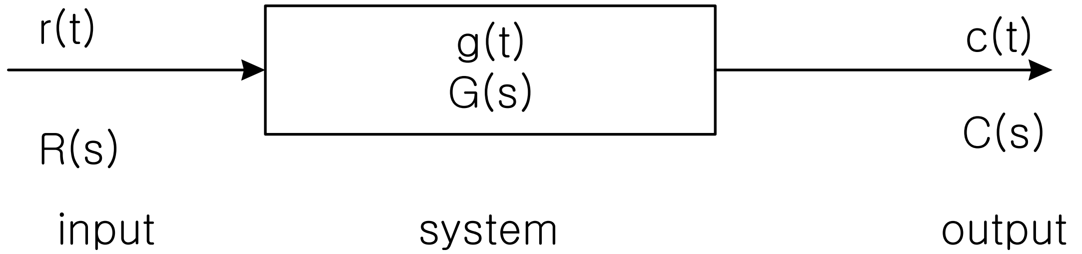
* Frequency Response Analysis

Recall : System representation by a block diagram



where G(s) is a transfer function

The Frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal. The sinusoid is a unique input signal and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs from the input waveform only in amplitude and phase angle.

If $r(t) = R_m \sin \omega t$ .... (1)

$then\ c(t) = C_m \sin(\omega t + \phi)$ .... (2)

We can also write the system output as

$$C(s) = G(s)R(s)$$

For the steady-state response for sinewaves,

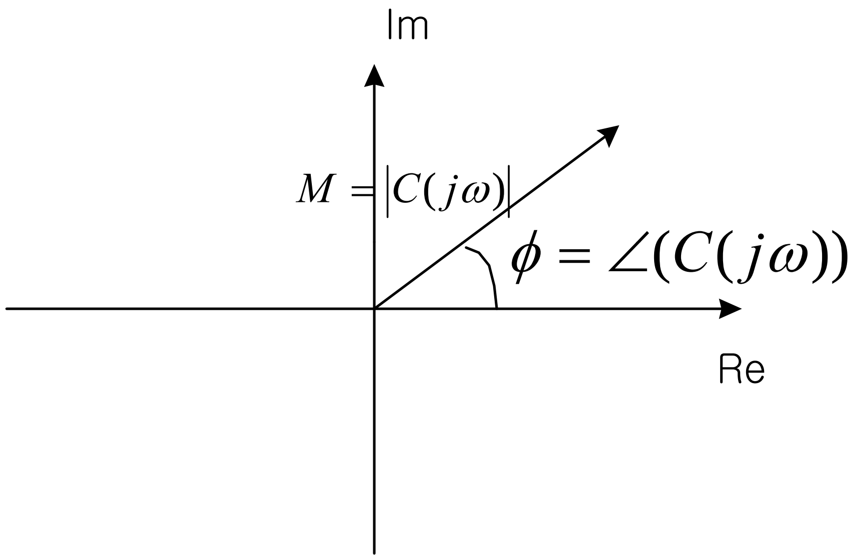
By replacing s with $j\omega$,

$$C(j\omega) = G(j\omega)R(j\omega)$$

The phasor form of $C(j\omega)$ is

$$C(j\omega) = Me^{j\phi} = M\angle\phi$$

$$= |C(j\omega)| \angle(C(j\omega))$$



$$Then \quad |C(j\omega)| = |G(j\omega)||R(j\omega)|$$

$$and \quad \angle(C(j\omega)) = \angle(G(j\omega)) + \angle(R(j\omega))$$

For Eqs (1) and (2)

$$\mathrm{M} = |C(j\omega)| = |R_m||G(j\omega)|$$

$$and \quad \phi = \angle(C(j\omega)) = \angle(G(j\omega)) + 0^\circ = \angle(G(j\omega))$$

The transfer function $G(j\omega)$ describes the system and is the most important parameter.

<u>*Polar Plot*</u> is a plot of the real versus the imaginary part of $G(j\omega)$

$$or \; G(j\omega) = |G(j\omega)|\angle\phi(\omega)$$

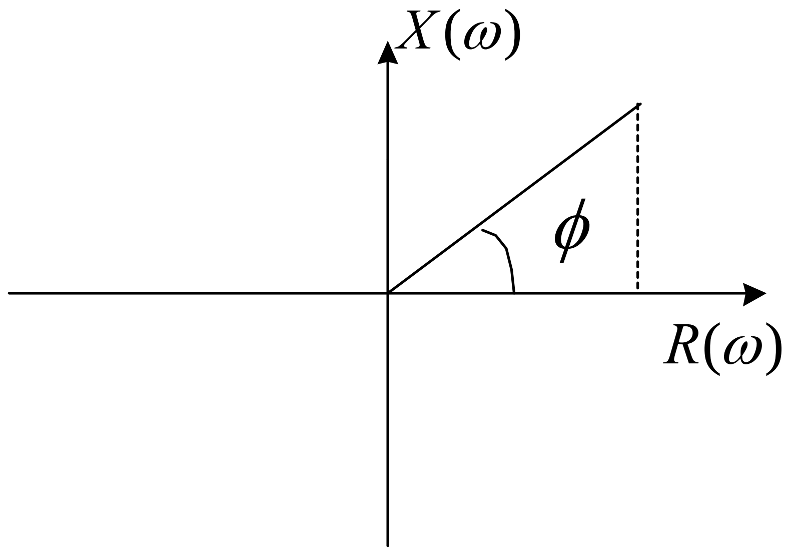
$$G(j\omega) = R(j\omega) + jX(j\omega)$$

