Prob. #1 (30 points)

1) The nationality of E. J. Routh is English (5 points)

2) - Laplace transform of $g(t)$

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt$$ (2 points)

- Inverse Laplace transform of $F(s)$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} e^{st} F(s)\,ds$$ (3 points)

3) The linearity of $\Phi(z)$, $z \in \mathbb{R}^n$

The function $\Phi(z)$ is said to be linear iff (5 points)

$$\Phi(\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1 \Phi(z_1) + \alpha_2 \Phi(z_2)$$

$\forall\ z_1, z_2 \in \mathbb{R}^n;\ \alpha_1, \alpha_2 \in \mathbb{R}$

4) For a given linear mapping Ω, if there exists a nonzero vector x s.t.

$$\Omega x = \lambda x, \quad \lambda \in \mathbb{C}$$ (5 points)

λ is called an eigen value of Ω, and x is called the eigen vector associated with λ.

5)
$$F(s) = \frac{10 e^{-2s}}{(s+1)(s+2)} = \left(\frac{A}{s+1} + \frac{B}{s+2}\right)e^{-2s}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 2A + B = 10 \end{cases} \Rightarrow \begin{cases} A = 10 \\ B = -10 \end{cases}$$ (5 points)

$$\Rightarrow F(s) = \left(\frac{10}{s+1} - \frac{10}{s+2}\right)e^{-2s}$$

6) Ch Cauchy integral formular

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z - z_0}$$ (5 points)

prob. #2. Given a system
(15 points)

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t) \quad (2.1)$$

a) Find the transfer function

Taking the Laplace transform both sides of (2.1)

$$s^2 Y(s) + 3s Y(s) + 2Y(s) = U(s)$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2} \quad \text{(5 points)}$$

b) $u(t) = \delta(t)$, find $y(0)$

$$\mathcal{L}\{u(t)\} = 1$$

$$Y(s) = T(s)U(s) = \frac{1}{s^2+3s+2} \quad \text{(2 points)}$$

$$y(0) = \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} \frac{s}{s^2+3s+2} = 0 \quad \text{(3 points)}$$

(Using the initial value theorem)

c) $u(t) = 2(t)$, find $y(\infty)$

$$\mathcal{L}\{u(t)\} = \frac{2}{s}$$

$$Y(s) = T(s)U(s) = \frac{2}{s(s^2+3s+2)} \quad \text{(2 points)}$$

Using the final value theorem

$$y(\infty) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s\frac{2}{s(s^2+3s+2)} = 1 \quad \text{(3 points)}$$

prob. 3 (15 points) Given a system

$$\dddot{y} + 3\ddot{y} + 2\dot{y} + y = 4\dddot{u} + \ddot{u} + 5\dot{u}$$

Find the state variable form!

Consider

$$\dddot{y} + 3\ddot{y} + 2\dot{y} + y = u \qquad (3.1)$$

Define:

$$\left.\begin{array}{l} x_1 = y \\ x_2 = \dot{y} \\ x_3 = \ddot{y} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -x_1 - 2x_2 - 3x_3 + u \end{array}\right.$$

(5 points)

Since the solution to u is y (3.1), the solution to $4\dddot{u} + \ddot{u} + 5\dot{u}$ will be

$$4\dddot{y} + \ddot{y} + 5\dot{y} = 4(-3\ddot{y} - 2\dot{y} - y + u) + \ddot{y} + 5\dot{y}$$

$$= -4y - 3\dot{y} - 11\ddot{y} + 4u$$

(8 points)

$$= -4x_1 - 3x_2 - 11x_3 + 4u$$

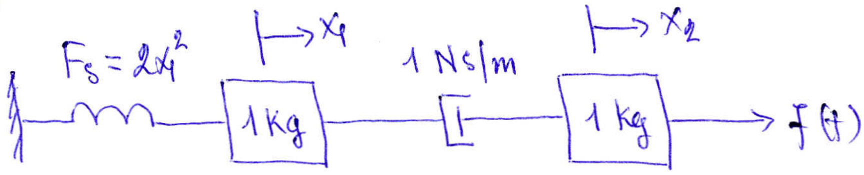
Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

