Prob. #1

(*) By writing the Laplace transform of $x(t)$ as $X(s)$ or $\mathcal{L}[x(t)] = X(s)$ we obtain $\mathcal{L}[\dot{x}] = sX(s) - x(0)$

$$\mathcal{L}[\ddot{x}] = s^2X(s) - sx(0) - \dot{x}(0).$$

And so the given differential equation becomes

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 4[sX(s) - X(0)] + 8X(s) = 0$$

By substituting the given initial conditions into this last equation, we obtain

$$[s^2X(s) - 0.s - 4] + 4[sX(s) - 0] + 8X(s) = 0$$

or

$$(s^2 + 4s + 8)X(s) = 4$$

Solving for $X(s)$, we have

$$X(s) = \frac{4}{s^2 + 4s + 8} = 2 \cdot \frac{2}{(s+2)^2 + 2^2}$$

The inverse Laplace transform of $X(s)$ gives

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[2 \cdot \frac{2}{(s+2)^2 + 2^2}\right] = 2\mathcal{L}^{-1}\left[\frac{2}{(s+2)^2 + 2^2}\right]$$

$$x(t) = 2e^{-2t}\sin 2t, \text{ for } t \geq 0$$

which is the solution of the given differential equation.

(**) The Laplace transform of: $\ddot{x}(t) + 4\dot{x}(t) + 8x(t) = 0$ is

$$(s^2 + 4s + 8) = 0$$

with roots:

$$s_1 = -2 + 2i$$

$$s_2 = -2 - 2i$$

Then: $x(t) = Ae^{s_1 t} + Be^{s_2 t} = Ae^{-2t}\sin 2t + Be^{-2t}\cos 2t$

with the given initial conditions:

$x(0) = B = 0$

$\dot{x}(0) = -2Ae^{-2t}\sin 2t + 2Ae^{-2t}\cos 2t - 2Be^{-2t}\cos 2t - 2Be^{-2t}\sin 2t$

$2A - 2B = 4$

$A = 2$

$\therefore \underline{\underline{x(t) = 2e^{-2t}\sin 2t}}$

which is the solution of the given differential equation.

***

$\ddot{x}(t) + 4\dot{x}(t) + 8x(t) = 0 \longrightarrow x(0) = 0$ and $\dot{x}(0) = 4$

define:

$y_1 = x(t) \longrightarrow \dot{y}_1 = \dot{x}(t) = y_2$

$y_2 = \dot{x}(t) \longrightarrow \dot{y}_2 = -8y_1 - 4y_2$

Therefore:

$$\dot{y}(t) = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\dot{y}(t) = A y(t)$

as $y(t) = \phi(t)x(0)$, where $\phi(t)$ is 2×2 state-transition matrix, for which

$\phi(t) = e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}]$

$\phi(0) = e^{A0} = I$

$$\Rightarrow sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 8 & s+4 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{1}{s^2+4s+8} \begin{bmatrix} s+4 & 1 \\ -8 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+4}{s^2+4s+8} & \frac{1}{s^2+4s+8} \\ \frac{-8}{s^2+4s+8} & \frac{s}{s^2+4s+8} \end{bmatrix}$$

$$(SI-A)^{-1} = \begin{bmatrix} \dfrac{s+2}{(s+2)^2+2^2} + \dfrac{2}{(s+2)^2+2^2} & \dfrac{1}{2}\dfrac{2}{(s+2)^2+2^2} \\ -4\dfrac{2}{(s+2)^2+2^2} & \dfrac{s+2}{(s+2)^2+2^2} - \dfrac{2}{(s+2)^2+2^2} \end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}\left[(SI-A)^{-1}\right]$$

$$= \begin{bmatrix} e^{-2t}\cos 2t + e^{-2t}\sin 2t & \frac{1}{2}e^{-2t}\sin 2t \\ -4e^{-2t}\sin 2t & e^{-2t}\cos 2t - e^{-2t}\sin 2t \end{bmatrix}$$