$$where \; R(j\omega) = \mathrm{Re}[G(j\omega)]$$

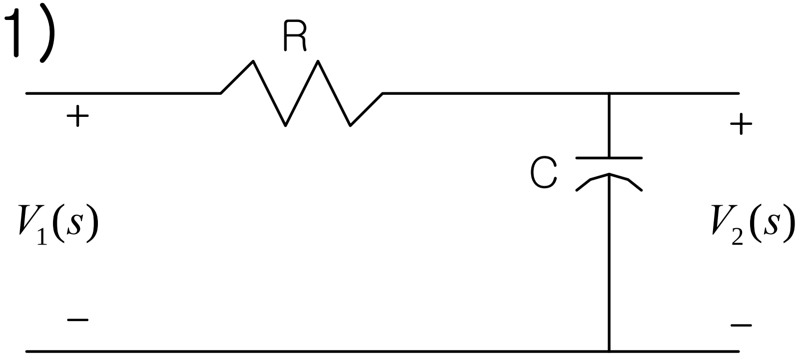
$$and \quad X(j\omega) = \mathrm{Im}[G(j\omega)]$$

$$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)}$$

$$and \; |G(j\omega)| = \sqrt{R^2(\omega) + X^2(\omega)}$$



Ex1)



$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC}$$

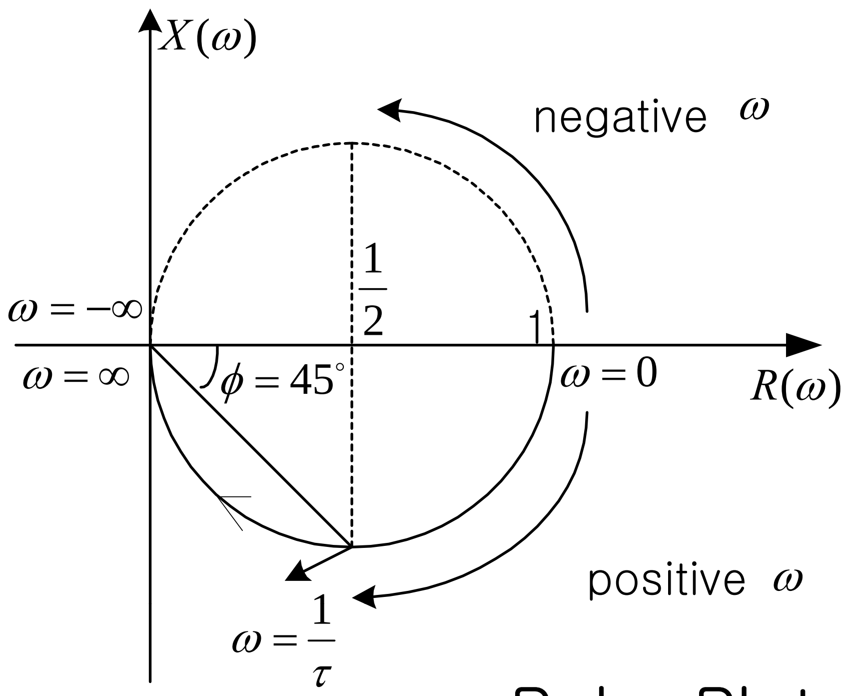
The sinusoidal steady-state transfer function is

$$G(j\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega\tau} = \frac{1 - j\omega\tau}{1 + \omega^2\tau^2} = \frac{1}{1 + \omega^2\tau^2} + j\frac{(-\omega\tau)}{1 + \omega^2\tau^2} = R(\omega) + jX(\omega)$$

$$|G(j\omega)| = \frac{1}{\sqrt{1 + (\omega\tau)^2}} = \left(1 + (\omega\tau)^2\right)^{-\frac{1}{2}} \text{..............(3)}$$

$$\angle\phi(\omega) = -\tan^{-1}(\omega RC) \text{ ....(4)} \quad when \quad \omega = \frac{1}{\tau} = \frac{1}{RC}$$

$$= -\tan^{-1}(\omega\tau)$$



$X(\omega)$
negative ω
$\frac{1}{2}$
$\omega = -\infty$
1
$\omega = \infty$
$\phi = 45^\circ$
$\omega = 0$
$R(\omega)$
positive ω
$\omega = \frac{1}{\tau}$

Polar Plot

* Bode Plot (Bode diagram or Bode chart)

Bode plots are plots of the magnitude, in dB, and the phase of a linear system's frequency response against frequency

For example.

Voltage gain(or transfer function) is

$$20\log_{10} Me^{j\phi} = \underbrace{20\log_{10} M}_{\text{Magnitude (dB)}} + \underbrace{j\phi\log_{10} e}_{\text{Phase(dB)}}$$

$$< note > \ power\ gain = 10\log_{10} Pe^{j\phi_p}$$

Ex2) with Ex1) plot Bode diagram

From Eq.(3),

the logarithmic gain is

$$20\log|G| = 20\log\left(1+(\omega\tau)^2\right)^{-\frac{1}{2}} = -10\log(1+(\omega\tau)^2)$$

For low frequencies $\left(\omega << \dfrac{1}{\tau}\right)$,

$$20\log|G| = -10\log(1) = 0dB,\ \ \omega << \frac{1}{\tau}$$

$$at\ \ \omega = \frac{1}{\tau}\ \text{(corner frequency)} = \text{-}\,3.01\,\text{dB}$$

$$\phi(j\omega) = -\tan^{-1}\omega\tau$$

Ex 3) $\left|G(j\omega)\right| = \dfrac{1}{\sqrt{1+(\omega\tau)^2}} = \left(1+(\omega\tau)^2\right)^{-\frac{1}{2}}$

$$20\log\left|G\right| = -10\log\left(1+(\omega\tau)^2\right)$$

$(a)\ \omega << \dfrac{1}{\tau}$ (For low frequencies)

