01$ N·s/m^2, and $g = 9.8$ m/s^2. The trolley and gantry move from the initial position $y(0) = z(0) = 0$ to the desired positions at $y_d = 4$ m and $z_d = 3$ m, respectively. The system is influenced by a boundary disturbance defined by $d(t) = 50\sin(5\pi t)$. The system responses are simulated using MATLAB, wherein the finite difference method is adopted to handle the differential equations. The space and time steps are $\Delta\xi = 0.1$ and $\Delta t = 0.00001$, respectively (i.e., $\xi = x/l$). We examine two cases for the beam: Extension and retraction. Additionally, the control performance of the designed control law is compared to that of the following laws proposed by Shah and Hong [13]:

$$f_y(t) = -k_{y1}e_y(t) - k_{y2}y(t) - k_{y3}w_{xxx}(0,t),$$
$$f_z(t) = -k_{z1}e_z(t) - k_{z2}z(t) - k_{z3}v_{xxx}(0,t), \quad (46)$$

where k_{y1}, k_{y2}, k_{y3}, k_{z1}, k_{z2}, and k_{z3} are control gains. The control law in [13] was developed for a flexible beam attached to a translating base, where the beam length was constant, and the axial motion of the beam was ignored. This control law also uses the two control forces of the trolley and gantry to position the base and suppress the transverse and lateral vibrations of the beam.

For the extension case, the beam's length is extended from $l_{\min} = 1.5$ m to $l_{\max} = 3$ m over 2 sec. Figs. 4 and 5 present the responses of the closed-loop system, where the disturbance is ignored. From these simulation results, we can conclude that the proposed control law in (22) and (23) can position the trolley and gantry and significantly suppress the vibration of the beam without disturbances. During the extension process, the proposed control law minimizes vibrations more effectively than the control law proposed in [13]. However, the control law in [13] also exhibits good control performance in this case. The vibration energy of the beam decreases during extension [28]. In other words, the axial motion of the beam in this case is a factor suppressing the vibration energy. Therefore, even though the control law in [13] was designed without considering the axial motion of the beam, it still effectively suppresses the vibration. The outstanding advantages of the proposed control law are highlighted in the retraction case, where the beam's length is reduced from $l_{\max} = 3$ m

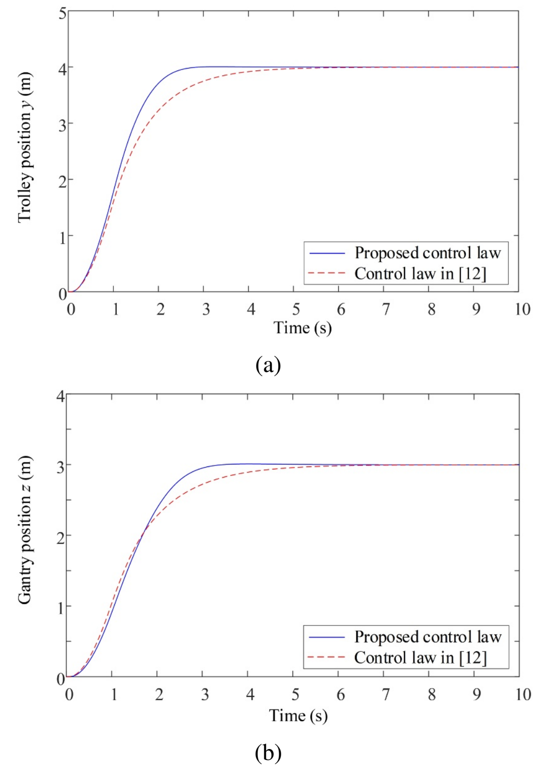


Fig. 4. Extension case: (a) Trolley position and (b) gantry position of the system, where the boundary disturbance is not considered.

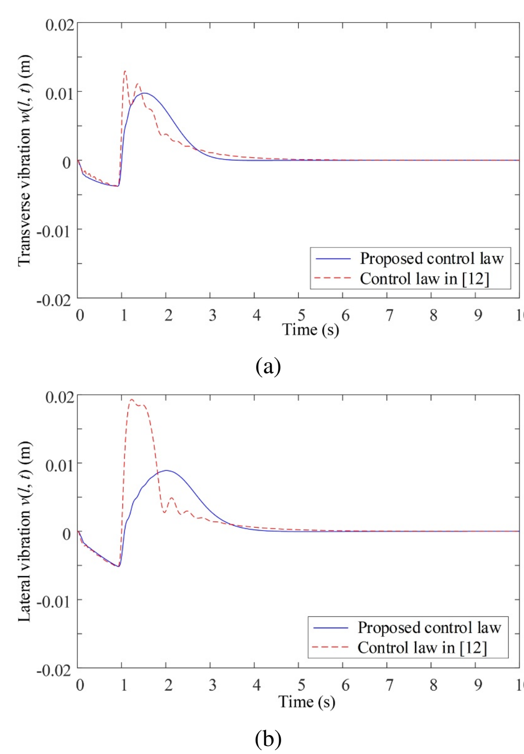


Fig. 5. Extension case: (a) Transverse vibration and (b) lateral vibration, where the boundary disturbance is not considered.

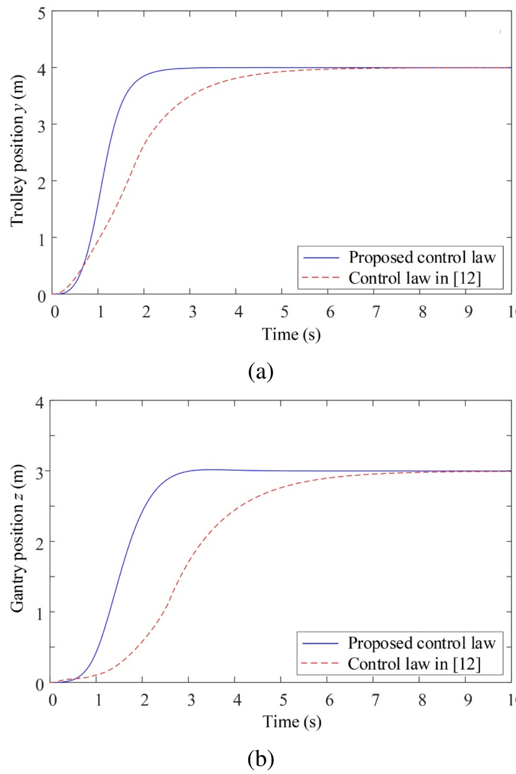


Fig. 6. Retraction case: (a) Trolley position and (b) gantry position of the system, where the boundary disturbance is not considered.

