Observer-Based Relative-Output Feedback Consensus of One-Sided Lipschitz Multi-Agent Systems Subjected to Switching Graphs

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***Abstract*—This article addresses a multiple Lyapunov functions approach for investigating an observer-based protocol for the consensus of nonlinear multiagent systems (MASs). Consensus protocol designs using relative-output feedback are developed by considering leader-following consensus and one-sided Lipschitz nonlinearity (OSL) under switching topologies. Switching occurrences are modeled using a relaxed average dwell time constraint. Consensus protocol designs have been provided for switching communication topologies that have a permanent or frequent directed spanning tree (DST) with respect to the leader. In contrast to conventional methods, a more practical and complicated output feedback approach for the consensus of OSL agents under switching topologies has been addressed. In addition, the scenario of frequent DST in a switching network is addressed for the first time to the best of our knowledge for the observer-based control protocol for MASs with OSL nonlinearities. Numerical simulations are provided for MASs to confirm observer-based protocol's proficiency at synchronization under switching graphs.**

***Index Terms*—Leader-following consensus, multiple Lyapunov functions, observer-based control, one-sided Lipschitz condition, switching topology.**

I. Introduction

IN THE consensus control of multiagent systems (MASs), a protocol has been designed that can help several agents work together to reach an agreement on certain objectives. In real situations, consensus schemes are applied in flocking and formation controls and have several applications such as the coordination of unmanned vehicles, attitude alignment, robotic systems, oscillators, military operations, and rescue operations [1]. A survey of consensus control, its applications, and convergence analysis for achieving an agreement was provided in [2]. A framework for consensus under directed graphs with uncertainty, the efficiency of switched systems, and spectral properties of topologies were examined in [3].

Due to the nonlinear nature of MASs, several works have been studied addressing the nonlinear effect: The event-triggered control of interconnected nonlinear systems was presented in [4] where an optimal control problem was solved by providing a cost function for each subsystem. The global consensus for Lipschitz MASs via the relative-state feedback by addressing robustness against communication links for a consensus region approach was discussed [5]. This work was further extended for input delay constraint in [6], for cyber-physical systems in [7], and LPV approach for adaptive distributed consensus [8]. One-sided Lipschitz (OSL) nonlinearity deals with a larger set of functions than Lipschitz nonlinearity. Consensus methods on the agreement of OSL systems were addressed in [9] for distributed consensus under directed graphs, in [10] for adaptive distributed consensus subject to undirected communications, and in [11] for input saturation with leader-following topologies.

It is worth noting that none of the aforementioned works on the consensus control of the nonlinear systems has discussed the role of switching topologies for attaining agreement. The work in [12] studied the consensus of linear systems with varying communication links, which was extended in [13] for the Lipschitz MASs. The consensus of OSL systems under switching topologies, based on local relative-state information, was recently reported in [14]. Multiple Lyapunov functions (MLFs) were adapted to attain a broader range of solutions for switching graphs in linear MASs [15]. A common disadvantage of these exceptional methods [5]–[15] is the consideration of state feedback rather than the output feedback realization, and we cannot readily access the information of all states, which can lead to an impractical scenario for state feedback. Accordingly, observer-based consensus controllers can play a vigorous role by estimating states through observers and using a relative of estimated states, rather than the actual ones for feedback.

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To overcome the drawback of the unavailability of agents' states, consensus protocols have been developed using observers and relative outputs. An output-feedback formation control of linear MASs was presented in [16]. The observer-based consensus of linear MASs with disturbance rejection was examined in [17], and the work was extended to switching networks in [18]. Output feedback agreement in Lipschitz MASs under switching networks was explored in [19] for under a permanent directed spanning tree (DST). As far as OSL systems are concerned, nonlinear observers [20], observer-based controllers, and observer-based consensus protocols [21] have been designed in the existing literature. The work of [21] designed a consensus protocol under switching topologies by highlighting the intermitted control paradigm. Notwithstanding, output feedback controllers for attaining an agreement between OSL MASs under switching networks have not been extensively reported in the previous work. The role of relative-output consensus control in providing a remedy for the violation of a DST in communication topology in addition to straightforward design methods is lacking in existing studies.

Previous works [22]–[24] considered the controls of classical switching problems by analyzing the switching of linear and Lipschitz-type nonlinear matrices. These classical theories cannot be straightforwardly extended to the consensus control of switching systems. The consensus control problem considers multiple agents with a switching communication topology, rather than matrices appearing in the system dynamics. Therefore, it needs specifically designed Lyapunov functions to deal with the switching graphs along with the knowledge of graph theory and the nature of network switching. The proposed consensus control approach employs communications of agents' outputs and estimated states through observers for implementing the proposed control law. It can increase the load on the communication network due to the transmission of both signals. In [23], an approach was described using only the exchange of outputs among agents. This approach is computationally simple in implementation and requires fewer communication resources for transmitting feedback information. However, the approach restricts agents' input and output matrices. In addition, such approaches are difficult in the computation of controller gains and can be vulnerable to measurement noise. Observer-based control methods are dynamic in nature and can filter the effects of unwanted signals while estimating states and using them as feedback.

In this work, novel methods for the observer-based consensus protocol realization by the application of MLFs are explored for nonlinear OSL agents under switching networks. Consensus protocol designs for the tracking of the leader by followers under switching networks are discussed, and matrix inequality-based solutions are developed to find the controller and observer gains. Two cases with respect to DST in a graph topology are addressed in this study. In the first scenario, it is assumed that a permanent DST exists in all switching topologies for all time. Meanwhile, the second case addresses a more practical situation for the consensus of MASs to deal with the violation of a DST and assumes the existence of a frequent (rather than permanent) spanning tree in the communication topology. Moreover, a relaxed switching constraint, namely, the ADT condition, is utilized in this work for both control situations. The existing method in [21] is limited to the permanent DST with dwell time consideration and does not consider all scenarios of OSL and quadratic inner-boundedness (QIB) parameters. In addition, Chu *et al.* [21] considered a slow switching topology whereas we have accounted for the arbitrary switching topologies. The proposed consensus controller designs are obtained by considering information about the sign of a combination of OSL and QIB parameters. Cone complementary linearization methods can be used to ascertain a realistic solution to the unpretentious matrix inequalities provided in our methods. Compared to [19], the presented approach is applicable to a larger class of nonlinear systems by employing more advanced MLFs and average dwell time (ADT) methodologies, provides a condition for heterogeneous agents (see Appendix A), and renders a way of dealing with the switching graphs in different agents. Moreover, Wen *et al.* [19] do not consider the case of a topology having a frequent DST. The main contributions of this article are as follows.

1) An observer-oriented consensus that uses MLFs and ADT methods for the leader-following scenario of OSL MASs under switching topologies with a permanent DST with respect to the leader has been revealed.
2) The proposed approach has been protracted to switching topologies without verifying a permanent DST due to communication faults and data losses. To the best of our knowledge, a relative-output feedback consensus under a frequent DST that is applicable to any combination of topologies with or without a DST has been probed for the first time for OSL nonlinear agents.
3) The concern of heterogeneous MASs due to the nonidentical nature of agents has also been catered for a more hardheaded observer-based control scenario.

Several complexities are considered in this work. First, it deals with the consensus of generalized Lipschitz nonlinear MASs, in which the design is complicated by different bounds and various parameter settings. Second, the observer-based consensus of nonlinear agents requires the computation of both the consensus controller and observer gains, in addition to the coupling parameters. Thus, the well-known separation principle cannot be applied in this case. To deal with this scenario, two independent terms are considered to verify the stability with some bounds. Third, this work applies MLFs and ADT, which are advanced concepts in the consensus control theory, requiring several Lyapunov functions, exponential analysis, Lyapunov redesign, and averaging methods. Fourth, the consideration of frequent DST requires the analysis of two MLFs to investigate and combine the stability and instability margins in order to ensure the overall stability of the consensus error. To epitomize the efficiency of our work, simulations comprised of six agents are presented for mobile systems and aircraft.

A column vector is $q \in \mathbb{R}^n$ and a $m \times n$ matrix is $Q \in \mathbb{R}^{m\times n}$, where $\mathbb{R}$ defines real numbers. The dot product between two vectors is $\langle q, a\rangle$, and the Kronecker product is denoted by $Q \otimes A$. The positive definiteness of a matrix T is denoted as $T > 0$, and a semipositive matrix is represented as $T \geq 0$. $\lambda_{\min}(Q)$ gives the minimum eigenvalue of Q.

II. System Description

Let the 0th agent be a leader and $1, \ldots, N$ be the followers for a total of $N+1$ agents. The relations for the ith agent are

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) + C\psi(x_i(t), t) \\ y_i(t) &= Dx_i(t), i = 0, 1, \ldots, N \end{aligned} \tag{1}$$

where the ith agent's state, output, and control input are denoted as $x_i(t) \in \mathbb{R}^n$, $y_i(t) \in \mathbb{R}^q$, and $u_i(t) \in \mathbb{R}^p$, respectively. The nonlinear dynamics are represented by $\psi(x_i(t), t)$. $A, C \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, and $D \in \mathbb{R}^{q\times n}$ are time-invariant components, and $u_0(t) = 0$ is taken throughout this work.