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \phi(t)\,x(0) = \phi(t)\begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} e^{-2t}\cos 2t + e^{-2t}\sin 2t & \frac{1}{2}e^{-2t}\sin 2t \\ -4e^{-2t}\sin 2t & e^{-2t}\cos 2t - e^{-2t}\sin 2t \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 2e^{-2t}\sin 2t \\ 4e^{-2t}\cos 2t - 4e^{-2t}\sin 2t \end{bmatrix}$$

$$\therefore \underline{\underline{x(t) = 2e^{-2t}\sin 2t}}$$

Prob. #2

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix} x$$

First find the eigenvalues and the eigenvectors:

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

The eigenvectors, x_i, satisfy: $0 = (\lambda_i I - A)x_i$. use $\det(\lambda_i I - A) = 0$ to find the eigenvalues, λ_i:

$$\det(\lambda I - A) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| = \begin{vmatrix} \lambda+3 & -1 \\ -1 & \lambda+3 \end{vmatrix} = \lambda^2 + 6\lambda + 8$$

from which the eigenvalues are $\lambda = -2$, and -4. then we have

$$A x_i = \lambda x_i$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or $\left.\begin{aligned} -3x_1 + x_2 &= -2x_1 \\ x_1 - 3x_2 &= -2x_2 \end{aligned}\right\}$ from which $x_1 = x_2$. Thus $x = \begin{bmatrix} c \\ c \end{bmatrix}$

Using the other eigenvalue, -4, we have : $x = \begin{bmatrix} c \\ -c \end{bmatrix}$

Then the eigenvectors are

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The transformation matrix P, whose columns consist of the eigenvectors, becomes $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

The diagonal system that is similar:

$$\dot{z} = P^{-1}APz + P^{-1}Bu$$
$$y = CPx + Du$$

where:

$$P^{-1}AP = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$P^{-1}B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

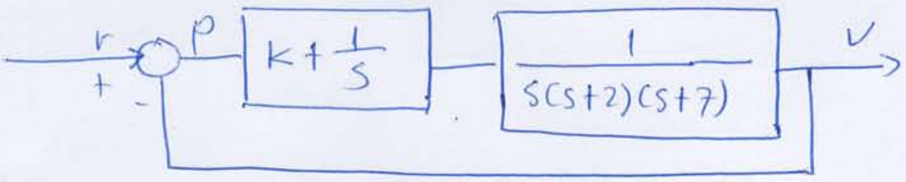
$$CP = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \end{bmatrix}$$

Then:

$$\dot{z} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} z + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & -1 \end{bmatrix} z$$

Prob. #3



The closed-loop transfer function is

$$p = r - v \longrightarrow V = \frac{ks+1}{s^2(s^2+7s+10)}\, p$$

$$\hookrightarrow p = \frac{s^2(s^2+7s+10)}{Ks+1}\, v$$

$$\frac{V}{r}(s) = \frac{Ks+1}{s^4+7s^3+10s^2+Ks+1}$$

3

The characteristic equation is

$$s^4+7s^3+10s^2+Ks+1=0$$

The array of coefficients becomes

s^4	1	10	1
s^3	7	K	0
s^2	$\frac{70-K}{7}$	1	
s^1	$K - \frac{49}{70-K}$		
s^0	1		

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For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

- $K > 0$,
- $K - \frac{49}{70-K} > 0$,
- $\frac{70-K}{7} > 0$

$$0{,}71 < K < 69{,}293$$

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$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

The array coefficients become

s^4	a_0	a_2	a_4
s^3	a_1	a_3	0
s^2	b_1	b_2	
s^1	c_1	c_2	
s^0	d_1		

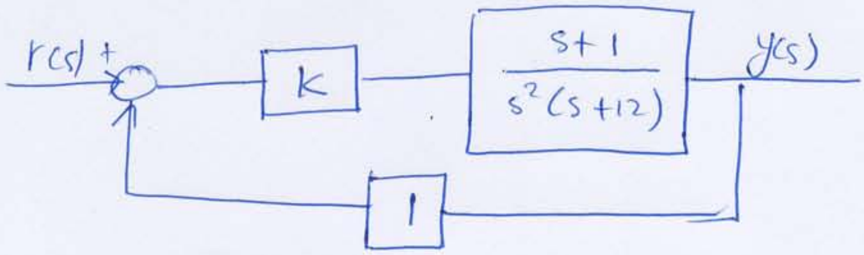
where,

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1};\quad b_2 = \frac{a_1a_4 - a_0 \cdot 0}{a_1}$$