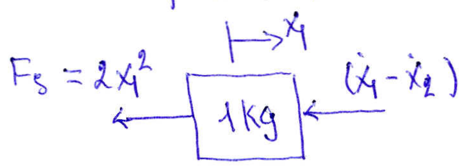
(2 points)

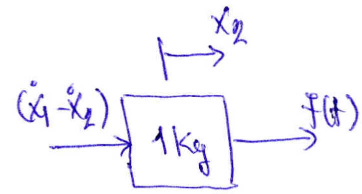
$$y = \begin{bmatrix} -4 & -3 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 4u$$

Prob. #4 (Method #1)
(15 points)



- Equations of motion





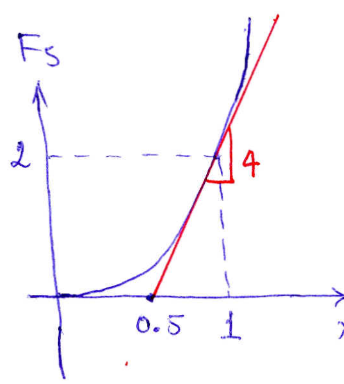
(3 points)

$$\ddot{x}_1 + (\dot{x}_1 - \dot{x}_2) + F_s(x_1) = 0 \qquad \ddot{x}_2 + (\dot{x}_2 - \dot{x}_1) = f(t)$$

- Linearization of the system at $x_1(t) = 1$

As shown in Figure, around $x_1 = 1$, the nonlinear spring force $F_s = 2x_1^2$ can be approximated by

$$F_s \approx -2 + 4x_1$$



(5 points)

Thus, the linearized equations of motion

$$\left.\begin{aligned} \ddot{x}_1 + (\dot{x}_1 - \dot{x}_2) + 4x_1 - 2 &= 0 \\ \ddot{x}_2 + (\dot{x}_2 - \dot{x}_1) &= f(t) \end{aligned}\right\} \text{(2 points)}$$

Define: $\left.\begin{aligned} x_3 &= \dot{x}_1 \\ \dot{x}_4 &= x_2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -x_3 + x_4 - 4x_1 + 2 \\ \dot{x}_4 &= x_3 - x_4 + f(t) \end{aligned}\right.$ (3 points)

The measurement ~~sys~~ signal y is $x_1(t) + x_2(t) \Rightarrow$

$$y = x_1 + x_2$$

In vector-matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ f(t) \end{bmatrix}$$

(2 points)

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(Method #2)

- Equations of motion

$$\ddot{x}_1 + (\dot{x}_1 - \dot{x}_2) + 2x_1^2 = 0 \qquad (1)$$

$$\ddot{x}_2 + (\dot{x}_2 - \dot{x}_1) = f(t) \qquad (2)$$

(3 points)

Let $(x_{10}, x_{20}) = (1, x_{20})$ be the operating point. It is necessary to find the force, f_0 that balances the system at (x_{10}, x_{20}).

Indeed, (1)+(2) becomes

$$\ddot{x}_1 + \ddot{x}_2 + 2x_1^2 = f(t) \qquad (3)$$

At the operating point (x_{10}, x_{20}), we have

$$2x_{10}^2 = f_0 \Rightarrow f_0 = 2\,(N)$$

(3 points)

Now, we define

$$\left.\begin{aligned} x_3 &= \dot{x}_1 \\ x_4 &= \dot{x}_2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -x_3 + x_4 - 2x_1^2 \\ \dot{x}_4 &= x_3 - x_4 + f(t) \end{aligned}\right. \qquad (4)$$

(3 points)

(4) can be expressed in the matrix form as

$$\dot{X} = F(X, f(t)) \qquad (5)$$

- Linearized system around the operating point $(x_{10}, x_{20}, x_{30}, x_{40}, f_0)$ has a form as

$$\delta \dot{X} = F(X_0, f_0) + \left.\frac{\partial F}{\partial X}\right|_{(X_0, f_0)} \delta X + \left.\frac{\partial F}{\partial f}\right|_{(X_0, f_0)} \delta f \qquad (6)$$

$$= \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \delta X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \delta f$$

(6 points)

$$\delta y = \delta x_1 + \delta x_2 = [1 \quad 1 \quad 0 \quad 0]\,\delta X \qquad (7)$$

(Method # 3)