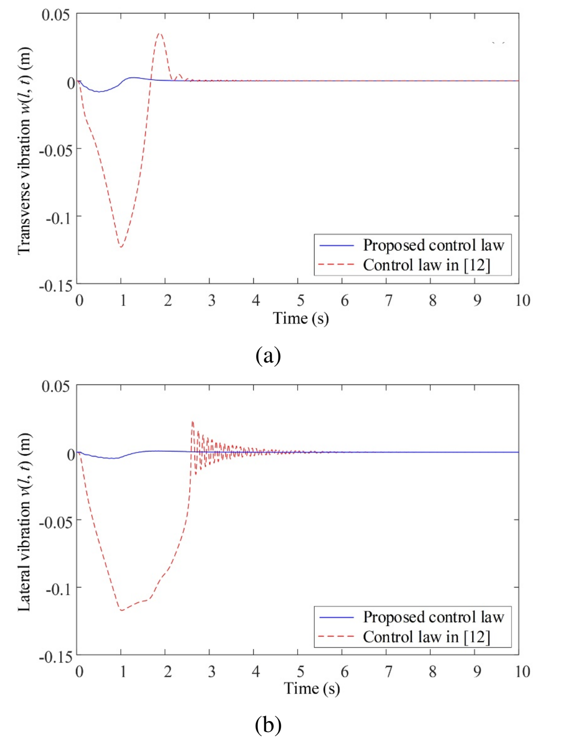


Fig. 7. Retraction case: (a) Transverse vibration and (b) lateral vibration, where the boundary disturbance is not considered.

to $l_{\min} = 1.5$ m over 2 sec. Because the vibration energy of the beam increases during retraction [31], the control law in [13] cannot handle vibrations during retraction (i.e., the red dashed lines in Figs. 6 and 7). The amplitude of the oscillations significantly increases during the first 2 sec (Fig. 7). In contrast, the proposed control law, which is designed to account for both the base motion and beam axial motion, can significantly attenuate the vibration of the beam (i.e., the solid blue line in Figs. 6 and 7). This result demonstrates that the designed vibration control law is necessary to guarantee the stability of the system, particularly in the retraction case.

The robustness of the proposed control law is illustrated in Figs. 8-13. Figs. 8 and 9 present the base positions and beam vibrations of the system under a boundary disturbance $d(t)$ during extension. The boundary disturbance significantly affects the transverse vibration (see the system's responses under the control law [12]), whereas its influence on lateral vibration is insignificant. Figs. 8(a) and 8(b) reveal that the proposed control law addresses the disturbance and guarantees the minimization of the transverse and lateral vibrations of the beam, respectively. The estimated bound of the disturbance is revealed in Fig. 10. These simulation results demonstrate the effectiveness of the proposed robust adaptive control method in the ex-

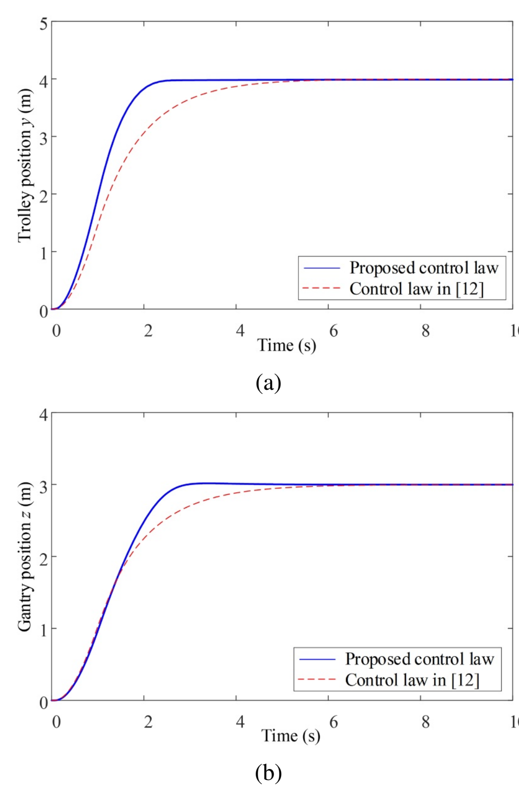
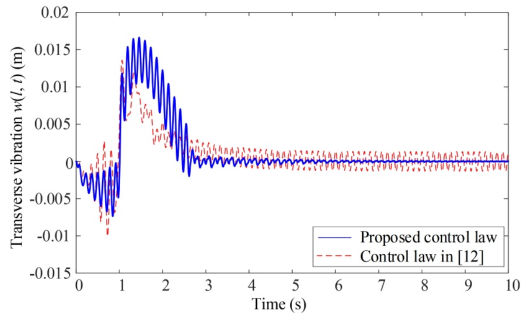
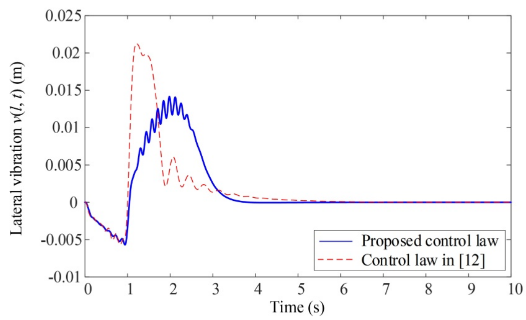


Fig. 8. Extension case: (a) Trolley position and (b) gantry position of the system, where the boundary disturbance is considered.



(a)



(b)

Fig. 9. Extension case: (a) Transverse vibration and (b) lateral vibration, where the boundary disturbance is considered.

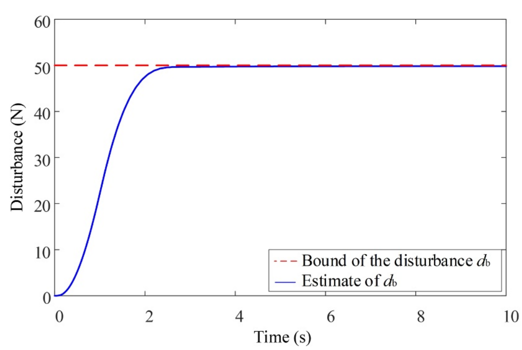


Fig. 10. Extension case: Bound of the disturbance and its estimate.

tension case. For the retraction case, Figs. 11 and 12 reveal that the trolley and gantry can track the desired positions and that the transverse and lateral vibrations converge to a small neighborhood around zero. In these figures, the dashed lines illustrate the bound of the signals calculated by (37)-(40). The convergence of the bound of disturbances is also shown in Fig. 13. Accordingly, the effectiveness of the designed boundary control method in the retraction case is proven.

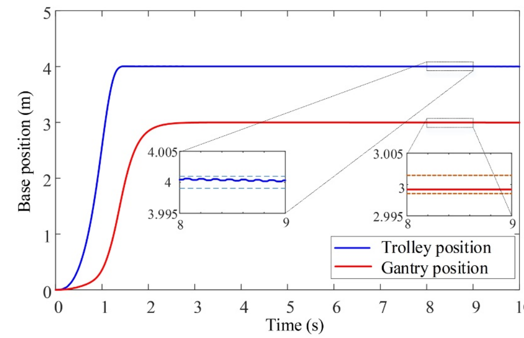
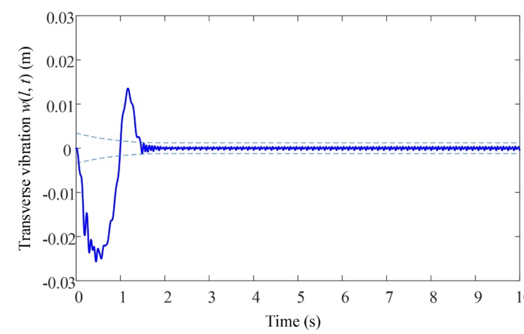
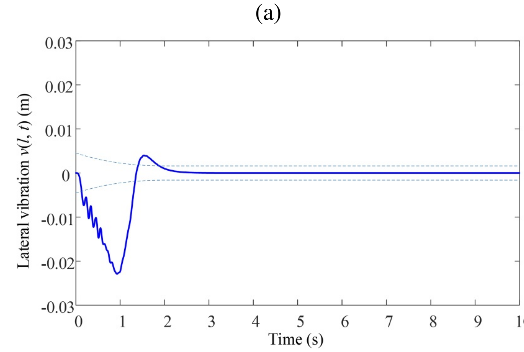


Fig. 11. Retraction case: Trolley position and gantry position of the system, where the boundary disturbance is considered.