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1};\quad c_2 = \frac{b_1 \cdot 0 - a_1 \cdot 0}{b_1} = 0$$

$$d_1 = \frac{c_1b_2 - b_1c_2}{c_1}$$

Prob. #4. Draw the root locus for the following system



Closed-loop transfer function

$$T(s) = \frac{k\,G(s)}{1 + k\,G(s)}, \quad \text{where} \quad G(s) = \frac{s+1}{s^2(s+12)}$$

- Poles: $(k=0)$ $P_1 = 0$; $P_2 = 0$; $P_3 = -12$
- Zero: $(k=\infty)$ $Z = -1$

- Number of branches: n = order of denominator - order of numerator

$$n = 3 - 1 = 2$$

- Asymptotes:

$$\sigma_A = \frac{0 + 0 - 12 - (-1)}{2} = -\frac{11}{2} = -5{,}5$$

$$\phi_A = \frac{(2k+1)\pi}{2} = \begin{cases} \frac{\pi}{2} & (k=0) \\ \frac{3\pi}{2} & (k=1) \end{cases}$$

- Departure angle from the poles (origin) is given by

$$0 - (\phi_d + \phi_d + 0) = 180 + \ell . 2\pi$$

$$\therefore \ \phi_d = \pm 90^\circ$$

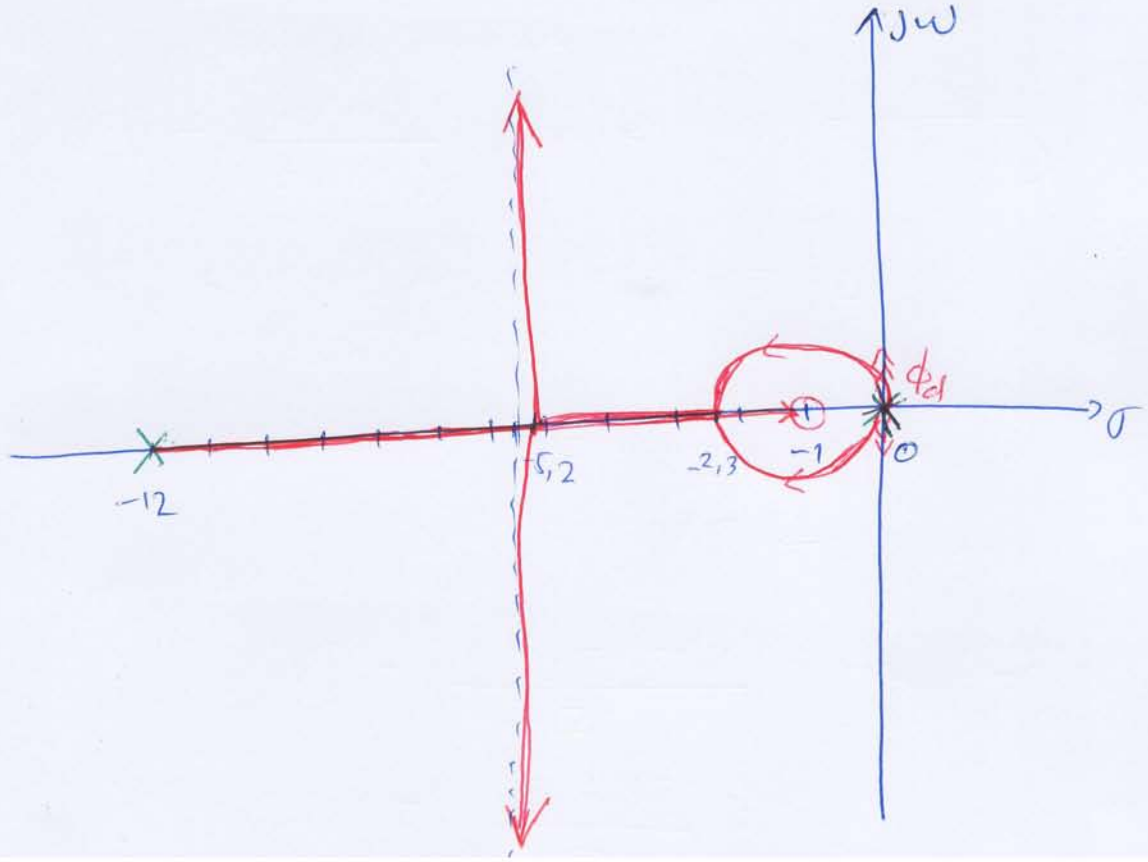
- Break-in (out) points

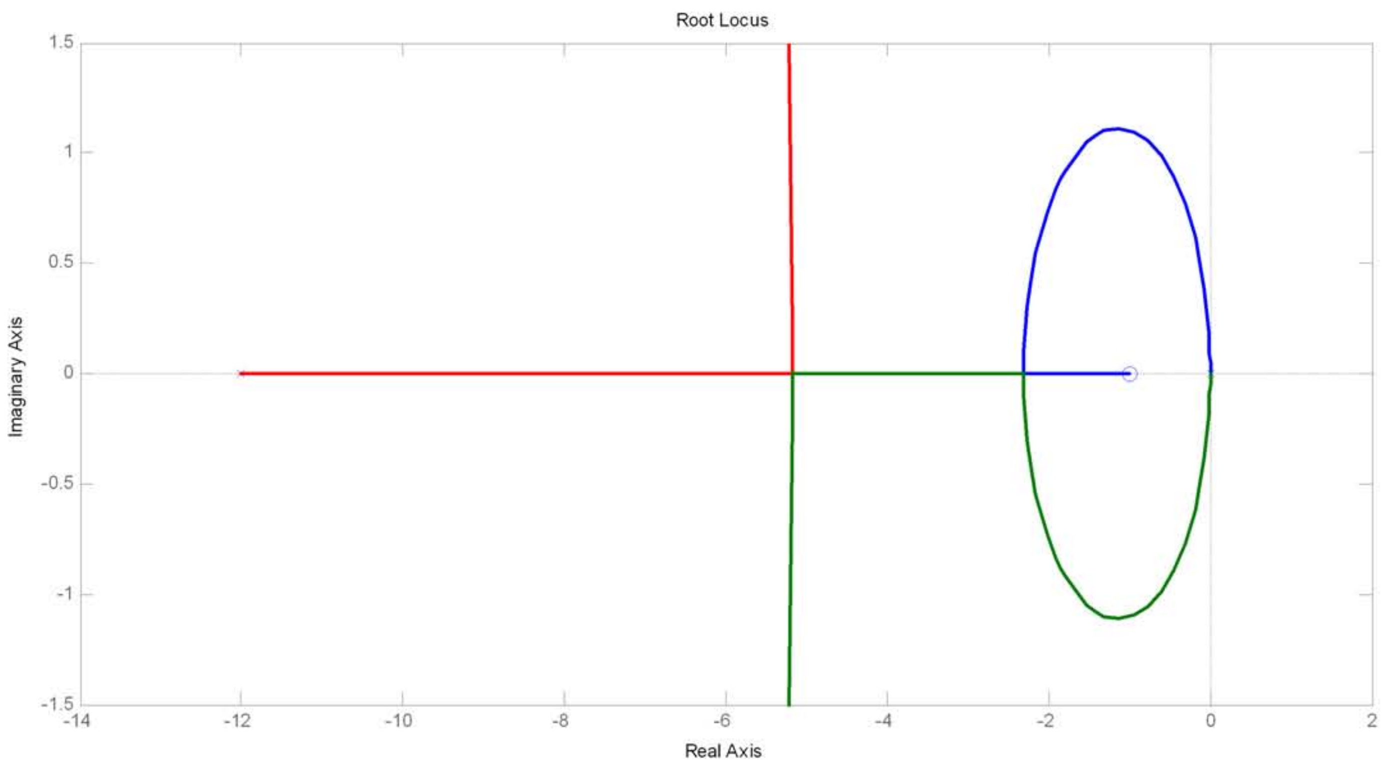
$$1 + kG(s) = 0 \Rightarrow 1 + k\frac{s+1}{s^2(s+12)} = 0$$

$$K = -\frac{s^3+12s^2}{s+1} \Rightarrow \frac{dK}{ds} = -\frac{(3s^2+24s)(s+1)-(s^3+12s^2)}{(s+1)^2} = 0$$