- Equations of motion

$$\ddot{x}_1 + (\dot{x}_1 - \dot{x}_2) + 2x_1^2 = 0$$
$$\ddot{x}_2 + (\dot{x}_2 - \dot{x}_1) = f(t)$$

(3 points)

Define: $\left.\begin{array}{l} x_3 = \dot{x}_1 \\ x_4 = \dot{x}_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = -x_3 + x_4 - 2x_1^2 \\ \dot{x}_4 = x_3 - x_4 + f(t) \end{array}\right.$ (3 points)

- Linearization of the system around $x_1 = 1$

Let $x_1 = 1 + \delta x_1$, and $\dot{x}_1 = \delta\dot{x}_1$

$$x_1^2 = x_1^2\Big|_{x_1=1} + \frac{\partial(x_1^2)}{\partial x_1}\Big|_{x_1=1}\delta x_1 = 1 + 2\delta x_1$$

(3 points)

Therefore, the state and output equations are

$$\delta\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = -x_3 + x_4 - 2(1 + 2\delta x_1)$$
$$\dot{x}_4 = x_3 - x_4 + f(t)$$
$$y = x_1 + x_2 = 1 + \delta x_1 + x_2$$

(2 points)

In vector-matrix form

$$\begin{bmatrix} \delta\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ f(t) \end{bmatrix}$$

(2 points)

$$y = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ f(t) \end{bmatrix}$$

where $f(t) = 2 + \delta f(t)$, since force in nonlinear spring is 2 N and must be balanced by 2 N force on damper. (2 points)

prob. #5 Given a system
(25 points)

$$\ddot{x} + 3\dot{x} + 2x = 5e^{-2t} + t, \quad x(0) = 2, \ \dot{x}(0) = 1$$

1) The output signal is $\dot{x}$. Obtain the state equation

Define: $u(t) = 5e^{-2t} + t$ —— (3 points)

$$\left.\begin{matrix} x_1 = x \\ x_2 = \dot{x} \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 3x_2 + u \end{matrix}\right\} \quad \text{(2 points)}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Bu$$

(5 points)

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2) Get the output signal y?

Taking the Laplace transform the above state equation

$$\left\{\begin{matrix} sX(s) - X(0) = AX(s) + BU(s) \\ Y(s) = CX(s) \end{matrix}\right. \Rightarrow$$

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0)$$

$$Y(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}x(0)$$

where $X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (2 points)

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$ (2 points)

$$C(sI - A)^{-1}B = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{s}{s^2 + 3s + 2}$$ (3 points)

$$C(sI-A)^{-1}x(0) = \frac{1}{s^2+3s+2}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}$$

$$= \frac{s-4}{s^2+3s+2}$$ (3 points)

$$U(s) = \mathcal{L}\left\{5e^{-2t}+t\right\} = \frac{5}{s+2} + \frac{1}{s^2}$$ (2 points)

$\Rightarrow$

$$Y(s) = \frac{s}{s^2+3s+2}\left[\frac{5}{s+2} + \frac{1}{s^2}\right] + \frac{s-4}{s^2+3s+2}$$

$$= \underbrace{\frac{5s}{(s+1)(s+2)^2}}_{Y_1(s)} + \underbrace{\frac{1}{s(s+1)(s+2)}}_{Y_2(s)} + \underbrace{\frac{s-4}{(s+1)(s+2)}}_{Y_3(s)}$$

$$Y_1(s) = \frac{A_1}{(s+2)^2} + \frac{A_2}{(s+2)} + \frac{A_3}{s+1} = \frac{10}{(s+2)^2} + \frac{5}{s+2} - \frac{5}{s+1}$$

$$Y_2(s) = \frac{B_1}{s} + \frac{B_2}{s+1} + \frac{B_3}{s+2} = \frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2}$$

$$Y_3(s) = \frac{C_1}{s+1} + \frac{C_2}{s+2} = \frac{-5}{s+1} + \frac{6}{s+2}$$

$$Y(s) = Y_1(s) + Y_2(s) + Y_3(s)$$

$$= \frac{10}{(s+2)^2} + \frac{0.5}{s} - \frac{11}{s+1} + \frac{11.5}{s+2}$$

$$y(t) = \dot{x}(t) = 10te^{-2t} + 0.5 - 11e^{-t} + 11.5e^{-2t}$$ (3 points)