Assumption 1: $\psi(x_i(t), t)$ with $\rho \in \mathbb{R}$ satisfies

$$(\psi(w,t) - \psi(z,t))^T (w-z) \leq \rho(w-z)^T(w-z) \, \forall w, z \in \mathbb{R}^n. \tag{2}$$

Assumption 2: $\psi(x_i(t), t)$ for $\varepsilon, \eta \in \mathbb{R}$ validates

$$\begin{aligned} &(\psi(w,t) - \psi(z,t))^T (\psi(w,t) - \psi(z,t)) \leq \varepsilon(w-z)^T \\ &\quad \times (w-z) + \eta(w-z)^T (\psi(w,t) - \psi(z,t)) \, \forall w, z \in \mathbb{R}^n. \end{aligned} \tag{3}$$

The OSL and QIB conditions in Assumptions 1–2 [that is, (2)–(3), respectively] cover a broader range of nonlinearities than the simple Lipschitz condition [14]. The later inequality is needed in the controller or observer synthesis due to its characteristics. The parameters ρ, η, and ε can be positive, negative, or zero, which expands the solution's feasibility region. For a Lipschitz nonlinear function, the OSL constant ρ in (2) can provide a smaller or equal value than the corresponding Lipschitz constant. Therefore, we can design a less conservative observer-based controller employing OSL nonlinearity with the support of the QIB condition. Employing the Cauchy–Schwarz inequality, we can prove that Lipschitz is a subset of the OSL inequality ($\psi^T e \leq \rho e^T e \Rightarrow |\psi^T e|^2 \leq \psi^T \psi e^T e \leq \rho |e^T e|^2 \Rightarrow \psi^T \psi \leq \rho e^T e$). Furthermore, substituting $\eta = 0$ in QIB (3) results in a simple Lipschitz condition. The limitation of (2)–(3) is that the design includes more parameter computations and complex computational design. To further clarify this, suppose $f(x)$ is an OSL function. Considering two dynamics $\dot{x} = f(x)$ and $\dot{\tilde{x}} = f(\tilde{x})$, we desire to attain synchronization between these systems. Given $e = x - \tilde{x}$, we have $\dot{e} = f(x) - f(\tilde{x})$. For energy function $V = 0.5e^T e$, we attain $\dot{V} = \dot{e}^T e = (f(x) - f(\tilde{x}))^T (x - \tilde{x})$, which represents the rate of energy change, on the left-hand side of the OSL condition. To have a stable system, we require $\dot{V} \leq 0$, which indicates $(f(x) - f(\tilde{x}))^T (x - \tilde{x}) \leq \rho(x - \tilde{x})^T (x - \tilde{x}) \leq 0$ under the OSL condition. Hence, the OSL condition represents a bound on the rate of energy change of the synchronization manifold.

The interaction between MASs is shown via $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, with collection of $N+1$ nodes via $\mathscr{V}$ and edges via $\mathscr{E}$. A transfer of information from node v_i to v_j is denoted by the edge $e_{ij} = (v_i, v_j)$. A condition ($e_{ij} = e_{ji}$) is true for undirected graphs, but not necessarily true for directed graphs. If information can be transferred from node v_i to v_j using intermediate nodes $v_{k_i}, i = 1, 2, \ldots, l$, then there exists a directed path from v_i to v_j. If the node v_i can transfer information to all other nodes using directed paths, then the graph owns a DST with respect to the node v_i. The adjacency matrix is $\mathcal{A}(t) = [a_{ij}(t)]_{(N+1)\times(N+1)}$. For $a_{ij} > 0$, we have $e_{ij} \in \mathscr{E}$. Laplacian matrix $\mathscr{L} = [l_{ij}]_{(N+1)\times(N+1)}$ can be partitioned as

$$\mathscr{L}^{\vartheta(t)} = \begin{bmatrix} 0 & 0_N^T \\ r^{\vartheta(t)} & \tilde{\mathscr{L}}^{\vartheta(t)} \end{bmatrix}. \tag{4}$$

For the time intervals $[t_q, t_{q+1}), q \in \mathbb{N}$, with $t_1 = 0$, $\tau_1 \geq t_{q+1} - t_q \geq \tau \geq 0$, τ symbolizes the conventional dwell time. The switching instances are taken to be $t_1, t_2, \ldots,$ and we can show it using a piecewise signal $\vartheta(t) : [0, +\infty) \to \{1, \ldots, m\}$. In practice, ADT is used as it has fewer constraints than the simple dwell time on the switching instance; it can allow switching to occur at different time instances. The ADT approach relaxes the switching constraint.

Definition 1: The parameter $\tau_a > 0$ is called the ADT for $\vartheta(t)$ if the following inequality holds for $Y_o \geq 0$

$$Y_{\vartheta(t)}(\tau, t) \leq Y_o + ((t-\tau)/\tau_a), \forall t > \tau \geq 0 \tag{5}$$

where $Y_{\vartheta(t)}(\tau, t)$ depicts the number of switching instances for $\vartheta(t)$ in (τ, t) and $Y_o \geq 0$ denotes the chatter bound.

Since we do not have access to the states of nodes, we need observers for agents. The observer dynamics are given as

$$\begin{aligned} \dot{\tilde{x}}_i(t) &= A\tilde{x}_i(t) + Bu_i(t) + C\psi(\tilde{x}_i(t), t) \\ &\qquad + \alpha \sum_{j=0}^{N} a_{ij}^{\vartheta(t)}(t) F(\delta_j(t) - \delta_i(t)) \\ \tilde{y}_i(t) &= D\tilde{x}_i(t), \quad i = 1, \ldots, N \end{aligned} \tag{6}$$

where the ith observer agent's state is $\tilde{x}_i(t) \in \mathbb{R}^n$, the output is denoted as $\tilde{y}_i(t) \in \mathbb{R}^q$, and the nonlinear dynamics are given as $\psi(\tilde{x}_i(t), t)$. The constant α and matrix $F \in \mathbb{R}^{n\times q}$ are unknown parameters and are needed to be encountered. The signal $\delta_i(t)$ is defined as the error between the outputs of the observer and system

$$\delta_i(t) = \tilde{y}_i(t) - y_i(t) = D(\tilde{x}_i(t) - x_i(t)). \tag{7}$$

In contrast with the conventional observers, the proposed observer (6) incorporates OSL nonlinearity, switching parameters due to $a_{ij}^{\vartheta(t)}(t)$, and information regarding $\delta_i(t)$ from other observers.

Assuming that the leader's state is known as it has to act as a reference signal for the followers and that the leader gets no information from its followers, we select $\tilde{x}_0(t) = x_0(t)$. The control protocol for the agreement between OSL systems is

$$u_i(t) = sK \sum_{j=0}^{N} a_{ij}^{\vartheta(t)}(t) (\tilde{x}_j(t) - \tilde{x}_i(t)), i = 1, \ldots, N \tag{8}$$

where s is the coupling constant and $K \in \mathbb{R}^{p\times n}$ is the protocol. If we define $x(t) = \left[x_1^T(t) \cdots x_N^T(t)\right]^T$ and $\tilde{x}(t) = \left[\tilde{x}_1^T(t) \cdots \tilde{x}_N^T(t)\right]^T$, then substituting (8) into (1) and using the Kronecker product, we obtain

$$\begin{aligned} \dot{x}(t) &= (I_N \otimes A) x(t) - s\left(\hat{\tilde{\mathscr{L}}}^{\vartheta(t)} \otimes BK\right) \hat{\tilde{x}}(t) \\ &\qquad + (I_N \otimes C) \Xi(x(t), t). \end{aligned} \tag{9}$$

Similarly, substituting (8) into (6) leads to

$$\dot{\hat{x}}(t) = (I_N \otimes A)\,\tilde{x}_i(t) - s\left(\hat{\tilde{\mathscr{L}}}^{\vartheta(t)} \otimes BK\right)\hat{\tilde{x}}(t) + (I_N \otimes C)\,\Gamma(\tilde{x}(t), t) - \alpha\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes F\right)\delta(t) \quad (10)$$

where $\delta(t) = \left[\delta_1^T(t) \cdots \delta_N^T(t)\right]^T$, $\hat{\tilde{x}}(t) = \left[\tilde{x}^T(t)\ \tilde{x}_0^T(t)\right]^T$, $\hat{\tilde{\mathscr{L}}}^{\vartheta(t)} = \left[r^{\vartheta(t)}\ \tilde{\mathscr{L}}^{\vartheta(t)}\right]$, $\Gamma(\tilde{x}(t),t) = \left[\psi^T(\tilde{x}_1(t),t) \cdots \psi^T(\tilde{x}_N(t),t)\right]^T$, and $\Xi(x(t),t) = \left[\psi^T(x_1(t),t) \cdots \psi^T(x_N(t),t)\right]^T$. Let us define the error between the states of agents and the leader as $e_i(t) = x_i(t) - x_0(t)$ and the error between agents' and observers' states as $\tilde{e}_i(t) = x_i(t) - \tilde{x}_i(t)$. With the error vectors as $e(t) = \left[e_1^T(t) \cdots e_N^T(t)\right]^T$ and $\tilde{e}(t) = \left[\tilde{e}_1^T(t) \cdots \tilde{e}_N^T(t)\right]^T$, we can find that

$$\left(\hat{\tilde{\mathscr{L}}}^{\vartheta(t)} \otimes BK\right)\hat{\tilde{x}}(t) = \left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes BK\right)(e(t) - \tilde{e}(t)). \quad (11)$$

By using the relations $\Omega(x(t),t) = \Xi(x(t),t) - 1_N \otimes \psi(x_0(t),t)$ and $\mathrm{O}(x(t),t) = \Xi(x(t),t) - \Gamma(\tilde{x}(t),t)$, the derivatives of $e(t)$ and $\tilde{e}(t)$ can be obtained as

$$\dot{\tilde{e}}(t) = \left[I_N \otimes A - \alpha\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes FD\right)\right]\tilde{e}(t) + (I_N \otimes C)\,\mathrm{O}(x(t),t) \quad (12)$$

$$\dot{e}(t) = \left[(I_N \otimes A) - s\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes BK\right)\right]e(t) + s\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes BK\right) \times \tilde{e}(t) + (I_N \otimes C)\,\Omega(x(t),t). \quad (13)$$

Combining (12)–(13) and $\hat{e}(t) = \left[\tilde{e}^T(t)\ e^T(t)\right]^T$, we have

$$\dot{\hat{e}}(t) = \hat{A}\hat{e}(t) + (I_{2N} \otimes C)\,\hat{\psi}(\hat{e}(t),t) \quad (14)$$

$$\hat{A} = \begin{bmatrix} I_N \otimes A - \alpha\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes FD\right) & 0 \\ s\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes BK\right) & (I_N \otimes A) - s\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes BK\right) \end{bmatrix}$$
$$\hat{\psi}(\hat{e}(t),t) = \begin{bmatrix} \mathrm{O}(x(t),t) \\ \Omega(x(t),t) \end{bmatrix}. \quad (15)$$

Assumption 3: The graphs $\mathscr{G}^{\vartheta(t)} \in \tilde{\mathscr{G}}$ always have a DST with respect to agent 0, where $\tilde{\mathscr{G}} = \{\mathscr{G}^1, \ldots, \mathscr{G}^m\}, m \geq 1$.