(a)



(b)

Fig. 12. Retraction case: (a) Transverse vibration and (b) lateral vibration, where the boundary disturbance is considered.

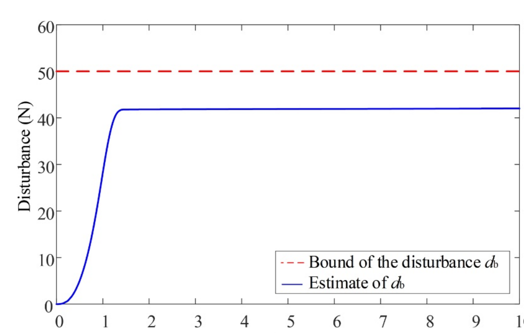


Fig. 13. Retraction case: Bound of the disturbance and its estimate.

5. CONCLUSION

This study addressed the position and vibration control problem of a variable-length beam attached to a translating base in the presence of an unknown disturbance. A dynamic model of a hybrid system consisting of a trolley, gantry, and flexible beam of time-varying length was developed using the Hamilton principle. To handle the unknown disturbance, the boundary control law in (22)-(23) and the adaptation law in (26) were designed based on the Lyapunov design method. Under this robust adaptive control law, the closed-loop system is uniformly ultimately bounded in the existence of the unknown disturbance. Numerical simulations were performed to verify the designed control law. The superior capabilities of the proposed robust adaptive control law for positioning control and vibration suppression were apparent in the simulation results.

The proposed method was developed for a cantilever beam with no additional mass at the tip; therefore, its application is limited in handling the load at the tip. When a payload at the tip is considered, the dynamics will be more involved. The proposed method will be extended to the varying length beam with unknown tip mass. Last but not least, the experiment will be conducted to verify the performance of the proposed control law.

APPENDIX A: PROOF OF LEMMA 4

Let γ_i, $i = 1, 2, ..., 6$, be positive design parameters. According to Lemmas 1 and 2, terms V_3 and V_4 can be evaluated as follows:

$$\left|\int_0^l \dot{y}\frac{Dw}{Dt}dx\right| \le \frac{1}{\gamma_1}l\dot{y}^2 + \gamma_1\int_0^l\left(\frac{Dw}{Dt}\right)^2 dx, \tag{A.1}$$

$$\left|\int_0^l w\dot{y}dx\right| \le \frac{l^4}{\gamma_2}\int_0^l w_{xx}^2 dx + \gamma_2 l\dot{y}^2, \tag{A.2}$$

$$\left|\int_0^l w\frac{Dw}{Dt}dx\right| \le \frac{l^4}{\gamma_2}\int_0^l w_{xx}^2 dx + \gamma_2\int_0^l\left(\frac{Dw}{Dt}\right)^2 dx, \tag{A.3}$$

$$(k_3 + a_2 l)\dot{y}e_y \le (k_3 + a_2 l)\dot{y}^2 + (k_3 + a_2 l)e_y{}^2, \tag{A.4}$$

$$\left|\int_0^l e_y\frac{Dw}{Dt}dx\right| \le \frac{1}{\gamma_3}le_y{}^2 + \gamma_3\int_0^l\left(\frac{Dw}{Dt}\right)^2 dx, \tag{A.5}$$

$$\left|\int_0^l \dot{z}\frac{Dv}{Dt}dx\right| \le \frac{1}{\gamma_4}l\dot{z}^2 + \gamma_4\int_0^l\left(\frac{Dv}{Dt}\right)^2 dx, \tag{A.6}$$

$$\left|\int_0^l v\dot{z}dx\right| \le \frac{l^4}{\gamma_5}\int_0^l v_{xx}^2 dx + \gamma_5 l\dot{z}^2, \tag{A.7}$$

$$\left|\int_0^l v\frac{Dv}{Dt}dx\right| \le \frac{l^4}{\gamma_5}\int_0^l v_{xx}^2 dx + \gamma_5\int_0^l\left(\frac{Dv}{Dt}\right)^2 dx, \tag{A.8}$$

$$(k_3 + b_2 l)\dot{z}e_z \le (k_3 + b_2 l)\dot{z}^2 + (k_3 + b_2 l)e_z^2, \tag{A.9}$$

$$\left|\int_0^l e_z\frac{Dv}{Dt}dx\right| \le \frac{1}{\gamma_6}le_z^2 + \gamma_6\int_0^l\left(\frac{Dv}{Dt}\right)^2 dx. \tag{A.10}$$

By using (42) to (A.5), the bounds for V are obtained as follows:

$$\begin{aligned}
V \ge{}& [k_1\rho A/2 - k_1\rho A\gamma_1 - a_1\rho A\gamma_2 - a_2\gamma_3] \\
&\times \int_0^l (Dw/Dt)^2 dx \\
&+ [k_1\rho A/2 - k_1\rho A\gamma_4 - b_1\rho A\gamma_5 - b_2\gamma_6] \\
&\times \int_0^l (Dv/Dt)^2 dx \\
&+ \left[(k_1+1)EI_y/2 - 2a_1\rho Al^4/\gamma_2\right]\int_0^l w_{xx}^2 dx \\
&+ \left[(k_1+1)EI_z/2 - 2b_1\rho Al^4/\gamma_4\right]\int_0^l v_{xx}^2 dx \\
&+ [k_2/2 - (k_3 + a_2 l + a_2 l/\gamma_3)]\,e_y^2 \\
&+ [m/2 + k_1\rho Al - k_1\rho Al/\gamma_1 - a_1\rho A\gamma_2 l - k_3 - a_2 l]\dot{y}^2 \\
&+ [k_4/2 - (k_5 + b_2 l + b_2 l/\gamma_6)]\,e_z^2 \\
&+ (M/2 + k_1\rho Al - k_1\rho Al/\gamma_4 - b_1\rho A\gamma_5 l - k_5 - b_2 l)\,\dot{z}^2 \\
\ge{}& \lambda_1\left[\int_0^l (Dw/Dt)^2 dx + \int_0^l (Dv/Dt)^2 dx + \int_0^l w_{xx}^2 dx\right. \\
&\left. + \int_0^l v_{xx}^2 dx + e_y^2 + \dot{y}^2 + e_z^2 + \dot{z}^2\right] \\
={}& \lambda_1 W_1,
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
V \le{}& (\rho A(1 + k_1/2 + k_1\gamma_1 + a_1\gamma_2) + a_2\gamma_3) \\
&\times \int_0^l (Dw/Dt)^2 dx \\
&+ (\rho A(1 + k_1/2 + k_1\gamma_4 + b_1\gamma_5) + b_2\gamma_6) \\
&\times \int_0^l (Dv/Dt)^2 dx \\
&+ (k_1+1)\int_0^l \left[P\left(w_x{}^2 + v_x{}^2\right)dx/2\right. \\
&\left. + \left(w_x{}^2 + v_x{}^2\right)^2 dx/4\right]dx \\
&+ \left[(k_1+1)EI_y/2 + 2a_1\rho Al^4/\gamma_2\right]\int_0^l w_{xx}^2 dx \\
&+ \left[(k_1+1)EI_z/2 + 2b_1\rho Al^4/\gamma_5\right]\int_0^l v_{xx}^2 dx \\
&+ (k_3 + a_2 l + a_2 l/\gamma_3 + k_2/2)\,e_y^2 \\
&+ (k_1\rho Al(1 + 1/\gamma_1) + a_1\rho A\gamma_2 l + k_3 + a_2 l \\
&+ m/2 + \rho Al)\dot{y}^2 \\
&+ (k_5 + b_2 l + b_2 l/\gamma_6 + k_4/2)\,e_z^2 \\
&+ (k_1\rho Al(1 + 1/\gamma_4) + b_1\rho A\gamma_5 l + k_5 + b_2 l \\
&+ M/2 + \rho Al)\dot{z}^2 \\
\le{}& \lambda_2\left[\tilde{d}_b{}^2 + \int_0^l (Dw/Dt)^2 dx + \int_0^l (Dv/Dt)^2 dx\right. \\
&+ \int_0^l P\left(w_x^2 + v_x^2\right)dx/2 + \int_0^l \left(w_x^2 + v_x^2\right)^2 dx/4 \\
&\left. + \int_0^l w_{xx}^2 dx + \int_0^l v_{xx}^2 dx + e_y^2 + e_z^2 + \dot{y}^2 + \dot{z}^2\right]
\end{aligned}$$