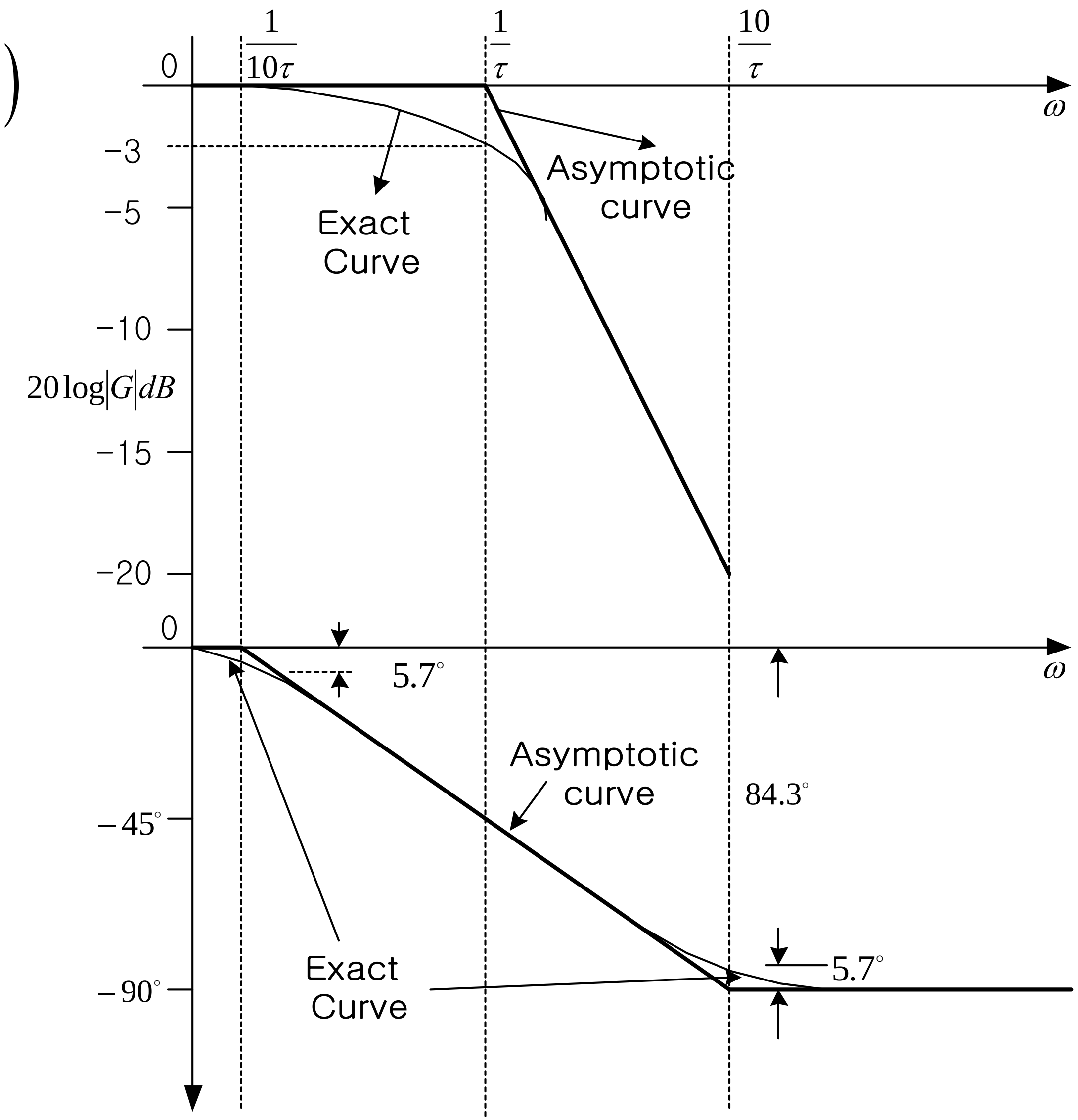
$$20\log\left|G\right| = -10\log(1) = 0\,dB$$