We need to find the parameters $\alpha > 0$, $s > 0$, $F \in \mathbb{R}^{n\times q}$, and $K \in \mathbb{R}^{p\times n}$ for observer-based protocol (6) and (8) to attain consensus for the nonlinear MASs (1) under switching topologies.

III. Observer-Based Consensus Control Design

Two cases are discussed in this section; the first deals with consensus in MASs with switching graph topologies that permanently have a DST and the second considers the frequent occurrence of a DST in graph topologies.

A. Consensus Control Under Permanent DST

Here, we find the parameters of observer-based protocol design for the consensus of systems (1) under switching networks. Knowing $u_0(t) = 0, \forall t \geq 0$, and (1), we obtain

$$\dot{x}_0(t) = Ax_0(t) + C\psi(x_0(t),t). \quad (16)$$

The followers (1) $(i = 1, \ldots, N)$ are required to track the leader dynamics (16). We consider a fundamental leader-following consensus problem with zero input at the leader, which can provide a way for dealing with nonzero input at the leader, similar to the error boundedness criterion in Appendix A.

Theorem 1: Under Assumptions 1–3 for constants $\upsilon_1 > 0$, $\upsilon_2 > 0, c_1 > 0, \upsilon_3 > 0, \upsilon_4 > 0, c_2 > 0, \beta > 0, \gamma > 0$ and positive-definite matrices $M \in \mathbb{R}^{n\times n}$, $J \in \mathbb{R}^{n\times n}$, suppose that a solution to inequality (17) along with any equation in (18)–(20) with respect to different scenarios of a scalar $\upsilon_3\rho + \upsilon_4\varepsilon$ exists for $\mathrm{X}_1 = \beta J + JA + A^TJ - c_1D^TD + 2(\upsilon_1\rho I_n + \upsilon_2\varepsilon I_n)$, $\mathrm{X}_2 = \gamma M + AM + MA^T - c_2BB^T$, $\mathrm{X}_3 = JC - \upsilon_1 I_n + \upsilon_2\eta I_n$, $\mathrm{X}_4 = C - \upsilon_3 M + \upsilon_4\eta M$, and $\mathrm{X}_5 = M\sqrt{2\upsilon_3\rho I_n + 2\upsilon_4\varepsilon I_n}$

$$\begin{bmatrix} \mathrm{X}_1 & \mathrm{X}_3 \\ * & -2\upsilon_2 I_n \end{bmatrix} \leq 0. \quad (17)$$

$$\text{i) If } \upsilon_3\rho + \upsilon_4\varepsilon > 0, \begin{bmatrix} \mathrm{X}_2 & \mathrm{X}_4 & \mathrm{X}_5 \\ * & -2\upsilon_4 I_n & 0 \\ * & * & -I_n \end{bmatrix} \leq 0. \quad (18)$$

$$\text{ii) If } \upsilon_3\rho + \upsilon_4\varepsilon = 0, \begin{bmatrix} \mathrm{X}_2 & \mathrm{X}_4 \\ * & -2\upsilon_4 I_n \end{bmatrix} \leq 0. \quad (19)$$

$$\text{iii) If } \upsilon_3\rho + \upsilon_4\varepsilon < 0, \begin{bmatrix} \mathrm{X}_2 - U & \mathrm{X}_4 \\ * & -2\upsilon_4 I_n \end{bmatrix} \leq 0 \quad (20)$$

where $U = -2M^T(\upsilon_3\rho I_n + \upsilon_4\varepsilon I_n)M$, and $\lambda_o = \min_{i=1,\ldots,m} \lambda_{\min}(\tilde{\mathscr{L}}^i + (\Phi^i)^{-1}(\tilde{\mathscr{L}}^i)^T\Phi^i)$. Φ^i can be selected via

$$\Phi^i = \begin{cases} I_n, & \text{if } \tilde{\mathscr{L}}^i + (\tilde{\mathscr{L}}^i)^T > 0, \\ \Sigma^i, & \text{otherwise,} \end{cases}$$

$$\Sigma^i\tilde{\mathscr{L}}^i + (\tilde{\mathscr{L}}^i)^T\Sigma^i > 0, \Sigma^i > \varepsilon_i I_N, \Sigma^i < I_N \quad (21)$$

where $0 < \varepsilon_i < 1$, which defines the conservatism index of the solution. Then, the protocol (8) ensures consensus of nonlinear agents (1) with $K = B^TM^{-1}$, $F = J^{-1}D^T$, $\alpha > c_1/\lambda_o$, and $s > c_2/\lambda_o$ for $\tau_a > \ln(\kappa_o)/min\{\beta, \gamma\}$, where $\kappa_o = \max_{i,j\in\{1,\ldots,m\}, i\neq j}\{\lambda_{\max}\{\Phi^i\}/\lambda_{\min}\{\Phi^j\}\}$.

Proof: We take MLFs for (12) and (13) as

$$V(t) = \tilde{e}^T(t)\left(\Phi^{\vartheta(t)} \otimes J\right)\tilde{e}(t) + \iota e^T(t)\left(\Phi^{\vartheta(t)} \otimes M^{-1}\right)e(t). \quad (22)$$

The time-derivative of (22) results in

$$\dot{V}(t) = \Delta + \iota\Lambda \quad (23)$$

$$\Delta = \tilde{e}^T(t)\left(\Phi^{\vartheta(t)} \otimes (JA + A^TJ) - 2\alpha\left(\Phi^{\vartheta(t)}\tilde{\mathscr{L}}^{\vartheta(t)} \otimes JFD\right)\right) \times \tilde{e}(t) + 2\tilde{e}^T(t)\left(\Phi^{\vartheta(t)} \otimes JC\right)\mathrm{O}(x(t),t) \quad (24)$$

$$\Lambda = 2se^T(t)\left(\Phi^{\vartheta(t)}\tilde{\mathscr{L}}^{\vartheta(t)} \otimes M^{-1}BK\right)\tilde{e}(t) + e^T(t) \times \left(\Phi^{\vartheta(t)} \otimes (M^{-1}A + A^TM^{-1})\right.$$

$$- 2s \left(\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BK\right)\right) e\left(t\right)$$
$$+ 2e^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \otimes M^{-1}C\right) \Omega\left(x\left(t\right),t\right). \tag{25}$$

We shall separately investigate the upper bounds on Δ and Λ. Using $F = J^{-1}D^{T}$ and $\alpha > c_1/\lambda_o$, from (24), we attain

$$\Delta \leq \tilde{e}^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \otimes \left(JA + A^{T}J - c_1 D^{T}D\right)\right) \tilde{e}\left(t\right)$$
$$+ 2\tilde{e}^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \otimes JC\right) \mathrm{O}\left(x\left(t\right),t\right). \tag{26}$$

Using (2) and (3), we can derive the following relationships:

$$2v_1 \mathrm{O}(x\left(t\right),t)^{T} \left(\Phi^{\vartheta(t)} \otimes I_n\right) \tilde{e}\left(t\right)$$
$$- 2v_1 \rho \tilde{e}(t)^{T} \left(\Phi^{\vartheta(t)} \otimes I_n\right) \tilde{e}\left(t\right) \leq 0 \tag{27}$$
$$2v_2 \mathrm{O}(x\left(t\right),t)^{T} \left(\Phi^{\vartheta(t)} \otimes I_n\right) \mathrm{O}\left(x\left(t\right),t\right) - 2v_2 \varepsilon \tilde{e}(t)^{T}$$
$$\times \left(\Phi^{\vartheta(t)} \otimes I_n\right) \tilde{e}\left(t\right) - 2v_2 \eta \tilde{e}(t)^{T}$$
$$\times \left(\Phi^{\vartheta(t)} \otimes I_n\right) \mathrm{O}\left(x\left(t\right),t\right) \leq 0. \tag{28}$$

Adding (26)–(28), it yields

$$\Delta \leq \tilde{e}^{T}(t)\left(\Phi^{\vartheta(t)} \otimes (\mathrm{X}_1 - \beta J)\right) \tilde{e}(t) + 2\tilde{e}^{T}(t)\left(\Phi^{\vartheta(t)} \otimes \mathrm{X}_3\right)$$
$$\times \mathrm{O}(x(t),t) - 2v_2 \mathrm{O}(x(t),t)^{T} \left(\Phi^{\vartheta(t)} \otimes I_n\right) \mathrm{O}(x(t),t). \tag{29}$$

For $\mathrm{Z} = \left[\, \tilde{e}^{T}(t)\ \mathrm{O}(x(t),t)^{T} \,\right]^{T}$, (29) is arranged as

$$\Delta \leq \mathrm{Z}^{T} \begin{bmatrix} \Phi^{\vartheta(t)} \otimes (\mathrm{X}_1 - \beta J) & \Phi^{\vartheta(t)} \otimes \mathrm{X}_3 \\ \Phi^{\vartheta(t)} \otimes \mathrm{X}_3^{T} & -2v_2 \Phi^{\vartheta(t)} \otimes I_n \end{bmatrix} \mathrm{Z}. \tag{30}$$