$$=\lambda_2 W_2, \tag{A.12}$$

where

$$\begin{aligned}\lambda_1 = \min\{&k_1\rho A/2 - k_1\rho A\gamma_1 - a_1\rho A\gamma_2 - a_2\gamma_3,\\ &k_1\rho A/2 - k_1\rho A\gamma_4 - b_1\rho A\gamma_5 - b_2\gamma_6,\\ &(k_1+1)EI_y/2 - 2a_1\rho Al^4/\gamma_2,\\ &(k_1+1)EI_z/2 - 2b_1\rho Al^4/\gamma_4,\\ &k_2/2 - (k_3 + a_2 l + a_2 l/\gamma_3),\\ &m/2 + \rho Alk_1(1-1/\gamma_1) - a_1\rho A\gamma_2 l - k_3 - a_2 l,\\ &k_4/2 - (k_5 + b_2 l + b_2 l/\gamma_6),\\ &M/2 + k_1\rho Al - k_1\rho Al/\gamma_4 - b_1\rho A\gamma_5 l - k_5 - b_2 l\},\end{aligned} \tag{A.13}$$

$$\begin{aligned}\lambda_2 = \max\{&\rho A(1 + k_1/2 + k_1\gamma_1 + a_1\gamma_2) + a_2\gamma_3,\\ &\rho A(1 + k_1/2 + k_1\gamma_4 + b_1\gamma_5) + b_2\gamma_6,\\ &EA(k_1+1), (k_1+1),\\ &(k_1+1)EI_y/2 + 2a_1\rho Al^4/\gamma_2,\\ &(k_1+1)EI_z/2 + 2b_1\rho Al^4/\gamma_5,\\ &k_3 + a_2 l + a_2 l/\gamma_3 + k_2/2,\\ &\rho Al(k_1 + k_1/\gamma_1 + a_1\gamma_2 + 1) + k_3 + a_2 l + m/2,\\ &k_5 + b_2 l + b_2 l/\gamma_6 + k_4/2,\\ &\rho Al(k_1 + k_1/\gamma_4 + b_1\gamma_5 + 1) + k_5 + b_2 l + M/2\}.\end{aligned} \tag{A.14}$$

If the control parameters k_i ($i = 1, 2, ..., 5$) and design parameters a_1, a_2, b_1, b_2, and γ_i ($i = 1, 2, ..., 6$) are selected such that

$$\lambda_1 > 0, \tag{A.15}$$

then $0 \le \lambda_1 W_1 \le V \le \lambda_2 W_2$. Therefore, Lemma 4 is proven. □

APPENDIX B: PROOF OF LEMMA 5

Because the flexible beam is an axially moving beam of time-varying length, the time rates of its vibration energy V_1, and the two auxiliary functions V_3 and V_4 should be determined based on the Reynolds transport theorem for a translating medium with variable length [17]. The time rate of V_1 is derived as follows:

$$\begin{aligned}\dot V_1 =& \rho A\int_0^l (\dot y + Dw/Dt)\left(\ddot y + D^2w/Dt^2\right)dx\\ &+\rho A\int_0^l (\dot z + Dv/Dt)\left(\ddot z + D^2v/Dt^2\right)dx\\ &+(k_1+1)\left[\int_0^l (DP/Dt)\left(w_x^2+v_x^2\right)dx/2\right.\\ &+\int_0^l Pw_x(Dw_x/Dt)\,dx + \int_0^l Pv_x(Dv_x/Dt)\,dx\\ &+\int_0^l EA\left(w_x^2/2+v_x^2/2\right)w_x(Dw_x/Dt)\,dx\\ &+\int_0^l EA\left(w_x^2/2+v_x^2/2\right)v_x(Dv_x/Dt)\,dx\\ &+EI_y\int_0^l w_{xx}(Dw_{xx}/Dt)\,dx\\ &\left.+EI_z\int_0^l v_{xx}(Dv_{xx}/Dt)\,dx\right].\end{aligned} \tag{B.1}$$

By integrating by parts for the last two terms and considering the fact that $(Dw/Dt)_x = (\dot y + Dw/Dt)_x$, (A.11) can be rearranged as follows:

$$\begin{aligned}\dot V_1 =& \rho A\int_0^l (\dot y + Dw/Dt)(\ddot y + D^2w/Dt^2)dx\\ &+\rho A\int_0^l (\dot z + Dv/Dt)(\ddot z + D^2v/Dt^2)dx\\ &+(k_1+1)\int_0^l [Pw_x + EAw_x(w_x^2+v_x^2)/2\\ &-EI_y w_{xxx}](\dot y + Dw/Dt)_x dx + (k_1+1)\int_0^l [Pv_x\\ &+EAv_x(w_x^2+v_x^2)/2 - EI_z v_{xxx}](\dot z + Dv/Dt)_x dx\\ &+(k_1+1)\int_0^l (DP/Dt)(w_x^2+v_x^2)dx/2\\ &-(k_1+1)EI_y\dot l w_{xx}(0,t)^2 - (k_1+1)EI_z\dot l v_{xx}(0,t)^2.\end{aligned} \tag{B.2}$$