$\Rightarrow s = 0, -5{,}2, -2{,}3$

- $j\omega$ crossings: There is no crossing point at $j\omega$ axis





prob. #5. Draw the Bode plot for

$$G(s) = \frac{2000(s+0.2)^2}{s^2(s+5)(s+20)}$$

i) G(s) can be rewritten as

$$G(s) = \frac{2000 \times 0.2 \times 0.2}{5 \times 20} \frac{\left(1+\frac{s}{0.2}\right)^2}{s^2\left(1+\frac{s}{5}\right)\left(1+\frac{s}{20}\right)}$$

$$= 0.8 \frac{\left(1+\frac{s}{0.2}\right)^2}{s^2\left(1+\frac{s}{5}\right)\left(1+\frac{s}{20}\right)}$$

$\Rightarrow$

$$20\log|G(s)| = \underset{(1)}{20\log 0.8} + \underset{(2)}{40\log\left|\left(1+\frac{s}{0.2}\right)\right|} - \underset{(3)}{40\log|s|}$$

$$- \underset{(4)}{20\log\left|\left(1+\frac{s}{5}\right)\right|} - \underset{(5)}{20\log\left|\left(1+\frac{s}{20}\right)\right|}$$

The Bode plots for both magnitude and phase are depicted as Fig. 5

ii) From the Bode plots, the gain margin (GM) and the phase margin (PM) are

$GM = \infty$

$PM = 30°$

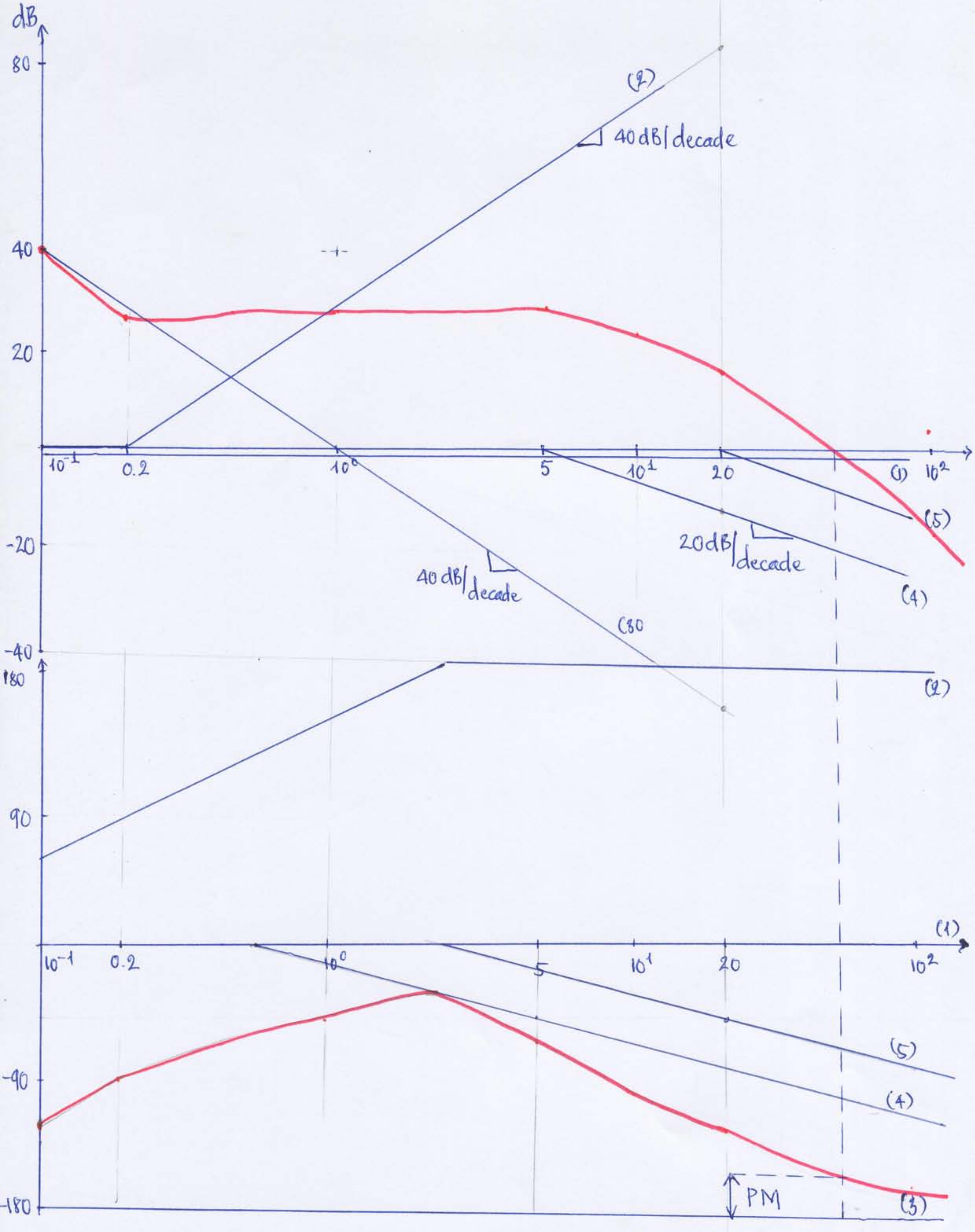
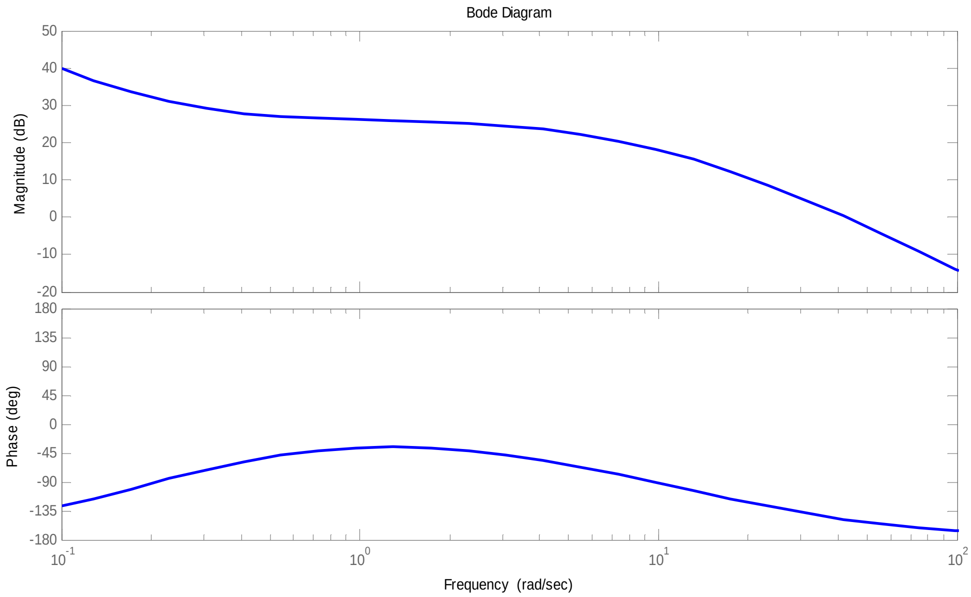
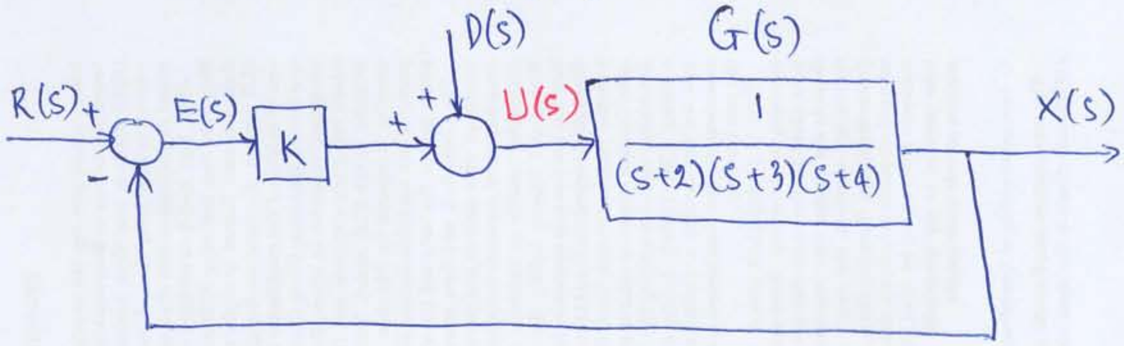

dB
80
40
20
-20
-40
180
90
-90
-180
10^{-1}
0.2
10^0
5
10^1
20
10^2
(2)
40dB/decade
40 dB/decade
20dB/decade
(1)
(5)
(4)
PM
(3)