$(b)\ \omega = \dfrac{1}{\tau}$

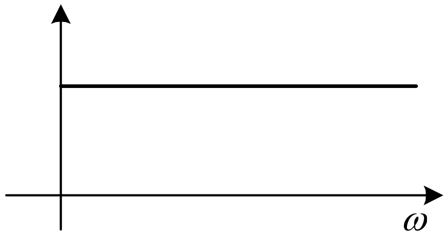
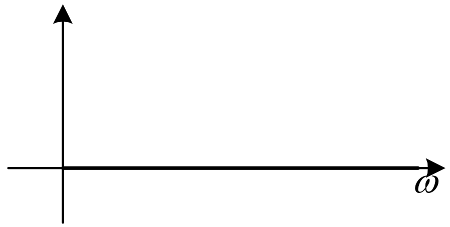
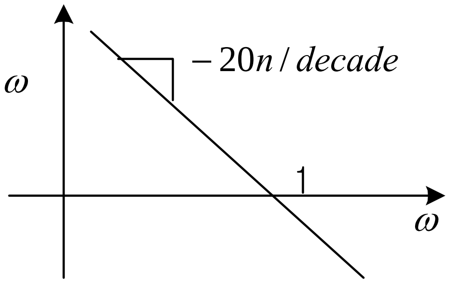
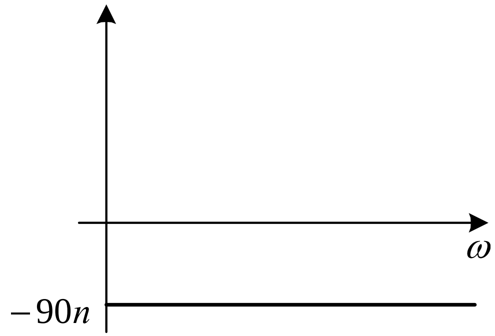
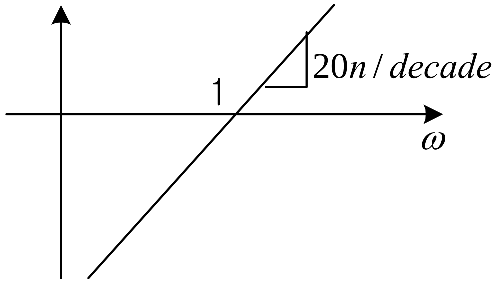
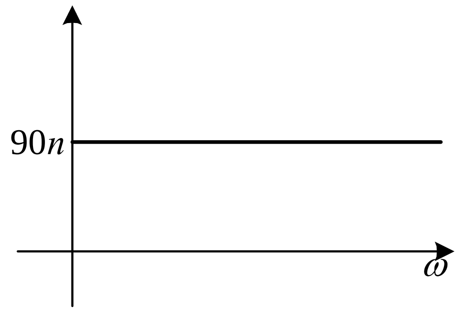
$$20\log\left|G\right| = -20\log(2) = -3.01\,dB$$

$(c)\ \omega >> \dfrac{1}{\tau}$ (For high frequencies)

$$20\log\left|G\right| = -20\log(\omega\tau)$$
$$= -20\log\omega - 20\log\tau$$

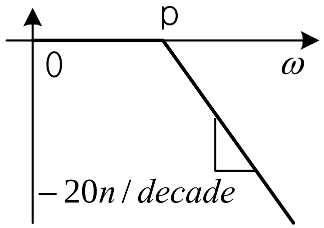


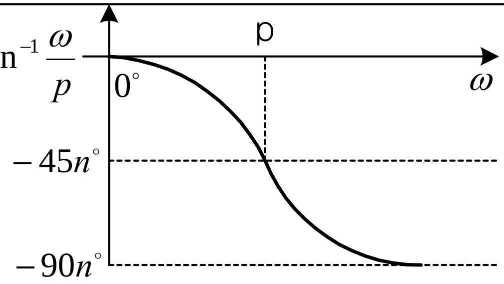
Asymtotic Properties

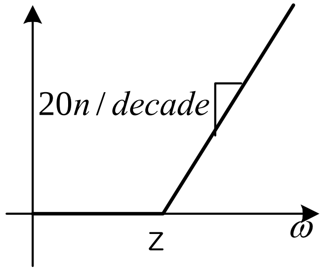
term	Magnitude , dB	Phase , degrees
1. constant gain K	$20\log K$ 	0° 
2. $pole\ at\ origin\,(j\omega)^{-n}$	$-20n\log\omega$ 	$-90n^{\circ}$ 
3. $Zero\ at\ origin\,(j\omega)^{n}$	$20n\log\omega$ 	$90n^{\circ}$ 

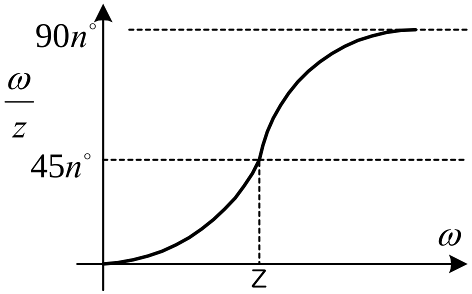
Asymtotic Properties

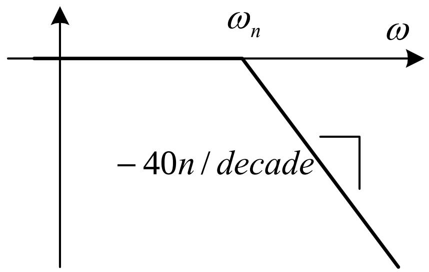
<table>
<tr><th>term</th><th>Magnitude , dB</th><th>Phase , degrees</th></tr>
<tr>
<td>4. Real $poles$ $\left(1+\frac{j\omega}{p}\right)^{-n}$</td>
<td>$-20n\log\sqrt{1+\frac{\omega^2}{p^2}}$


</td>
<td>$-n\tan^{-1}\frac{\omega}{p}$


</td>
</tr>
<tr>
<td>5. Real $zeros$ $\left(1+\frac{j\omega}{z}\right)^{n}$</td>
<td>$20n\log\sqrt{1+\frac{\omega^2}{z^2}}$


</td>
<td>$n\tan^{-1}\frac{\omega}{z}$


</td>
</tr>
<tr>
<td>6. $Complex\ conjugate$ $poles$ $\left[1+j2\varsigma\frac{\omega}{\omega_n}-\left(\frac{\omega}{\omega_n}\right)^2\right]^{-n}$</td>
<td>$-20n\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^{\ 2}}\right)^2+\frac{4\varsigma^2\omega^2}{\omega_n^{\ 2}}}$


</td>
<td>$-n\tan^{-1}\frac{2\varsigma\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^{\ 2}}}$</td>
</tr>
</table>