Substituting the dynamic model of the beam in (6) and (9) into (A.12) and using the boundary conditions in (8) and (11) yields

$$\begin{aligned}\dot V_1 =& -c_w\int_0^l w_t^2 dx - c_w\dot l\int_0^l w_t w_x dx - \dot y c_w\int_0^l w_t dx\\ &+\dot y EI_y w_{xxx}(0,t) - c_v\int_0^l v_t^2 dx - c_v\dot l\int_0^l v_t v_x dx\\ &-\dot z c_v\int_0^l v_t dx + \dot z EI_z v_{xxx}(0,t)\\ &+k_1\int_0^l [Pw_x + EAw_x(w_x^2+v_x^2)/2 - EI_y w_{xxx}]\\ &\times(\dot y + Dw/Dt)_x dx\\ &+k_1\int_0^l [Pv_x + EAv_x(w_x^2+v_x^2)/2 - EI_z v_{xxx}]\\ &\times(\dot z + Dv/Dt)_x dx\\ &+(k_1+1)\int_0^l (DP/Dt)(w_x^2+v_x^2)dx/2\\ &-(k_1+1)EI_y\dot l w_{xx}^2(0,t) - (k_1+1)EI_z\dot l v_{xx}^2(0,t).\end{aligned} \tag{B.3}$$

By using the dynamic model of the trolley and gantry, (A.13) can be rewritten as

$$\begin{aligned}\dot V_1 =& -c_w\int_0^l w_t^2 dx - c_w\dot l\int_0^l w_t w_x dx + \dot y f_y - m\dot y\ddot y - \rho A\dot l\dot y^2\\ &-c_v\int_0^l v_t^2 dx - c_v\dot l\int_0^l v_t v_x dx + \dot z f_z - M\dot z\ddot z - \rho A\dot l\dot z^2\end{aligned}$$

$$
\begin{aligned}
&+k_1\int_0^l [Pw_x+EAw_x(w_x^2+v_x^2)/2\\
&-EI_y w_{xxx}](\dot{y}+Dw/Dt)_x dx+k_1\int_0^l [Pv_x\\
&+EAv_x(w_x^2+v_x^2)/2-EI_z v_{xxx}](\dot{z}+Dv/Dt)_x dx\\
&+(k_1+1)\int_0^l (DP/Dt)(w_x^2+v_x^2)dx/2\\
&-(k_1+1)EI_y\dot{l}w_{xx}^2(0,t)-(k_1+1)EI_z\dot{l}v_{xx}^2(0,t)\\
&+\dot{y}d. \qquad \text{(B.4)}
\end{aligned}
$$

The time rate of V_2 can be calculated using the time derivative as follows:

$$\dot{V}_2=m\dot{y}\ddot{y}+k_2e_y\dot{y}+M\dot{z}\ddot{z}+k_4e_z\dot{z}. \qquad \text{(B.5)}$$

The time rate of V_3 is derived as follows:

$$
\begin{aligned}
\dot{V}_3=&\,k_1\rho A\int_0^l(\dot{y}+Dw/Dt)(\ddot{y}+D^2w/Dt^2)dx+k_1\rho Al\dot{y}\ddot{y}\\
&+(k_1\rho A\dot{l}+k_3+a_2l)\dot{y}^2+k_3\ddot{y}e_y\\
&+\rho Aa_1\int_0^l (Dw/Dt)^2dx\\
&+\rho Aa_1\int_0^l w(\ddot{y}+D^2w/Dt^2)dx\\
&+a_2\dot{l}e_y\dot{y}+(\rho Aa_1+a_2)\int_0^l \dot{y}(Dw/Dt)dx\\
&+a_2\int_0^l e_y(\ddot{y}+D^2w/Dt^2)dx. \qquad \text{(B.6)}
\end{aligned}
$$

By substituting the equation of motion corresponding to the transverse vibration in (6) into (B.1) and integrating by parts, (B.1) can be rewritten as follows:

$$
\begin{aligned}
\dot{V}_3=&-k_1\dot{y}c_w\int_0^l w_t dx-k_1c_w\int_0^l w_t^2dx\\
&-k_1c_w\dot{l}\int_0^l w_tw_xdx+k_1\int_0^l[(\dot{y}+Dw/Dt)(Pw_x\\
&+EAw_x(w_x^2+v_x^2)/2-EI_yw_{xxx})_x]dx+k_3\ddot{y}e_y\\
&+k_1\rho Al\dot{y}\ddot{y}+(k_1\rho A\dot{l}+k_3+a_2l)\dot{y}^2\\
&+\rho Aa_1\int_0^l(Dw/Dt)^2dx-c_wa_1\int_0^l ww_tdx\\
&-a_1\int_0^l Pw_x^2dx-a_1EA\int_0^l w_x^2(w_x^2+v_x^2)dx/4\\
&+a_2\dot{l}e_y\dot{y}-a_1EI_y\int_0^l w_{xx}^2dx\\
&+(\rho Aa_1+a_2)\int_0^l\dot{y}(Dw/Dt)dx\\
&-c_wa_2\int_0^l e_yw_tdx/\rho A+a_2e_yEI_yw_{xxx}(0,t)/\rho A. \qquad \text{(B.7)}
\end{aligned}
$$

Similarly, the time rate of V_4 is obtained as follows:

$$
\begin{aligned}
\dot{V}_4=&-k_1\dot{z}c_v\int_0^l v_tdx-k_1c_v\int_0^l v_t^2dx\\
&-k_1c_v\dot{l}\int_0^l v_tv_xdx+k_1\int_0^l[(\dot{z}+Dv/Dt)(Pv_x\\
&+EAv_x(w_x^2+v_x^2)/2-EI_zv_{xxx})_x]dx\\
&+k_1\rho Al\dot{z}\ddot{z}+(k_1\rho A\dot{l}+k_5+b_2l)\dot{z}^2\\
&+k_5\ddot{z}e_z+\rho Ab_1\int_0^l(Dv/Dt)^2dx\\
&-c_vb_1\int_0^l vv_tdx-b_1\int_0^l Pv_x^2dx\\
&-b_1EA\int_0^l v_x^2(w_x^2+v_x^2)dx/4+b_2\dot{l}e_z\dot{z}\\
&-b_1EI_z\int_0^l v_{xx}^2dx\\
&+(\rho Ab_1+b_2)\int_0^l\dot{z}(Dv/Dt)dx\\
&-c_vb_2\int_0^l e_zv_tdx/\rho A+b_2e_zEI_zv_{xxx}(0,t)/\rho A. \qquad \text{(B.8)}
\end{aligned}
$$