For $0 < \tilde{\beta} \ll \beta$, one attains

$$\Delta < -\left(\beta + \tilde{\beta}\right) \tilde{e}^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \otimes J\right) \tilde{e}\left(t\right) \tag{31}$$

if (17) holds. Using (2) and (3) reveals that

$$2v_3 \Omega(x\left(t\right),t)^{T} \left(\Phi^{\vartheta(t)} \otimes I_n\right) \tilde{e}\left(t\right)$$
$$- 2v_3 \rho \tilde{e}(t)^{T} \left(\Phi^{\vartheta(t)} \otimes I_n\right) \tilde{e}\left(t\right) \leq 0 \tag{32}$$
$$2v_4 \Omega(x\left(t\right),t)^{T} \left(\Phi^{\vartheta(t)} \otimes I_n\right) \Omega\left(x\left(t\right),t\right) - 2v_4 \varepsilon \tilde{e}(t)^{T}$$
$$\times \left(\Phi^{\vartheta(t)} \otimes I_n\right) \tilde{e}\left(t\right) - 2v_4 \eta \tilde{e}(t)^{T}$$
$$\times \left(\Phi^{\vartheta(t)} \otimes I_n\right) \Omega\left(x\left(t\right),t\right) \leq 0. \tag{33}$$

Applying $\mathrm{X}_6 = M^{-1}A + A^{T}M^{-1} - c_2 M^{-1}BB^{T}M^{-1} + 2v_4 \varepsilon I_n + 2v_3 \rho I_n, \mathrm{X}_7 = M^{-1}C + v_4 \eta I_n - v_3 I_n, K = B^{T}M^{-1}$, $s > c_2/\lambda_o$, (32) and (33) into (25), we obtain

$$\Lambda \leq 2se^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1}\right) \tilde{e}\left(t\right)$$
$$+ e^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \otimes \mathrm{X}_6\right) e\left(t\right) + 2e^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \otimes \mathrm{X}_7\right) \Omega$$
$$\times \left(x\left(t\right),t\right) - 2v_4 \times \Omega(x\left(t\right),t)^{T} \left(\Phi^{\vartheta(t)} \otimes I_n\right) \Omega(x\left(t\right),t). \tag{34}$$

Taking $\Gamma = \left[\, e^{T}(t)\ \Omega(x(t),t)^{T} \,\right]^{T}$, (34) leads to

$$\Lambda \leq \Gamma^{T} \begin{bmatrix} \Phi^{\vartheta(t)} \otimes \mathrm{X}_6 & \Phi^{\vartheta(t)} \otimes \mathrm{X}_7 \\ \Phi^{\vartheta(t)} \otimes \mathrm{X}_7^{T} & -2v_4 \Phi^{\vartheta(t)} \otimes I_n \end{bmatrix} \Gamma$$
$$+ 2se^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1}\right) \tilde{e}\left(t\right). \tag{35}$$

Under $0 < \tilde{\gamma} \ll \gamma$, one can attain

$$\Lambda \leq 2se^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1}\right) \tilde{e}\left(t\right)$$
$$- \left(\gamma + \tilde{\gamma}\right) e^{T}\left(t\right) \left(\Phi^{\vartheta(t)} \otimes M^{-1}\right) e\left(t\right) \tag{36}$$

if any of (18)–(20) hold, which can be attained using the congruence transformation of $diag(M, I_n)$, the Schur complement for $v_3\rho + v_4\varepsilon > 0$, substituting $v_3\rho + v_4\varepsilon = 0$ and using $U = -2M^{T}(v_3\rho I_n + v_4\varepsilon I_n)M$ for $v_3\rho + v_4\varepsilon < 0$. Using (31) and (36) into (23) leads to

$$\dot{V}(t) < -\beta \tilde{e}^{T}(t)\left(\Phi^{\vartheta(t)} \otimes J\right)\tilde{e}(t) - \iota\gamma e^{T}(t)\left(\Phi^{\vartheta(t)} \otimes M^{-1}\right)e(t)$$
$$+ \hat{e}^{T}(t) \begin{bmatrix} -\tilde{\beta}\Phi^{\vartheta(t)} \otimes J & \iota s \Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1} \\ * & -\iota\tilde{\gamma}\Phi^{\vartheta(t)} \otimes M^{-1} \end{bmatrix} \hat{e}\left(t\right). \tag{37}$$

To make $\begin{bmatrix} -\tilde{\beta}\Phi^{\vartheta(t)} \otimes J & \iota s \Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1} \\ * & -\iota\tilde{\gamma}\Phi^{\vartheta(t)} \otimes M^{-1} \end{bmatrix} < 0$, we take the Schur complement and knowing that $-\iota\tilde{\gamma}\Phi^{\vartheta(t)} \otimes M^{-1} < 0$. As a result, we reach

$$\tilde{\beta}\Phi^{\vartheta(t)} \otimes J > \iota\left(s\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1}\right)^{T}$$
$$\times \left(\tilde{\gamma}\Phi^{\vartheta(t)} \otimes M^{-1}\right)^{-1} \left(s\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1}\right). \tag{38}$$

As $\left(s\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1}\right)^{T} \left(\tilde{\gamma}\Phi^{\vartheta(t)} \otimes M^{-1}\right)^{-1} \left(s\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1}\right) \geq 0$ and $\tilde{\beta}\ \Phi^{\vartheta(t)} \otimes J > 0$, we obtain that all the eigenvalues of $(\tilde{\beta}\ \Phi^{\vartheta(t)} \otimes J)^{-1}\ (s\Phi^{\vartheta(t)} \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1})^{T}\ (\tilde{\gamma}\Phi^{\vartheta(t)} \otimes M^{-1})\ (s\Phi^{\vartheta(t)}\ \tilde{\mathfrak{X}}^{\vartheta(t)} \otimes M^{-1}BB^{T}M^{-1})$ are real and not less than zero. We can define a parameter ι such that

$$\iota < \min_{i \in \{1,\ldots,m\}} \left\{ (1/\lambda_{\max}) \left(\tilde{\beta}\Phi^{i} \otimes J\right)^{-1} \right.$$
$$\times \left(s\Phi^{i}\tilde{\mathfrak{X}}^{i} \otimes M^{-1}BB^{T}M^{-1}\right)^{T}$$
$$\left. \times \left(\tilde{\gamma}\Phi^{i} \otimes M^{-1}\right) \left(s\Phi^{i}\tilde{\mathfrak{X}}^{i} \otimes M^{-1}BB^{T}M^{-1}\right) \right\}. \tag{39}$$

According to (39), (38) holds and we attain

$$\dot{V}\left(t\right) < -\min\left\{\beta, \gamma\right\} V\left(t\right). \tag{40}$$

For the interval $(0, t)$, the number of switching occurrences can be given as $Y_{\vartheta(t)}(0, t)$ for a piecewise signal $\vartheta(t)$. Switching instances are t_Ξ, $\Xi = 1, \ldots, Y_{\vartheta(t)}(0,t)$. For $\Xi > 1$, $V(t) \leq \kappa_o^{Y_{\vartheta(t)}(0,t)} e^{-\min\{\beta,\gamma\}t} V(0)$; using Definition 1, this leads to

$$V(t) \leq e^{Y_o \ln(\kappa_o)} e^{-(\min\{\beta,\gamma\} - (\ln(\kappa_o)/\tau_a))\min\{\beta,\gamma\}t} V(0). \tag{41}$$

We require $\min\{\beta,\gamma\} - (\ln(\kappa_o)/\tau_a) > 0$, which makes our ADT constraint $\tau_a > \ln(\kappa_o)/\min\{\beta,\gamma\}$. We choose t at random and (41) yields that consensus is achieved, i.e., $e(t) \to 0$ as $t \to \infty$. □

Remark 1: Consensus control of one-sided Lipschitz agents using distributed algorithms has remained a topic of concern as seen in [9] and [10]. An agreement between OSL systems for dealing with switching topologies has been emphasized in [14]. An observer-based consensus of linear switching networked systems has been considered in [18]. In this article, we tackle the issue of observer-based consensus of more generic OSL systems in a switching network, by application of less restrictive MLFs methodology and provide design methods for finding parameters for our control and observer protocols, based on the output feedback.

Remark 2: In [19], an observer-based control protocol is designed for systems that have Lipschitz nodes and switching topologies using MLFs. The drawback of this method is that it uses Lipschitz nonlinearity, while one-sided Lipschitz functions cover a broader category of nonlinearities. In Theorem 1, we use OSL and QIB constraints, both of which cover a more general systems' class than the traditional one. The OSL condition can also be applied to provide less conservative controllers because OSL constants can have a smaller magnitude than Lipschitz constants.

Remark 3: In [14], the consensus of OSL MASs linked in a switching network was explored. The drawback of this work is that it used relative state feedback rather than an output feedback scheme, which makes it difficult to implement due to sensor constraints. Another issue with the work is that it does not use the MLFs approach that would facilitate the establishment of less conservative controllers. In addition, it provides a dwell time that needs to be strictly followed. Realistically, it is difficult to control the time of switching instances, so the ADT solution is implemented in our study. In Theorem 1, we provide the MLFs approach to solve the observer-based consensus of OSL MASs and provide ADT that makes implementing the developed protocol less conservative and more realistic.

Remark 4: The use of MLFs and ADT is an advanced subject in the consensus control under switching graphs. For the present scenario, MLFs theory has been employed by accounting for both consensus and state estimations errors by taking the matrix $\Phi^{\vartheta(t)}$. In the dynamics (14), both the consensus error and estimation error are coupled. Furthermore, separation of inequalities for the observer and consensus controller is required to estimate a tractable upper bound on the derivative of MLFs. The other factors on OSL nonlinear dynamics and observer-based consensus control also affect the upper bound on the MLFs. Finally, the nonlinearity bounds in (27)–(28) have been shaped to achieve the elimination of $\Phi^{\vartheta(t)}$. The ADT approach is based on MLFs, which can be used to relax the switching of networks.

A variant of the proposed approach in Theorem 1 is provided in Appendix A to deal with heterogeneous portions of agents.