Fig. 5


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
50
40
30
20
10
0
-10
-20
180
135
90
45
0
-45
-90
-135
-180
10^-1
10^0
10^1
10^2

prob. # 6. Consider the following system



a) Write the differential eq. that relates r(t) and d(t) to x(t)

Transfer function

$$\frac{X(s)}{U(s)} = G(s) \text{ , where } U(s) = D(s) + K[R(s) - X(s)]$$

$$\Rightarrow \frac{X(s)}{D(s) + K[R(s) - X(s)]} = G(s) = \frac{1}{(s+2)(s+3)(s+4)}$$

$$\Rightarrow X(s)[s^3 + 9s^2 + 26s + (24+K)] = KR(s) + D(s)$$

$$\therefore \dddot{x}(t) + 9\ddot{x}(t) + 26\dot{x}(t) + (24+K)x(t) = Kr(t) + d(t)$$

(b), (c) Steady state error

The transform of the output

$$X(s) = E(s)KG(s) + D(s)G(s) \text{ , where and}$$

$$X(s) = R(s) - E(s)$$

The Transform of the error

$$E(s) = \frac{1}{1+KG(s)}R(s) - \frac{G(s)}{1+KG(s)}D(s)$$

Apply the final value theorem

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s}{1+KG(s)}R(s) - \lim_{s\to 0}\frac{sG(s)}{1+KG(s)}D(s)$$

$$= e_r(\infty) + e_d(\infty)$$

where

$$e_r(\infty) = \lim_{s \to 0} \frac{s}{1 + KG(s)} R(s) \qquad (*)$$

$$e_d(\infty) = -\lim_{s \to 0} \frac{sG(s)}{1 + KG(s)} D(s) = \cancel{\frac{1}{\lim_{s\to 0}\frac{1}{G(s)} + \lim_{s \to 0} K}} \qquad (**)$$

b) Steady state error in responding to a constant command signal

$$r(t) = A \Rightarrow R(s) = \frac{A}{s}$$

Apply $(*)$ $\Rightarrow$

$$e_r(\infty) = \lim_{s \to 0} \frac{A}{1 + K\frac{1}{(s+2)(s+3)(s+4)}} = \frac{A}{1 + \frac{K}{24}} = \frac{24A}{24 + K}$$

In order to $e_r(\infty)$ less than 5% of the command signal, it requires that

$$\frac{24A}{24+K} < 0.05A \Rightarrow K > 456$$

c) Steady state error due to a constant disturbance

$$d(t) = B \Rightarrow D(s) = \frac{B}{s}$$

Apply $(**)$

$$e_d(\infty) = -\lim_{s \to 0} \frac{sG(s)}{1 + KG(s)} \frac{B}{s} = -\cancel{1}\frac{B}{\lim_{s \to 0}\frac{1}{G(s)} + K}$$

$$= -\frac{B}{\lim_{s \to 0} \frac{1}{\frac{1}{(s+2)(s+3)(s+4)}} + K} = \frac{B}{24 + K}$$

In order to $e_d(\infty)$ is less than 1% of B, it yields

$$\frac{B}{24+K} < 0.01B \Rightarrow K > 76$$

d) Range of K that produce stability for the system

The charateristic equation

$$s^3 + 9s^2 + 26s + 24 + K = 0$$

Routh table

$$\begin{array}{lll} s^3 & 1 & 26 \\ s^2 & 9 & 24+K \\ s^1 & \frac{9\times 26-(24+K)}{9} & 0 \\ s^0 & 24+K & \end{array}$$