From (A.14), (A.15), (B.2), and (B.3), and the dynamic model corresponding to the trolley and gantry, the time derivative of V can be derived as follows:

$$\dot{V}=\dot{V}_w+\dot{V}_v+\dot{V}_{wv}, \qquad \text{(B.9)}$$

where

$$
\begin{aligned}
\dot{V}_w=&\,k_3e_yd/m+(k_1\rho Al/m+1)\dot{y}d\\
&-(k_1+1)c_w\int_0^l w_t^2dx/2\\
&-[(k_1+1)c_w/2-\rho Aa_1]\int_0^l(Dw/Dt)^2dx\\
&-a_1EI_y\int_0^l w_{xx}^2dx\\
&+k_1EI_y(1-\rho Al/m)\dot{y}w_{xxx}(0,t)\\
&+(k_2-k_3\rho A\dot{l}/m)e_y\dot{y}+(1+k_1\rho Al/m)\dot{y}f_y\\
&+(k_1\rho A\dot{l}+k_3+a_2l-\rho A\dot{l}-k_1(\rho A)^2l\dot{l}/m)\dot{y}^2\\
&+EI_y(a_2e_y/\rho A-k_3/m)e_yw_{xxx}(0,t)\\
&+k_3e_yf_y/m-(k_1+1)EI_y\dot{l}w_{xx}^2(0,t)\\
&+k_1c_w(\rho Al/m-1)\int_0^l\dot{y}w_tdx\\
&+(\rho Aa_1+a_2)\int_0^l\dot{y}(Dw/Dt)dx\\
&-c_wa_1\int_0^l ww_tdx\\
&+c_w(k_3/m-a_2/\rho A)\int_0^l e_yw_tdx+a_2\dot{l}e_y\dot{y}\\
&+(k_1+1)\int_0^l(c_w\dot{l}^2+DP/Dt)w_x^2dx/2+\xi_2\tilde{d}_b\dot{\tilde{d}}_b, \qquad \text{(B.10)}
\end{aligned}
$$

$$\dot{V}_v=-(k_1+1)c_v\int_0^l v_t^2dx/2-b_1EI_z\int_0^l v_{xx}^2dx$$

$$
\begin{aligned}
&-[(k_1+1)c_v/2-\rho Ab_1]\int_0^l (Dv/Dt)^2dx \\
&+k_1EI_z(1-\rho Al/M)\dot{z}v_{xxx}(0,t) \\
&+(k_4-k_5\rho A\dot{l}/M)e_z\dot{z} \\
&+(k_1\rho A\dot{l}+k_5+b_2l-\rho A\dot{l}-k_1(\rho A)^2l\dot{l}/M)\dot{z}^2 \\
&+(1+k_1\rho Al/M)\dot{z}f_z+k_5e_zf_z/M \\
&+EI_z(b_2/\rho A-k_5/M)e_zv_{xxx}(0,t) \\
&-(k_1+1)EI_z\dot{l}v_{xx}^2(0,t) \\
&+k_1c_v(\rho Al/M-1)\int_0^l \dot{z}v_tdx \\
&+(\rho Ab_1+b_2)\int_0^l \dot{z}(Dv/Dt)dx-c_vb_1\int_0^l vv_tdx \\
&+c_v(k_5/M-b_2/\rho A)\int_0^l e_zv_tdx+b_2\dot{l}e_z\dot{z} \\
&+(k_1+1)\int_0^l (c_v\dot{l}^2+DP/Dt)v_x^2dx/2, \qquad \text{(B.11)}
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_{wv}=&-a_1\int_0^l Pw_x^2dx-a_1EA\int_0^l w_x^2(w_x^2+v_x^2)dx/4 \\
&-b_1\int_0^l Pv_x^2dx-b_1EA\int_0^l v_x^2({w_x}^2+v_x^2)dx/4. \qquad \text{(B.12)}
\end{aligned}
$$

We now evaluate $\dot{V}_w$, $\dot{V}_v$, and $\dot{V}_{wv}$. Let δ_i ($i=1, 2, ..., 10$) be the positive design parameters. By applying Lemmas 1 and 2 to the terms in (B.5), $\dot{V}_w$ can be evaluated as follows:

$$
\begin{aligned}
\dot{V}_w\le&-c_w[-\delta_1k_1(1-\rho Al/m)-\delta_3a_1 \\
&-\delta_4|k_3/m-a_2/\rho A|+(k_1+1)/2]\int_0^l w_t^2dx \\
&-[(k_1+1)c_w/2-\rho Aa_1 \\
&-\delta_2(\rho Aa_1+a_2)]\int_0^l (Dw/Dt)^2dx \\
&-[a_1EI_y-a_1c_wl^4/\delta_3 \\
&-(k_1+1)(c_w\dot{l}^2+P_{\mathrm{Dmax}})l^2/2]\int_0^l w_{xx}^2dx \\
&+(1+k_1\rho Al/m)\dot{y}f_y \\
&+k_1EI_y(1-\rho Al/m)\dot{y}w_{xxx}(0,t) \\
&+(k_2-k_3\rho A\dot{l}/m)e_y\dot{y} \\
&+[k_1\rho A\dot{l}+k_3+a_2l-\rho A\dot{l} \\
&+(\rho Aa_1+a_2)l/\delta_2+k_1c_w(1-\rho Al/m)l/\delta_1 \\
&+a_2|\dot{l}|\delta_5-k_1(\rho A)^2l\dot{l}/m]\dot{y}^2 \\
&+[c_w|k_3/m-a_2/\rho A|l/\delta_4 \\
&+a_2|\dot{l}|/\delta_5]e_y^2+k_3e_yf_y/m \\
&-(k_1+1)EI_y\dot{l}w_{xx}^2(0,t) \\
&+EI_y(a_2e_y/\rho A-k_3/m)e_yw_{xxx}(0,t) \\
&+k_3e_yd/m+(k_1\rho Al/m+1)\dot{y}d+\xi_2\tilde{d}_b\dot{\hat{d}}_b. \qquad \text{(B.13)}
\end{aligned}
$$

Under the boundary control law in (22) and adaptation law in (26), (B.8) can be rewritten as follows:

$$
\begin{aligned}
\dot{V}_w\le&-c_w[(k_1+1)/2-\delta_1k_1(1-\rho Al/m)-\delta_3a_1 \\
&-\delta_4|k_3/m-a_2/\rho A|]\int_0^l w_t^2dx-[(k_1+1)c_w/2 \\
&-\rho Aa_1-\delta_2(\rho Aa_1+a_2)]\int_0^l (Dw/Dt)^2dx \\
&-[a_1EI_y-a_1c_wl^4/\delta_3 \\
&-(k_1+1)(c_w\dot{l}^2+P_{\mathrm{Dmax}})l^2/2]\int_0^l w_{xx}^2dx-[k_6 \\
&-k_3-a_2l\ -k_1(\rho A\dot{l}+c_wl/\delta_1)(1-\rho Al/m) \\
&+\rho A\dot{l}-a_2|\dot{l}|\delta_5-(\rho Aa_1+a_2)l/\delta_2]\dot{y}^2 \\
&-[-c_w|k_3-ma_2|l/\delta_4m\rho A(k_3/(m+k_1\rho Al)) \\
&\times(k_2+k_3k_6/(m+k_1\rho Al)-\rho A\dot{l}k_3/m) \\
&-a_2|\dot{l}|/\delta_5]e_y^2+EI_y[(a_2/\rho A-k_3/m) \\
&-k_1k_3(1-\rho Al/m)/(m+k_1\rho Al)]e_yw_{xxx}(0,t) \\
&-(k_1+1)EI_y|g\dot{l}|\dot{l}w_{xx}^2(0,t)/(2|g|+\mathrm{sgn}(\dot{l}g)g) \\
&+\xi_1\xi_2d_b^2/2-\xi_1\xi_2\tilde{d}_b^2/2-\xi_1\xi_2\hat{d}_b^2/2. \qquad \text{(B.14)}
\end{aligned}
$$