B. Consensus Control Under Frequent DSTs

In this subsection, consensus control is designed for systems where there is a break in the communication links between agents, resulting in topologies frequently possessing a DST with respect to the leader. To recover the failures of links between agents, communication reinstatement protocols are applied. A graph topology with a DST present is usually required for consensus control, but it cannot be ensured due to communication failures. Therefore, considering a graph with frequent DST is more practical than always having a DST in the topology. This results in a less conservative option that also includes the permanent DST case.

Assumption 4: $\mathscr{G}^{\vartheta(t)}$ is split into $\widehat{\mathscr{G}} \in \{\mathscr{G}^1, \ldots, \mathscr{G}^{\tilde{m}}\}$ and $\breve{\mathscr{G}} \in \{\mathscr{G}^{\tilde{m}+1}, \ldots, \mathscr{G}^{m}\}$, where the former set has a DST and the latter set lacks a DST, where $\tilde{m} \leq m$ and $\tilde{m} \in \mathbb{N}$.

Let $\widehat{T}_{t_k}(t)$ and $\breve{T}_{t_k}(t)$ denote the total time for which the graph topology is from the sets $\widehat{\mathscr{G}}$ and $\breve{\mathscr{G}}$, respectively. The following assumption has been affirmed for these times.

Assumption 5: $\breve{T}_{t_k}(t) \leq \mu_o \widehat{T}_{t_k}(t) \, \forall t \geq t_k, \mu_o \geq 0, t_k \geq 0$.

Theorem 2: Under Assumptions 1, 2, 4, and 5 for scalars $v_1 > 0, v_2 > 0, v_3 > 0, v_4 > 0, c_1 > 0, c_2 > 0, \beta_1 > 0, \beta_2 \geq 0$, $\gamma_1 > 0$, and $\gamma_2 \geq 0$ and the matrix inequalities are given as $M > 0, J > 0$, suppose that a solution to inequalities (42) and any of scenarios i, ii, and iii with respect to different cases of scalar $v_3\rho + v_4\varepsilon$ exist for $\coprod_1 = JA + A^TJ + 2(v_1\rho I_n + v_2\varepsilon I_n) - c_1 D^TD + \beta_1 J$, $\coprod_2 = JA + A^TJ + 2(v_1\rho I_n + v_2\varepsilon I_n) - \alpha\hat{\hat{\lambda}}_o D^TD - \beta_2 J$, $\coprod_3 = AM + MA^T + \gamma_1 M - c_2 BB^T$, $\coprod_5 = JC - v_1 I_n + v_2\eta I_n$, $\coprod_4 = AM + MA^T - \gamma_2 M - s\hat{\hat{\lambda}}_o BB^T$, $\coprod_6 = C - v_3 M + v_4\eta M$, and $\coprod_7 = \sqrt{2v_3\rho I_n + 2v_4\varepsilon I_n}M$

$$\begin{bmatrix} \coprod_1 & \coprod_5 \\ * & -2v_2 I_n \end{bmatrix} \leq 0, \begin{bmatrix} \coprod_2 & \coprod_5 \\ * & -2v_2 I_n \end{bmatrix} \leq 0. \tag{42}$$

1) If $v_3\rho + v_4\varepsilon > 0$, then

$$\begin{bmatrix} \coprod_3 & \coprod_6 & \coprod_7 \\ * & -2v_4 I_n & 0 \\ * & * & -I_n \end{bmatrix} \leq 0, \begin{bmatrix} \coprod_4 & \coprod_6 & \coprod_7 \\ * & -2v_4 I_n & 0 \\ * & * & -I_n \end{bmatrix} \leq 0. \tag{43}$$

2) If $v_3\rho + v_4\varepsilon = 0$, then

$$\begin{bmatrix} \coprod_3 & \coprod_6 \\ * & -2v_4 I_n \end{bmatrix} \leq 0, \begin{bmatrix} \coprod_4 & \coprod_6 \\ * & -2v_4 I_n \end{bmatrix} \leq 0. \tag{44}$$

3) If $v_3\rho + v_4\varepsilon < 0$, then

$$\begin{bmatrix} \coprod_3 - U & \coprod_6 \\ * & -2v_4 I_n \end{bmatrix} \leq 0, \begin{bmatrix} \coprod_4 - U & \coprod_6 \\ * & -2v_4 I_n \end{bmatrix} \leq 0$$
where $U = -2M^T(v_3\rho I_n + v_4\varepsilon I_n)M$.
$$\tag{45}$$

We can determine $\hat{\lambda}_o = \min_{i=1,..,\tilde{m}} \lambda_{\min}(\tilde{\mathscr{L}}^i + (\hat{\Phi}^i)^{-1}(\tilde{\mathscr{L}}^i)^T\hat{\Phi}^i)$ and $\hat{\hat{\lambda}}_o = \min_{i=\tilde{m}+1,..,m} \lambda_{\min}(\tilde{\mathscr{L}}^i + (\Pi)^{-1}(\tilde{\mathscr{L}}^i)^T\Pi)$, and $\hat{\Phi}^i$

can be computed by solving the following condition:

$$\hat{\Phi}^i = \begin{cases} I_n, & \text{if } \tilde{\mathscr{X}}^i + \left(\tilde{\mathscr{X}}^i\right)^T > 0, \\ \Sigma^i, & \text{otherwise,} \end{cases}$$

$$\Sigma^i \tilde{\mathscr{X}}^i + \left(\tilde{\mathscr{X}}^i\right)^T \Sigma^i > 0, \Sigma^i > \varepsilon_i I_N, \Sigma^i < I_N \tag{46}$$

where the conservatism index for $i = 1, \ldots, \tilde{m}$ is $0 < \varepsilon_i < 1$ and $\Pi = (1/\tilde{m}) \sum_{i=1}^{\tilde{m}} \hat{\Phi}^i$. Then, the protocol (8) ensures that nonlinear agents (1) for $K = B^T M^{-1}$, $F = J^{-1} D^T$, $\alpha > c_1/\hat{\lambda}_o$ and $s > c_2/\hat{\lambda}_o$ achieve consensus with ADT given as $\hat{\tau}_a > \ln(\hat{\kappa}_o)/min\{\beta_1, \gamma_1\}$ for $\hat{\kappa}_o = \max_{i,j \in \{1,\ldots,m\}, i \neq j} \{\lambda_{\max} \{\hat{\Phi}^i\}/\lambda_{\min}\{\hat{\Phi}^j\}\}$, $\mu_o < (\min\{\beta_1, \gamma_1\} - \ln(\hat{\kappa}_o)/\hat{\tau}_a)/(\max\{\beta_2, \gamma_2\} + \ln(\hat{\kappa}_o)/\hat{\tau}_a)$.

Proof: Suppose that the communication between agents of the graph $\mathcal{G} \in \widehat{\mathcal{G}} \cup \breve{\mathcal{G}}$ changes over time via the switching signal $\tilde{\vartheta}(t) : [t_o, +\infty) \to \{1, \ldots, \tilde{m}+1\}$,where $\tilde{\vartheta}(t) \in \{1, \ldots, \tilde{m}\} \in \widehat{\mathcal{G}}$ are topologies with DST present and $\tilde{\vartheta}(t) = \tilde{m}+1 \in \breve{\mathcal{G}}$ represents graph topology with absent DST. Let $\hat{\Phi}^{\tilde{m}+1} = \Pi$ and constructing MLFs for $t \geq 0$ as

$$V(t) = \tilde{e}^T(t)\left(\Phi^{\tilde{\vartheta}(t)} \otimes J\right)\tilde{e}(t) + \iota e^T(t)\left(\Phi^{\tilde{\vartheta}(t)} \otimes M^{-1}\right) e(t). \tag{47}$$

For a time interval $(t_\Xi, t_{\Xi+1})$ such that $t_\Xi \geq t_o, \Xi \in \mathbb{N}$. At previous switching instant, if the switching signal is $\tilde{\vartheta}(t_\Xi) \in \{1, \ldots, \tilde{m}\}$, the following inequality can be obtained:

$$\begin{aligned} \dot{V}(t) < &-\beta_1 \tilde{e}^T(t)\left(\Phi^{\tilde{\vartheta}(t)} \otimes J\right)\tilde{e}(t) \\ &- \hat{\iota}\gamma_1 e^T(t)\left(\Phi^{\tilde{\vartheta}(t)} \otimes M^{-1}\right) \times e(t) + \hat{e}^T(t) \\ &\begin{bmatrix} -\tilde{\beta}_1 \Phi^{\tilde{\vartheta}(t)} \otimes J & \hat{\iota} s \Phi^{\tilde{\vartheta}(t)} \tilde{\mathscr{X}}^{\tilde{\vartheta}(t)} \otimes M^{-1}BB^T M^{-1} \\ * & -\hat{\iota}\tilde{\gamma}_1 \Phi^{\tilde{\vartheta}(t)} \otimes M^{-1} \end{bmatrix} \hat{e}(t). \end{aligned} \tag{48}$$

Using (48), we can derive (42)–(45), and the subsequent inequality by following the condition similar to Theorem 1:

$$V(t) \leq e^{-\min\{\beta_1, \gamma_1\}(t - t_\Xi)} V(t_\Xi), t \in [t_\Xi, t_{\Xi+1}). \tag{49}$$

Under $\tilde{\vartheta}(t_\Xi) = \tilde{m}+1$, we have

$$V(t) = \tilde{e}^T(t)(\Pi \otimes J)\tilde{e}(t) + \hat{\iota} e^T(t)\left(\Pi \otimes M^{-1}\right) e(t). \tag{50}$$

Taking the derivative of (50), we have

$$\dot{V}(t) = \Theta + \hat{\iota} \times \Upsilon. \tag{51}$$

Using a similar analysis as in Theorem 1, (51) leads to

$$\begin{aligned} \Theta \leq\ & \tilde{e}^T(t)\left(\Pi \otimes \left(\coprod\nolimits_2 + \beta_2 J\right)\right)\tilde{e}(t) + 2\tilde{e}^T(t)\left(\Pi \otimes \coprod\nolimits_5\right) \\ &\times \mathrm{O}(x(t), t) - 2v_2 \mathrm{O}(x(t), t)^T (\Pi \otimes I_n)\, \mathrm{O}(x(t), t). \end{aligned} \tag{52}$$