Problem) An amplifier has the voltage transfer function H(s) :

$$H(s)=\frac{10^8 s(s+100)}{(s+10)(s+10^3)(s+10^4)}$$

a) Determine the poles and zeros.

b) Draw the asymptotic Bode plots for both magnitude and phase

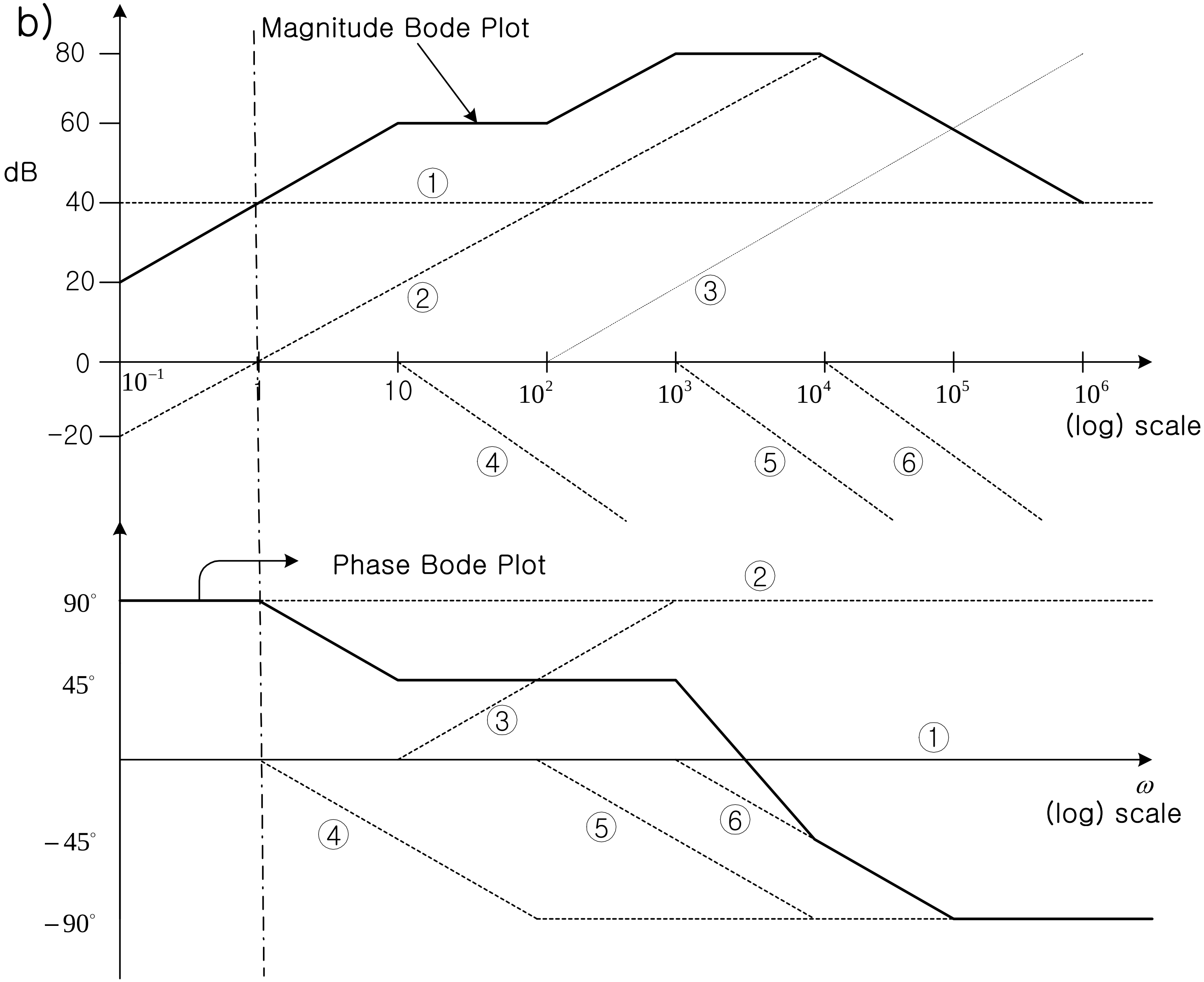
Solution) $$H(s)=\frac{\frac{10^8\times 10^2}{10\times 10^3\times 10^4}s\left(1+\frac{s}{100}\right)}{\left(1+\frac{s}{10}\right)\left(1+\frac{s}{10^3}\right)\left(1+\frac{s}{10^4}\right)}=\frac{100s\left(1+\frac{s}{100}\right)}{\left(1+\frac{s}{10}\right)\left(1+\frac{s}{10^3}\right)\left(1+\frac{s}{10^4}\right)}$$

$$20\log|H(s)|=20\left[\underset{①}{\log 100}+\underset{②}{\log s}+\underset{③}{\log\left(1+\frac{s}{100}\right)}-\underset{④}{\log\left(1+\frac{s}{10}\right)}-\underset{⑤}{\log\left(1+\frac{s}{10^3}\right)}-\underset{⑥}{\log\left(1+\frac{s}{10^4}\right)}\right]$$