$\dot{V}_v$ can be evaluated similarly. Additionally, we have

$$
\begin{aligned}
\dot{V}_{wv}\le&-2\min(a_1,b_1)\int_0^l P\left(w_x^2+v_x^2\right)dx/2 \\
&-\min(a_1,b_1)\int_0^l EA\left(w_x^2+v_x^2\right)^2dx/4. \qquad \text{(B.15)}
\end{aligned}
$$

Noted that $\dot{l}w_{xx}(0,t)^2|g\dot{l}|/(2|g|+\mathrm{sgn}(\dot{l}g)g)\ge 0$ and $\dot{l}v_{xx}(0,t)^2|h\dot{l}|/(2|h|+\mathrm{sgn}(\dot{l}h)h)\ge 0$. According to (B.9) and (B.10), if the control and design parameters are selected to satisfy

$$
\left(\frac{a_2}{\rho A}-\frac{k_3}{m}\right)-\frac{k_1k_3}{m+k_1\rho Al}\left(1-\frac{\rho Al}{m}\right)=0, \qquad \text{(B.16)}
$$

$$
\left(\frac{b_2}{\rho A}-\frac{k_5}{M}\right)-\frac{k_1k_5}{M+k_1\rho Al}\left(1-\frac{\rho Al}{M}\right)=0, \qquad \text{(B.17)}
$$

$$
\begin{aligned}
&\frac{(k_1+1)}{2}-\delta_1k_1\left(1-\frac{\rho Al}{m}\right)-\delta_3a_1 \\
&\quad-\delta_4\left|\frac{k_3}{m}-\frac{a_2}{\rho A}\right|\ge 0, \qquad \text{(B.18)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(k_1+1)}{2}-\delta_6k_1\left(1-\frac{\rho Al}{M}\right)-\delta_8b_1 \\
&\quad+\delta_9\left|\frac{k_5}{M}-\frac{b_2}{\rho A}\right|\ge 0, \qquad \text{(B.19)}
\end{aligned}
$$

then $\dot{V}$ can be evaluated as follows:

$$
\dot{V}\le-\lambda_3W_2+\varepsilon, \qquad \text{(B.20)}
$$

where $\varepsilon=\xi_1\xi_2d_b^2/2\ge 0$ and

$$
\lambda_3=\min\Big\{(k_1+1)c_w/2-\rho Aa_1-\delta_2\left(\rho Aa_1+a_2\right),
$$

$$
\begin{aligned}
&2\min(a_1,b_1),\\
&(k_1+1)c_v/2-\rho Ab_1-\delta_7(\rho Ab_1+b_2),\\
&\xi_1\xi_2/2,\\
&a_1EI_y-a_1c_wl^4/\delta_3-(k_1+1)(c_w\dot{l}^2+P_{\mathrm{Dmax}})l^2/2,\\
&b_1EI_z-c_vb_1l^4/\delta_8-(k_1+1)(c_v\dot{l}^2+P_{\mathrm{Dmax}})l^2/2,\\
&k_6-k_1(\rho A\dot{l}+c_wl/\delta_1)(1-\rho Al/m)-k_3-a_2l\\
&+\rho A\dot{l}-(\rho Aa_1+a_2)l/\delta_2-a_2|\dot{l}|\delta_5,\\
&k_7-k_1(\rho A\dot{l}+c_vl/\delta_6)(1-\rho Al/M)-k_5-b_2l\\
&+\rho A\dot{l}-(\rho Ab_1+b_2)l/\delta_7-b_2|\dot{l}|\delta_{10},\\
&\frac{k_3}{m+k_1\rho Al}\left(k_2+\frac{k_3k_6}{m+k_1\rho Al}-\rho A\dot{l}k_3/m\right)\\
&-(c_w|k_3-ma_2|l/\delta_4m\rho A+a_2|\dot{l}|/\delta_5),\\
&\frac{k_5}{M+k_1\rho Al}\left(k_4+\frac{k_5k_7}{M+k_1\rho Al}-\rho A\dot{l}k_5/M\right)\\
&-(c_v|k_5-Mb_2|l/\delta_9M\rho A+b_2|\dot{l}|/\delta_{10})\Big\}. \quad \text{(B.21)}
\end{aligned}
$$

The control gains and design parameters are selected to satisfy the following condition:

$$\lambda_3 \geq 0. \quad \text{(B.22)}$$

By using Lemma 4 and (B.15), $\dot{V}$ can be evaluated as follows: $\dot{V} \leq -\lambda V + \varepsilon$, where $\lambda = \lambda_3/\lambda_2$. Accordingly, Lemma 5 is proven.

CONFLICT OF INTEREST

The authors declare that there is no competing financial interest or personal relationship that could have appeared to influence the work reported in this paper.

REFERENCES

[1] K.-S. Hong and U. H. Shah, *Dynamics and Control of Industrial Cranes*, Springer, Singapore, 2019.

[2] H. Huang, W. He, J. Wang, L. Zhang, and Q. Fu, "An all servo-driven bird-like flapping-wing aerial robot capable of autonomous flight," *IEEE/ASME Transactions on Mechatronics*, vol. 27, no. 6, pp. 5484-5494, 2022.

[3] J.-S. Huang, R.-F. Fung, and C.-R. Tseng, "Dynamic stability of a cantilever beam attached to a translational/rotational base," *Journal of Sound and Vibration*, vol. 224, no. 2, pp. 221-242, 1999.

[4] G.-P. Cai, J.-Z. Hong, and S. X. Yang, "Dynamic analysis of a flexible hub-beam system with tip mass," *Mechanics Research Communications*, vol. 32, no. 2, pp. 173-190, 2005.

[5] T. R. Kane, R. R. Ryan, and A. K. Banerjeer, "Dynamics of a cantilever beam attached to a moving base," *Journal of Guidance, Control, and Dynamics*, vol. 10, no. 2, pp. 139-151, 1987.

[6] W. Yao, Y. Guo, Y. F. Wu, and J. Guo, "Robust adaptive dynamic surface control of multi-link flexible joint manipulator with input saturation," *International Journal of Control, Automation, and Systems,* vol. 20, no. 2, pp. 577-588, 2022.

[7] S. Hanagud and S. Sarkar, "Problem of the dynamics of a cantilevered beam attached to a moving base," *Journal of Guidance, Control, and Dynamics*, vol. 12, no. 3, pp. 438-441, 1989.

[8] S. Park, W. K. Chung, Y. Youm, and J. W. Lee, "Natural frequencies and open-loop responses of an elastic beam fixed on a moving cart and carrying an intermediate lumped mass," *Journal of Sound and Vibration*, vol. 230, no. 3, pp. 591-615, 2000.

[9] S. Park and Y. Youm, "Motion of a moving elastic beam carrying a moving mass-analysis and experimental verification," *Journal of Sound and Vibration*, vol. 240, no. 1, pp. 131-157, 2001.

[10] D. Wu, T. Endo, and F. Matsuno, "Exponential stability of two Timoshenko arms for grasping and manipulating an object," *International Journal of Control Automation and System*, vol. 19, no. 3, pp. 1328-1339, 2021.

[11] K. Yang and L. Zhao, "Command-filter-based backstepping control for flexible joint manipulator systems with full-state constrains," *International Journal of Control, Automation, and Systems*, vol. 20, no. 7, pp.2231-2238, 2022.