Similarly, for $\coprod_8 = M^{-1}A + A^T M^{-1} - s\hat{\tilde{\lambda}}_o M^{-1}BB^T M^{-1} + 2v_4 \varepsilon I_n + 2v_3 \rho I_n$ and $\coprod_9 = M^{-1}C + v_4 \eta I_n - v_3 I_n$, it attains

$$\begin{aligned} \Upsilon \leq\ & 2se^T(t)\left(\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right)\tilde{e}(t) + e^T(t) \\ &\times \left(\Pi \otimes \coprod\nolimits_8\right) e(t) + 2e^T(t)\left(\Pi \otimes \coprod\nolimits_9\right)\Omega(x(t), t)^T \\ &- 2v_4 \Omega(x(t), t)^T (\Pi \otimes I_n)\, \Omega(x(t), t)^T. \end{aligned} \tag{53}$$

By adding and subtracting $(\beta_2 - \tilde{\beta}_2)\tilde{e}^T(t)(\Pi \otimes J)\tilde{e}(t)$, in (52) such that $\tilde{\beta}_2$ is much smaller than β_2, and considering inequalities (42) and (52), one obtains

$$\Theta < \left(\beta_2 - \tilde{\beta}_2\right)\tilde{e}^T(t)(\Pi \otimes J)\tilde{e}(t), \tag{54}$$

if (42) holds. Similarly, using (53), we attain (43) using matrix manipulation, using the Schur complement, (44) by substituting $v_3\rho + v_4\varepsilon = 0$ and (45) by substituting $U = -2M^T(v_3\rho I_n + v_4 \varepsilon I_n)M$ in the same way as in Theorem 1. Let $\tilde{\gamma}_2$ be much smaller than γ_2, which leads to

$$\begin{aligned} \Upsilon \leq\ & 2se^T(t)\left(\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right)\tilde{e}(t) \\ &- (\gamma_2 - \tilde{\gamma}_2)\, e^T(t)\left(\Pi \otimes M^{-1}\right) e(t). \end{aligned} \tag{55}$$

Substituting (54) and (55) into relation (51) reveals

$$\begin{aligned} \dot{V}(t) <\ & \beta_2 \tilde{e}^T(t)(\Pi \otimes J)\tilde{e}(t) + \hat{\iota}\gamma_2 e^T(t)\left(\Pi \otimes M^{-1}\right) e(t) \\ &+ \hat{e}^T(t)\begin{bmatrix} -\tilde{\beta}_2 \Pi \otimes J & \hat{\iota} s \Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1} \\ * & -\hat{\iota}\tilde{\gamma}_2 \Pi \otimes M^{-1} \end{bmatrix} \hat{e}(t). \end{aligned} \tag{56}$$

Knowing that $-\hat{\iota}\tilde{\gamma}_2 \Pi \otimes M^{-1} < 0$, we can derive that

$$\begin{aligned} \tilde{\beta}_2 \Pi \otimes J >\ & \hat{\iota}\left(s\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right)^T \left(\tilde{\gamma}_2 \Pi \otimes M^{-1}\right)^{-1} \\ &\times \left(s\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right) \end{aligned} \tag{57}$$

by assuring $\begin{bmatrix} -\tilde{\beta}_2 \Pi \otimes J & \hat{\iota} s \Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1} \\ * & -\hat{\iota}\tilde{\gamma}_2 \Pi \otimes M^{-1} \end{bmatrix} < 0$.

Since $\left(s\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right)^T \left(\tilde{\gamma}_2 \Pi \otimes M^{-1}\right)^{-1} \times \left(s\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right) \geq 0$ and $\tilde{\beta}_2 \Pi \otimes J > 0$, it follows that $\left(\tilde{\beta}_2 \Pi \otimes J\right)^{-1} \left(s\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right)^T \left(\tilde{\gamma}_2 \Pi \otimes M^{-1}\right) \times \left(s\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right) \geq 0$, selecting $\hat{\iota}$ from the following:

$$\begin{aligned} \hat{\iota} < & \left\{(1/\lambda_{\max})\left(\tilde{\beta}_2 \Pi \otimes J\right)^{-1}\left(s\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right)^T \right. \\ &\left. \times \left(\tilde{\gamma}_2 \Pi \otimes M^{-1}\right)\left(s\Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1}\right)\right\}. \end{aligned} \tag{58}$$

For a suitable selection of the parameter $\hat{\iota}$, $\hat{e}^T(t)$ $\begin{bmatrix} -\tilde{\beta}_2 \Pi \otimes J & \hat{\iota} s \Pi \tilde{\mathscr{X}}^{\vartheta(t)} \otimes M^{-1}BB^T M^{-1} \\ * & -\hat{\iota}\tilde{\gamma}_2 \Pi \otimes M^{-1} \end{bmatrix} \hat{e}(t)$ in (56) can be neglected. By applying relations (49), (56), $V(t_{\Xi+1}) \leq \hat{\kappa}_o$

$\lim\limits_{t \to t_{\Xi+1}} V(t)\ \forall \Xi \in \mathbb{N}$, and Definition 1, it implies

$$V(t) \le \hat{\kappa}_o^{[Y_o + ((t-t_k)/\hat{\tau}_a)]} e^{\max\{\gamma_2, \beta_2\} \breve{T}_{t_k}(t)} - \min\{\beta_1, \gamma_1\} \widehat{T}_{t_k}(t) V(t_k) \tag{59}$$

for any given $t \ge t_k$. Using Assumption 5, $\mu_o < (\min\{\beta_1, \gamma_1\} - \ln(\hat{\kappa}_o)/\hat{\tau}_a)/(\max\{\beta_2, \gamma_2\} + \ln(\hat{\kappa}_o)/\hat{\tau}_a)$ and $\hat{\tau}_a > \ln(\hat{\kappa}_o)/\min\{\beta_1, \gamma_1\}$, we obtain

$$\begin{aligned} V(t) &\le e^{Y_o \ln(\hat{\kappa}_o)} e^{-\chi \widehat{T}_{t_o}(t)} V(t_o) \\ &\le e^{Y_o \ln(\hat{\kappa}_o)} e^{-(\chi/1+\mu_o)(t-t_o)} V(t_o) \end{aligned} \tag{60}$$

where $\chi > 0$. The inequality (60) demonstrates that the consensus of systems is achieved. □

Remark 5: Theorem 1 and the approach of [19] require a permanent DST from the leader in a network topology for observer-based agreement in OSL agents. In contrast, the approach in Theorem 2 can be applied to counteract the performance degradation due to the loss of a DST for a time interval. In real situations, communication links cannot be retained due to packet losses and disturbances, leading to the violation of a DST in the switching communications. Therefore, the approach of Theorem 2 can be applied to a more practical situation of communication between agents and can be used by following the constraints on ADT and μ_o for an agreement under a frequent DST in switching topologies. To the best of our knowledge, MLFs-oriented observer-based consensus of OSL agents using output feedback under switching topologies that often accommodates a DST is developed here for the first time.

The observer (6) cannot be applied to estimate the leader's state because the leader does not receive any information from its followers. The observer employs the term $\alpha \sum_{j=0}^{N} a_{ij}^{\vartheta(t)}(t) F(\delta_j(t) - \delta_i(t))$ to update the state $\tilde{x}_i(t)$. However, this term is zero for the leader because the leader receives no information from its followers. Note that this term is significant as the observer updates its state using information from neighboring agents. A different Luenberger-type observer for the leader can be used to deal with this restriction and the leader can transmit its estimated state rather than the actual state. The leader agent acts as a reference signal generator and gets no information from its followers. Practically, the leader's state can be unknown; however, our protocol requires the leader's state to track those of its followers. If the leader's state is unavailable, a Luenberger-type observer can be used to estimate it. Let $\tilde{x}_0(t)$ and $\tilde{y}_0(t)$ be the observer state and observer output of the leader agent, respectively. Then, we have Luenberger-type observer as

$$\begin{aligned} \dot{\tilde{x}}_0(t) &= A\tilde{x}_0(t) + C\psi(x_0(t), t) - L(\tilde{y}_0(t) - y_0(t)) \\ \tilde{y}_0(t) &= D\tilde{x}_0(t) \end{aligned}$$

where gain L can be selected as in [20] and [25].

Remark 6: We can experience data packet loss and communication faults in practical applications, which result in the absence of DST in the graph topology. In this scenario, our control signal switches between unstable and stable behavior of consensus error. We have employed two Lyapunov functions in Theorem 2, see (47) and (50), such that the stable mode of the system dominates the unstable mode. Provided that the switching constraints (ADT τ_a and the ratio of activation time of non-DST to DST graphs μ_o) hold, the overall consensus error becomes stable.

Remark 7: The ADT of a switching system can be decreased by increasing the value of $\{\beta, \beta_1\}$ and $\{\gamma, \gamma_1\}$ depending on Theorems 1–2. To obtain a higher ratio of the activation time for non-DST graphs to DST graphs (a bigger μ_o), we can decrease the values of β_2 and γ_2 in Theorems 2.