$a)$ *Poles at* $s=-10,\ -10^3,\ -10^4$

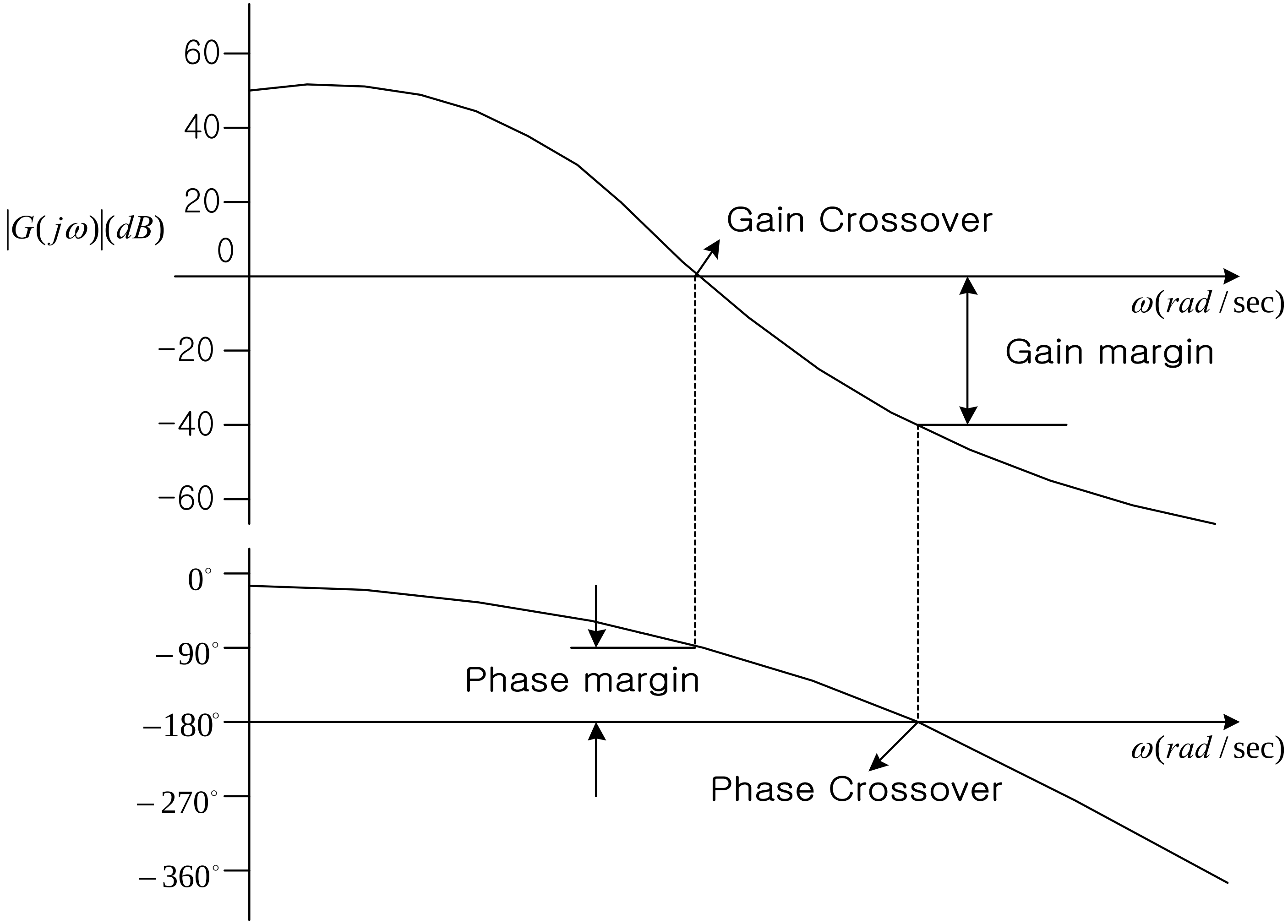
Zeros at $s=0,\ -100$

b)



Magnitude Bode Plot
80
60
dB
40
20
0
−20
10⁻¹
1
10
10²
10³
10⁴
10⁵
10⁶
(log) scale
①
②
③
④
⑤
⑥
Phase Bode Plot
90°
45°
−45°
−90°
ω
(log) scale

* Stability Analysis with the Bode Plot



1. The gain margin is positive and the system is stable if the magnitude of $G(j\omega)$ at the phase crossover is negative in dB. That is, the gain margin is measured below the 0-dB axis. If the gain margin is measured above the 0-dB axis, the gain margin is negative and the system is unstable.

2. The phase margin is positive and the system is stable if the phase of $G(j\omega)$ is greater than -180° at the gain crossover. That is the phase margin is measured above the -180° axis. If the phase margin is measured below the -180° axis, the phase margin is negative, and the system is stable.

10.3 Nyquist Criterion

Derivation of the Nyquist Criterion

Consider the system of Figure 10.20. The Nyquist criterion can tell us how many closed-loop poles are in the right half-plane. Before deriving the criterion, let us establish four important concepts that will be used during the derivation: (1) the relationship between the poles of $1+G(s)H(s)$ and the poles of $G(s)H(s)$; (2) the relationship between the zeros of $1+G(s)H(s)$ and the poles of the closed-loop transfer function, $T(s)$; (3) the concept of mapping points; and (4) the concept of mapping contours.