[12] S. Park, B. K. Kim, and Y. Youm, "Single-mode vibration suppression for a beam–mass–cart system using input preshaping with a robust internal-loop compensator," *Journal of Sound and Vibration*, vol. 241, no. 4, pp. 693-716, 2001.

[13] U. H. Shah and K.-S. Hong, "Active vibration control of a flexible rod moving in water: Application to nuclear refueling machines," *Automatica*, vol. 93, pp. 231-243, 2018.

[14] Y. Liu, X. Chen, Y. Mei, and Y. Wu, "Observer-based boundary control for an asymmetric output-constrained flexible robotic manipulator," *Science China Information Sciences*, vol. 65, no. 3, pp. 1-3, 2022.

[15] M. A. Eshag, L. Ma, Y. Sun, and K. Zhang, "Robust boundary vibration control of uncertain flexible robot manipulator with spatiotemporally-varying disturbance and boundary disturbance," *International Journal of Control, Automation, and System*, vol. 19, no. 2, pp. 788-798, 2021.

[16] Y. Liu, F. Guo, X. He, and Q. Hui, "Boundary control for an axially moving system with input restriction based on disturbance observers," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 49, no. 11, pp. 2242-2253, 2018.

[17] Z. Han, Z. Liu, W. Kang, and W. He, "Boundary feedback control of a nonhomogeneous wind turbine tower with exogenous disturbances," *IEEE Transactions on Automatic Control*, vol. 67, no. 4, 1952-1959, 2021.

[18] A. Tavasoli, "Well-posedness and exponential stability of two-dimensional vibration model of a boundary-controlled curved beam with tip mass," *International Journal of Systems Science*, vol. 49, no. 13, pp. 2847-2860, 2018.

[19] K. J. Yang, K.-S. Hong, W. S., Yoo, and F. Matsuno, "Model reference adaptive control of a cantilevered flexible beam," *JSME International Journal Series C Mechanical Systems, Machine Elements and Manufacturing*, vol. 46, no. 2, pp. 640-651, 2003.

[20] Y. Liu, X. Chen, Y. Wu, H. Cai, and H. Yokoi, "Adaptive neural network control of a flexible spacecraft subject to input nonlinearity and asymmetric output constraint," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 33, no. 11, pp. 6226-6234, 2021.

[21] Y. Liu, Y. Mei, H. Cai, C. He, T. Liu, and G. Hu, "Asymmetric input-output constraint control of a flexible variable-length rotary crane arm," *IEEE Transactions on Cybernetics*, vol. 52, no. 10, pp. 10582-10591, 2022.

[22] Y. Liu, W. Zhan, M. Xing, Y. Wu, R. Xu, and X. Wu, "Boundary control of a rotating and length-varying flexible robotic manipulator system," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 52, no. 1, pp. 377-386, 2022.

[23] Y. Liu, Y. Fu, W. He, and Q. Hui, "Modeling and observer-based vibration control of a flexible spacecraft with external disturbances," *IEEE Transactions on Industrial Electronics*, vol. 66, no. 11, pp. 8648-8658, 2018.

[24] U. H. Shah, K.-S. Hong, and S.-H. Choi, "Open-loop vibration control of an underwater system: Application to refueling machine," *IEEE/ASME Transactions on Mechatronics*, vol. 22, no. 4, pp. 1622-1632, 2017.

[25] P.-T. Pham, G.-H. Kim, Q. C. Nguyen, and K.-S. Hong, "Control of a non-uniform flexible beam: Identification of first two modes," *International Journal of Control, Automation, and Systems*, vol. 19, no. 11, pp. 3698-3707, 2021.

[26] J. Lin and W. S. Chao, "Vibration suppression control of beam-cart system with piezoelectric transducers by decomposed parallel adaptive neuro-fuzzy control," *Journal of Vibration and Control*, vol. 15, no. 12, pp. 1885-1906, 2009.

[27] J. Zhang, Y. Liu, Z. Zhao, T. Zou, and K.-S. Hong, "Vibration control for a nonlinear three-dimensional suspension cable with input and output constraints," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 52, no. 12, pp. 7821-7832, 2022.

[28] P. T. Pham, Q. C. Nguyen, M. Yoon, and K.-S. Hong, "Vibration control of a nonlinear cantilever beam operating in the 3D space," *Scientific Reports*, vol. 12, no. 1, 13811, 2022.

[29] K.-S. Hong, L.-Q. Chen, P.-T. Pham, and X.-D. Yang, *Control of Axially Moving Systems*, Springer, Singapore, 2022.

[30] P.-T. Pham and K.-S. Hong, "Dynamic models of axially moving systems: A review," *Nonlinear Dynamics*, vol. 100, no. 1, pp. 315-349, 2020.

[31] W. D. Zhu and J. Ni, "Energetics and stability of translating media with an arbitrarily varying length," *Journal of Vibration and Acoustics*, vol. 122, no. 3, pp. 295-304, 2000.

[32] K.-S. Hong and P.-T. Pham, "Control of axially moving systems: A review," *International Journal of Control, Automation and Systems*, vol. 17, no. 12, pp. 2983-3008, 2019.

[33] R. F. Fung, J. H. Lin, and C. M. Yao, "Vibration analysis and suppression control of an elevator string actuated by a PM synchronous servo motor," *Journal of Sound and Vibration*, vol. 206, no. 3, pp. 399-423, 1997.

[34] B. Kim and J. Chung, "Residual vibration reduction of a flexible beam deploying from a translating hub," *Journal of Sound and Vibration*, vol. 333, no. 16, pp. 3759-3775, 2014.

[35] W. D. Zhu, J. Ni, and J. Huang, "Active control of translating media with arbitrarily varying length," *Journal of Vibration and Acoustics*, vol. 123, no. 3, pp. 347-358, 2001.

[36] C.-S. Kim and K.-S. Hong, "Boundary control of container cranes from the perspective of controlling an axially moving string system," *International Journal of Control, Automation and Systems*, vol. 7, no. 3, pp. 437-445, 2009.

[37] Q. H. Ngo, K.-S. Hong, and I. H. Jung, "Adaptive control of an axially moving system," *Journal of Mechanical Science and Technology*, vol. 23, no. 11, pp. 3071-3078, 2010.

[38] P.-T. Pham, G.-H. Kim, and K.-S. Hong, "Vibration control of a Timoshenko cantilever beam with varying length," *International Journal of Control, Automation, and Systems*, vol. 20, no. 1, pp. 175-183, 2022.

[39] X. Xing, J. Liu, and Z. Liu, "Dynamic modeling and vibration control of a three-dimensional flexible string with variable length and spatiotemporally varying parameters subject to input constraints," *Nonlinear Dynamics*, vol. 95, no. 2, pp. 1395-1413, 2018.

[40] X. Xing and J. Liu, "Vibration and position control of overhead crane with three-dimensional variable length cable subject to input amplitude and rate constraints," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 7, pp. 4127-4138, 2021.

[41] C. D. Rahn, *Mechanical Control of Distributed Noise and Vibration*, Springer, New York, 2001.

[42] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1959.

[43] K.-S. Hong and J. Bentsman, "Direct adaptive control of parabolic systems: Algorithm synthesis and convergence and stability analysis," *IEEE Transactions on Automatic Control*, vol. 39, no. 10, pp. 2018-2033, 1999.

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