Using Algorithm 1 in [14] for Theorem 2, we can construct a mesh of various values of the gains v_3 and v_4. The values of v_3 and v_4, taken from the mesh, can be used to verify the sign of $v_3\rho + v_4\varepsilon$ (positive, negative, or zero). The conditions (42)–(44) can be solved using the LMI techniques when the relations $v_3\rho + v_4\varepsilon = 0$ and $v_3\rho + v_4\varepsilon > 0$ are satisfied. For $v_3\rho + v_4\varepsilon < 0$, a feasible solution can be found using the optimization approach and cone complementarity linearization (CCL) schema [14]. Using the CCL approach, the feasibility solution of (45) for $U = -2M^T(v_3\rho I_n + v_4\varepsilon I_n)M$ can be attained via the following optimization problem:

$$\text{Minimize Trace}\left(0.5\bar{U}U + \bar{M}M - \bar{U}\left(M^T(v_3\rho I_n + v_4\varepsilon I_n) M\right)\right)$$

$$\begin{aligned} &\begin{bmatrix} \coprod_3 - U & \coprod_6 \\ * & -2v_4 I_n \end{bmatrix} \le 0, \\ &\begin{bmatrix} \coprod_4 - U & \coprod_6 \\ * & -2v_4 I_n \end{bmatrix} \le 0, \quad \begin{bmatrix} M & I_n \\ * & \bar{M} \end{bmatrix} \ge 0 \\ &\begin{bmatrix} U & I_n \\ * & \bar{U} \end{bmatrix} \ge 0, \quad \begin{bmatrix} -2(v_3\rho I_n + v_4\varepsilon I_n) & \bar{M} \\ * & \bar{U} \end{bmatrix} \ge 0 \end{aligned} \tag{61}$$

where $\bar{U}$ and $\bar{M}$ are the inverses of U and M, respectively. However, the results in Theorems 1–2 can become infeasible if ρ and ε have higher values (for $v_3\rho + v_4\varepsilon > 0$). In the future, a more computationally complex linear (or nonlinear) parameter varying method can be developed to improve the feasibility of control methods. In the future, our work can be extended to the event-triggered approach as presented in [4] and finite-time consensus as in [24].

Remark 8: In contrast to our previous works [9], [10], and [11], the present work deals with the consensus control of nonlinear MASs for dealing with the switching topologies. In contrast with state feedback methods [9], [10], [11], and [14], a relative-output feedback consensus protocol has been investigated in the present work. The proposed relative-output feedback method has been developed by employing observers for state estimation, and the estimated states are used for attaining the consensus of MASs. The present problem of the output feedback approach for the nonlinear agents becomes non-trivial owing to the estimation of states, feedback of estimated states, and switching topologies (for permanent and frequent DST via MLFs) over a network. Note that the straightforward separation principle cannot be used for the design of observer-based consensus protocol because of OSL nonlinear dynamics of MASs and complex switching in the network.

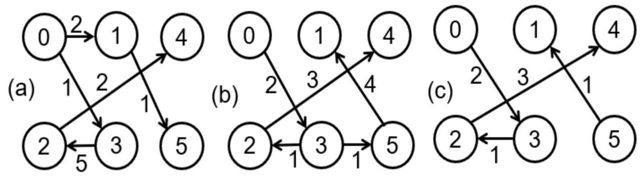

Fig. 1. Graph topologies: (a) G^1, (b) G^2, and (c) G^3.

IV. Simulation Results

Example 1: The evolution of states [16] for moving particles is

$$\dot{x}_{i1}(t) = x_{i1}(t) - x_{i2}(t) + u_i(t) - x_{i1}(t)\left(x_{i1}^2(t) + x_{i2}^2(t)\right)$$
$$\dot{x}_{i2}(t) = x_{i1}(t) + x_{i2}(t) + u_i(t) - x_{i2}(t)\left(x_{i1}^2(t) + x_{i2}^2(t)\right)$$
$$y_i(t) = x_{i1}.$$

The OSL and QIB constants for the nonlinear function in the above equations are calculated as $\rho = 0$, $\varepsilon = 0$, and $\eta = -40$ [26]. We take the chatter-bound of the switching instant as $Y_o = 0$ and the constants $c_1 = 1, c_2 = 1$ and $v_i = 1, i = 1, \ldots, 4$ are accounted to design the consensus protocol. First, we consider graphs G^1, and G^2, where we have a permanent occurrence of DST in our graph topologies as shown in Fig. 1(a) and (b). The conservatism indexes for these topologies in Fig. 1(a) and (b) are chosen as $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.3$, respectively. Using the eigenvalues of Φ^i (calculated by applying the information from graphs) helps explore the constant $\kappa_o = 1.9484$. The solution for matrices M and J using the inequalities of Theorem 1 are revealed as

$$M = 10^{-4}\begin{bmatrix} 268 & 40 \\ 40 & 239 \end{bmatrix}, J = 10^{-2}\begin{bmatrix} 31.03 & 4.35 \\ 4.35 & 13.64 \end{bmatrix}$$

with the controller gains and coupling weights as

$$K = \begin{bmatrix} 31.8260 \; 36.4793 \end{bmatrix}, \; F = \begin{bmatrix} 3.3730 \; -1.0750 \end{bmatrix}^T,$$
$$\alpha = 2.5791 > c_1/\lambda_o, \; s = 2.5791 > c_2/\lambda_o,$$

where $\lambda_o = 0.3877$. To validate the exponential stabilities of the controller and observer scenarios, the constants are taken as $\beta = 2$ and $\gamma = 3$, and the ADT constraint is formed as $\tau_a > 0.3335$. For simulations, $\tau_a = 0.5$ is applied (the graphs G^1 and G^2 are active for the time intervals $[l-1, l-0.5)$ and $[l-0.5, l)$, respectively, where $l \in \mathbb{N}$) and the efficiency of the proposed consensus protocol for Theorem 1 is demonstrated in Figs. 2 and 3. The states of the nonlinear follower agents track the reference of the leader states. Note that these simulation results have been attained using output feedback for switching graphs in the case of highly complex OSL nonlinear agents.

Now, we consider the more complicated frequent occurrence of a DST in the graph topologies G^1, G^2, and G^3 as given in Fig. 1(a)–(c), respectively. The conservative indices for graphs G^1, and G^2 are taken as $\varepsilon_1 = 0.07$ and $\varepsilon_2 = 0.25$. For the corresponding value of $\hat{\Phi}^i$, we find the constant $\hat{\kappa}_o = 9.9335$.

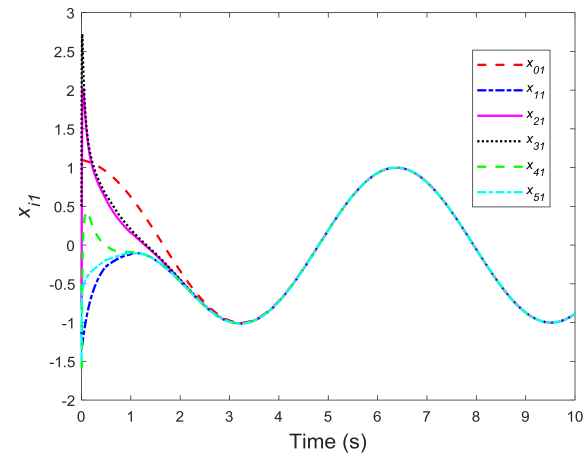

Fig. 2. States x_{i1} for mobile agents with $i = 0, 1, 2, 3, 4, 5$ for graph owning a permanent DST.

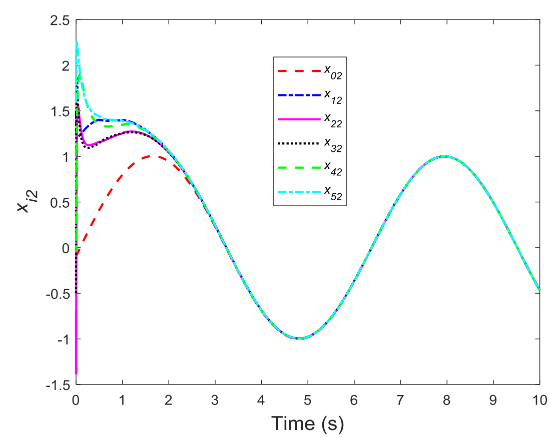

Fig. 3. States x_{i2} for mobile agents with $i = 0, 1, 2, 3, 4, 5$ for graph owning a permanent DST.

The solution using Theorem 2 results in

$$M = 10^{-4}\begin{bmatrix} 211 & 14 \\ 14 & 190 \end{bmatrix}, J = 10^{-4}\begin{bmatrix} 1726 & 338 \\ 338 & 621 \end{bmatrix},$$

with the corresponding controller and observer parameters

$$K = \begin{bmatrix} 44.0358 \; 49.3486 \end{bmatrix}, \; F = \begin{bmatrix} 6.4859 \; -3.5320 \end{bmatrix}^T,$$
$$\alpha = 2.3456 > c_1/\lambda_o, \; s = 2.3456 > c_2/\lambda_o$$

where $\lambda_o = 0.4263$ and $\hat{\hat{\lambda}}_o = 0.023$. By taking $\beta_1 = 1.3$, $\beta_2 = .7, \gamma_1 = 8$, and $\gamma_2 = 3$ the constraints obtained on ADT is $\hat{\tau}_a > 1.7661$. The value $\mu_o = 0.0367$ is achieved when $\hat{\tau}_a = 2$ and the results using the proposed consensus protocol of Theorem 2 are established in Figs. 4 and 5.

Due to the consideration of G^3 , the overall communication scheme does not possess a DST in the communication topology and the resultant scenario becomes more complicated than the previous simulation case. However, the protocol given in (8) can be applied for the observer-based consensus of OSL systems under switching topologies. The time taken for consensus is $t = 3$s and $t = 6$s. Our results for both cases are comparable

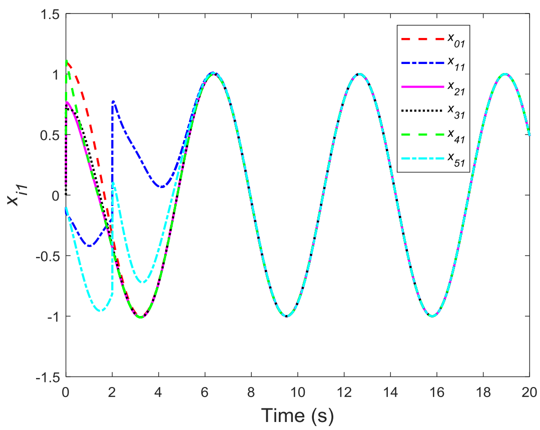

Fig. 4. States x_{i1} for mobile agents with $i = 0, 1, 2, 3, 4, 5$ for graphs owning frequent DST.