Requirement for stability

$$\frac{9\times 26-(24+K)}{9} > 0 \Rightarrow K < 210$$

$$24 + K > 0 \Rightarrow K > -24$$

$$\Rightarrow -24 < K < 210$$

It is obvious that (b), (c) and (d) can not be satisfied simultaneously

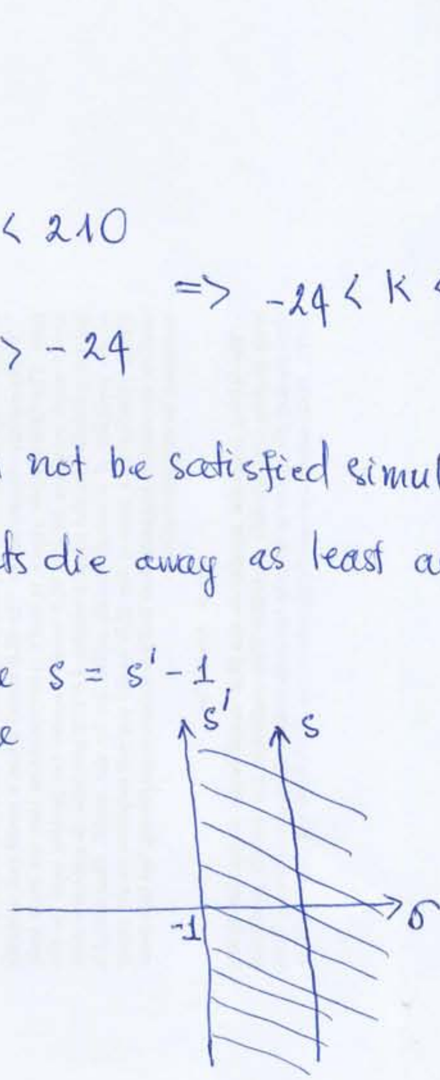
e) Find the range of K that have all transients die away as least as fast as e^{-t}

- Change the coordinate in s-plane become $s = s' - 1$
- All the poles lie in the left-half of s'-plane

The characteristic eq. becomes

$$(s'-1)^3 + 9(s'-1)^2 + 26(s'-1) + 24 + K = 0$$

$$\Rightarrow s'^3 + 6s'^2 + 11s' + 6 + K = 0$$



Routh table

s'^3	1	11
s'^2	6	$6+K$
s'^1	$\frac{6\times 11-(6+K)}{6}$	0
s'^0	$6+K$	

Requirement for stability

$$\frac{6\times 11-(6+K)}{6} > 0 \Rightarrow K < 60$$

$$6+K > 0 \Rightarrow K > -6$$

Hence, $-6 < K < 60$

f) For maximum value of K $(K=60)$, the characteristic eq. is

$$s'^3 + 6s'^2 + 11s' + 66 = 0$$

Routh table

s'^3	1	11
s'^2	6	66
s'^1	0	0
s'^0	0	

Auxiliary eq.

$$6s^2 + 66 = 0 \quad \text{or}$$

$$s^2 + 11 = 0$$

Hence, the characteristic Eq. is rewritten as

$$(s'^2+11)(s'+6) = 0 \Rightarrow \begin{cases} s'_1 = -6 \\ s'_2 = j\sqrt{11} \\ s'_3 = -j\sqrt{11} \end{cases}$$

- For $K = 60$, $r(t) = t$, $d(t) = 1$, the differential eq. of (a) becomes

$$\dddot{x} + 9\ddot{x} + 26\dot{x} + 84x = 60t + 1$$

The general solution is

$$x(t) = x_1(t) + x_2(t)$$

where $x_1(t)$ is the general solution of the homogeneous eq., and $x_2(t)$ is the particular solution of the nonhomogeneous eq.

- Homogeneous eq.

$$\dddot{x} + 9\ddot{x} + 26\dot{x} + 84x = 0$$

The roots of the characteristic eq. are -7, and $-1 \pm j\sqrt{11}$

$$\therefore \; x_1(t) = Ae^{-7t} + Be^{-t}\cos(\sqrt{11}t) + Ce^{-t}\sin(\sqrt{11}t)$$

- It is obvious that the particular solution of the nonhomogeneous eq. has the form as

$$x_2(t) = \alpha t + \beta$$

By substituting $x_2(t)$ into the nonhomogeneous eq., we obtaine

$$\alpha = \frac{5}{7}; \; \beta = \frac{41}{196}$$

Finally, the ~~re~~ general solution of the nonhomogeneous eq. is

$$x(t) = \frac{5}{7}t + Ae^{-7t} + Be^{-t}\cos(\sqrt{11}t) + Ce^{-t}\sin(\sqrt{11}t) + \frac{41}{196}$$