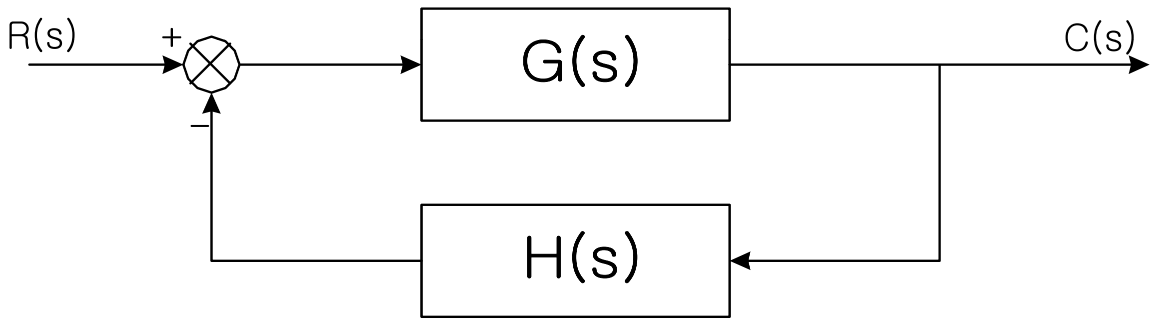


Figure 10.20 Closed-loop control system

Letting $G(s) = \frac{N_G}{D_G}$ (10.37a) , $H(s) = \frac{N_H}{D_H}$ (10.37b)

We find $G(s)H(s) = \frac{N_G N_H}{D_G D_H}$ (10.38a)

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H} \qquad (10.38b)$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G D_H}{D_G D_H + N_G N_H} \qquad (10.38c)$$

From Eqs. (10.38), we conclude that (1) the poles of 1+G(s)H(s) are the same as the poles of G(s)H(s), the open-loop system, and (2) the zeros of 1+G(s)H(s) are the same as the poles of T(s), the closed-loop system.

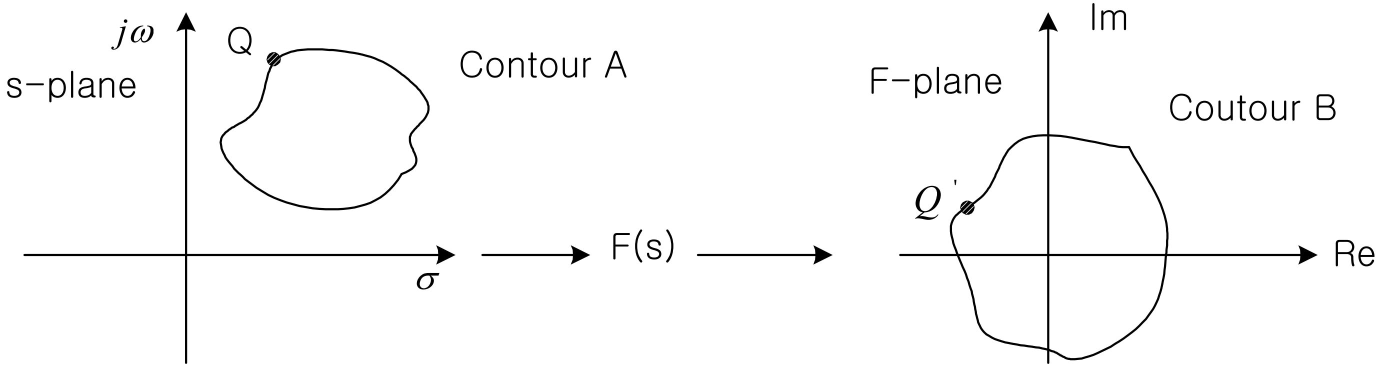
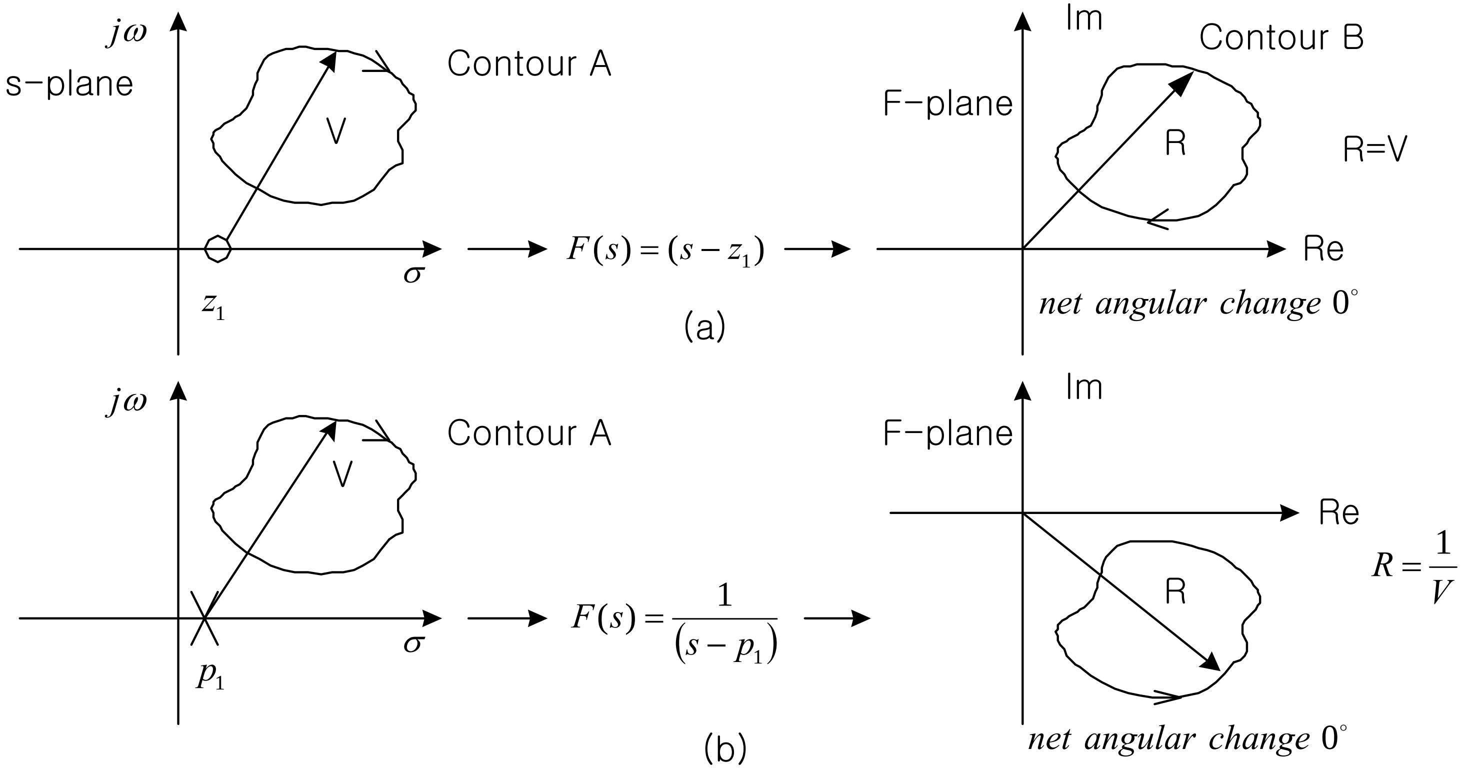
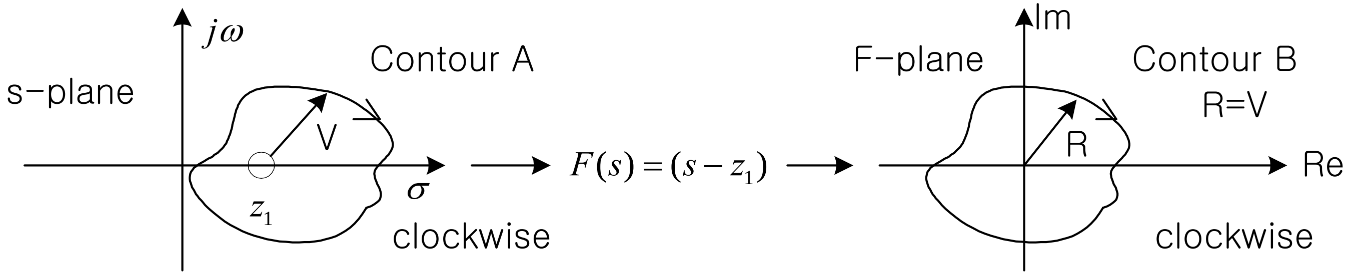
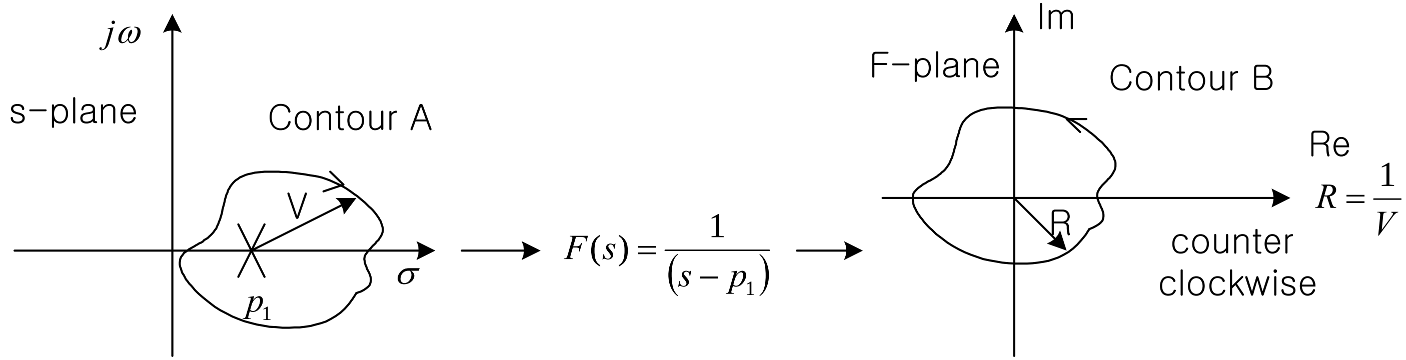