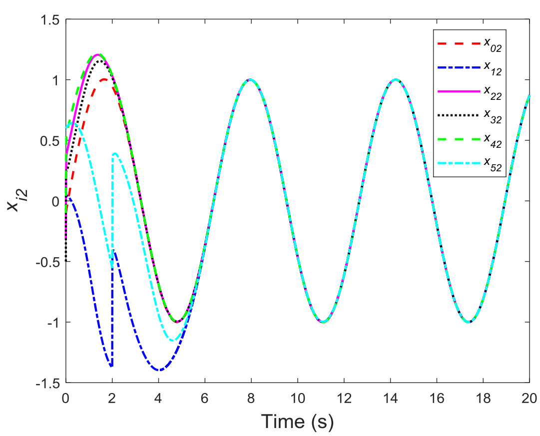

Fig. 5. States x_{i2} for mobile agents with $i = 0, 1, 2, 3, 4, 5$ for graphs owning frequent DST.

to [16], where consensus was achieved in time $t = 5$s using the relative-state feedback approach. The stated mobile agents attain agreement by applying the proposed output feedback-oriented consensus protocol, even in the violation of a DST.

Example 2: Consider six aircrafts F-18 [17] communicating via the same graphs as in Fig. 1, where θ is attack angle, ω being pitch rate, p represents position input, and v denotes the nozzle position input. The system dynamics can be written as

$$\dot{\theta} = -1.175\theta + 0.9871\omega - 0.194p - 0.03593v - \theta\left(\theta^2 + \omega^2\right)$$

$$\dot{\omega} = -8.458\theta + 0.8776\omega - 19.29p - 3.803v - \omega\left(\theta^2 + \omega^2\right).$$

With the value of θ and ω as output, we get $D = I_2$. The solution of matrices following Case 2 of Theorem 2 for $\beta_1 = 0.6$, $\beta_2 = 0.3$, $\gamma_1 = 1$, and $\gamma_2 = 0.4$ are

$$M = 10^{-4}\begin{bmatrix} 1798 & -0.1463 \\ -1463 & 4683 \end{bmatrix}, J = 10^{-4}\begin{bmatrix} 15614 & 648 \\ 648 & 1283 \end{bmatrix}$$

with the corresponding controller and observer parameters as

$$K = \begin{bmatrix} -46.3679 & -55.6763 \\ -9.1241 & -10.9711 \end{bmatrix}, F = \begin{bmatrix} 0.6541 & -0.3302 \\ -0.3302 & 7.9612 \end{bmatrix},$$

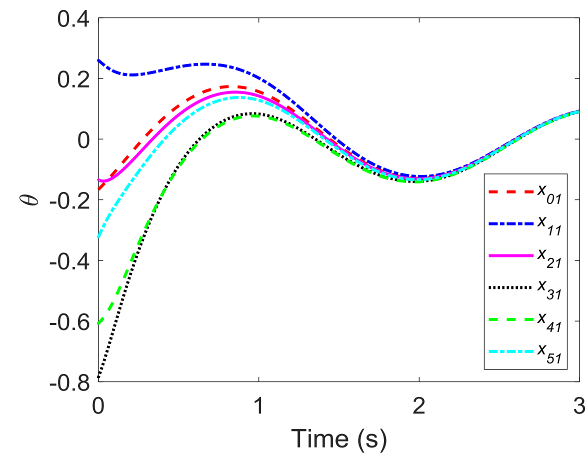

Fig. 6. States θ for aircraft with $i = 0, 1, 2, 3, 4, 5$ for graphs owning frequent DST.

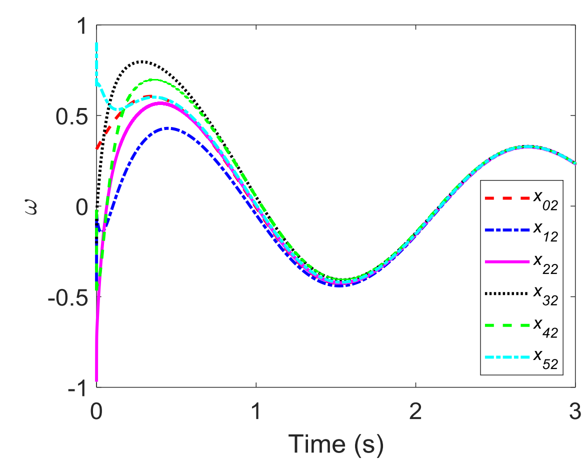

Fig. 7. States ω for aircraft with $i = 0, 1, 2, 3, 4, 5$ for graphs owning frequent DST.

$$\alpha = 2.3456 > c_1/\lambda_o, \; s = 2.3456 > c_2/\lambda_o$$

where $\lambda_o = 0.4263$ and $\hat{\bar{\lambda}}_o = 0.0219$. The ADT constraint is $\hat{\tau}_a > 3.8185$. We select $\mu_o = 0.028$ for $\hat{\tau}_a = 4$. The states θ and ω are plotted in Figs. 6 and 7, respectively, with our method. A few other practical examples where we can apply our proposed controller protocol (8) with observer (6) are YF-22 UAV [12], a mechanical revolving model [21], and spacecraft [17].

V. Conclusion

This study considered the observer-based protocol design with the help of MLFs for the consensus of OSL nonlinear systems over switching graphs. For switching networks, a relaxed ADT constraint was considered that can relax the conventional dwell time to be followed at every instant. By accounting for the switching network always having a DST from the leader, a given protocol was formed for selecting the parameters of the observer, controller, coupling weights, and ADT. The proposed scheme was stretched to a situation in which the topology has a frequent DST from the leader and a communication restoration mechanism is applied for reestablishment from failures. With respect to the previous works, we have considered MLFs and observer-based protocols in the study for the consensus of OSL systems under switching topologies. Simulations of mobile systems were carried out, moving in the Cartesian coordinate, to depict the efficiency of the developed consensus protocols under different switching topology situations. Future work can be considered for fully distributed protocols with adaptation laws to deal with the global information of λ_o on Laplacian matrices. In this regard, the gains α and s can be adapted rather than choosing $\alpha > c_1/\lambda_o$ and $s > c_2/\lambda_o$.

Appendix A
Extension to Heterogeneous Agents

Consider the heterogeneous agents as

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) + \psi_i(x_i(t), t) \\ y_i(t) &= Dx_i(t), i = 0, 1, \ldots, N \end{aligned} \tag{A.1}$$

where $\psi_i(x_i(t), t) = C\psi(x_i(t), t) + \varpi_i(x_i(t), t)$.

Assumption A.1: Let $\|\varpi_i(x_i(t), t)\| \leq \bar{\varpi}$ for $\bar{\varpi} > 0$.

Under $\varpi = [\,\varpi_1(x_1(t), t)^T \cdots \varpi_N(x_N(t), t)^T\,]^T$, we have

$$\begin{aligned} \dot{\tilde{e}}(t) = &\left[I_N \otimes A - \alpha\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes FD\right)\right]\tilde{e}(t) \\ &+ (I_N \otimes C)\,\psi((x(t) - \tilde{x}(t)), t) + \tilde{\varpi} \end{aligned} \tag{A.2}$$

$$\begin{aligned} \dot{e}(t) = &\left[(I_N \otimes A) - s\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes BK\right)\right]e(t) + s\left(\tilde{\mathscr{L}}^{\vartheta(t)} \otimes BK\right) \\ &\tilde{e}(t) + (I_N \otimes C)\,\psi((x(t) - x_0(t)), t) + \varpi_e \end{aligned} \tag{A.3}$$

$\tilde{\varpi} = [\varpi_i(x_i(t), t)]_{N\times 1}, i = 1, \ldots\ \varpi_e = \tilde{\varpi} - 1_N \otimes \varpi_0(\tilde{x}_0(t), t)$.

Theorem A.1: Under Assumptions 1–4 for constants $v_1 > 0, v_2 > 0, c_1 > 0, v_3 > 0, v_4 > 0, c_2 > 0, \beta > 0, \gamma > 0$ and matrices $M > 0, J > 0$, suppose that both a solution to inequality (A.4) and any of (A.5)–(A.7) with respect to different scenarios of a scalar $v_3\rho + v_4\varepsilon$ exists

$$\begin{bmatrix} \mathrm{X}_1 + J & \mathrm{X}_3 \\ * & -2v_2 I_n \end{bmatrix} \leq 0 \tag{A.4}$$

i. If $v_3\rho + v_4\varepsilon > 0$, $$\begin{bmatrix} \mathrm{X}_2 + M & \mathrm{X}_4 & \mathrm{X}_5 \\ * & -2v_4 I_n & 0 \\ * & * & -I_n \end{bmatrix} \leq 0. \tag{A.5}$$

ii. If $v_3\rho + v_4\varepsilon = 0$, $$\begin{bmatrix} \mathrm{X}_2 + M & \mathrm{X}_4 \\ * & -2v_4 I_n \end{bmatrix} \leq 0. \tag{A.6}$$

iii. If $v_3\rho + v_4\varepsilon < 0$, $$\begin{bmatrix} \mathrm{X}_2 + M - U & \mathrm{X}_4 \\ * & -2v_4 I_n \end{bmatrix} \leq 0. \tag{A.7}$$

Then, the protocol (8) ensures the uniformly ultimately bounded consensus of nonlinear agents in (A.1) with $K = B^T M^{-1}$, $F = J^{-1}D^T$, $\alpha > c_1/\lambda_o$, and $s > c_2/\lambda_o$ for $\tau_a > \ln(\kappa_o)/\min\{\beta, \gamma\}$, $\kappa_o = \max_{i,j\in\{1,\ldots,m\}, i\neq j}\{\lambda_{\max}\{\Phi^i\}/\lambda_{\min}\{\Phi^j\}\}$.

Proof: The proof is omitted for brevity.

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