Figure 10.21
Mapping contour A through function F(s) to contour B

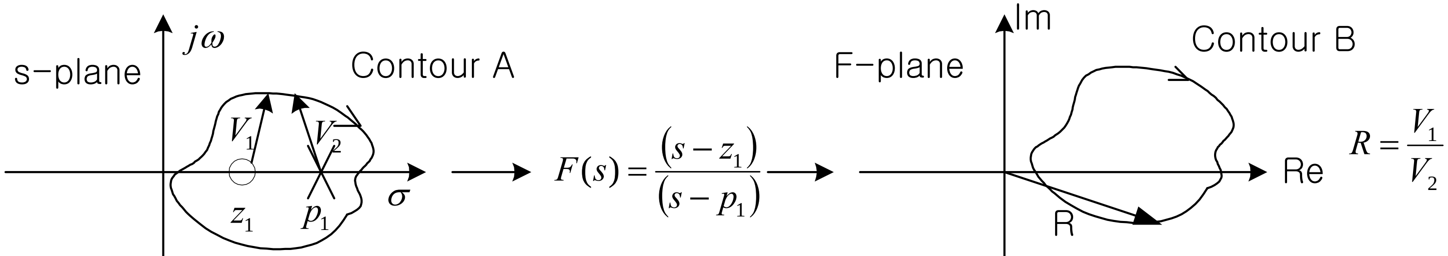




A zero z_1 is enclosed by contour A (c) encircles the origin 360°



A pole p_1 is enclosed by contour A (d) encircles the origin 360°



(e)

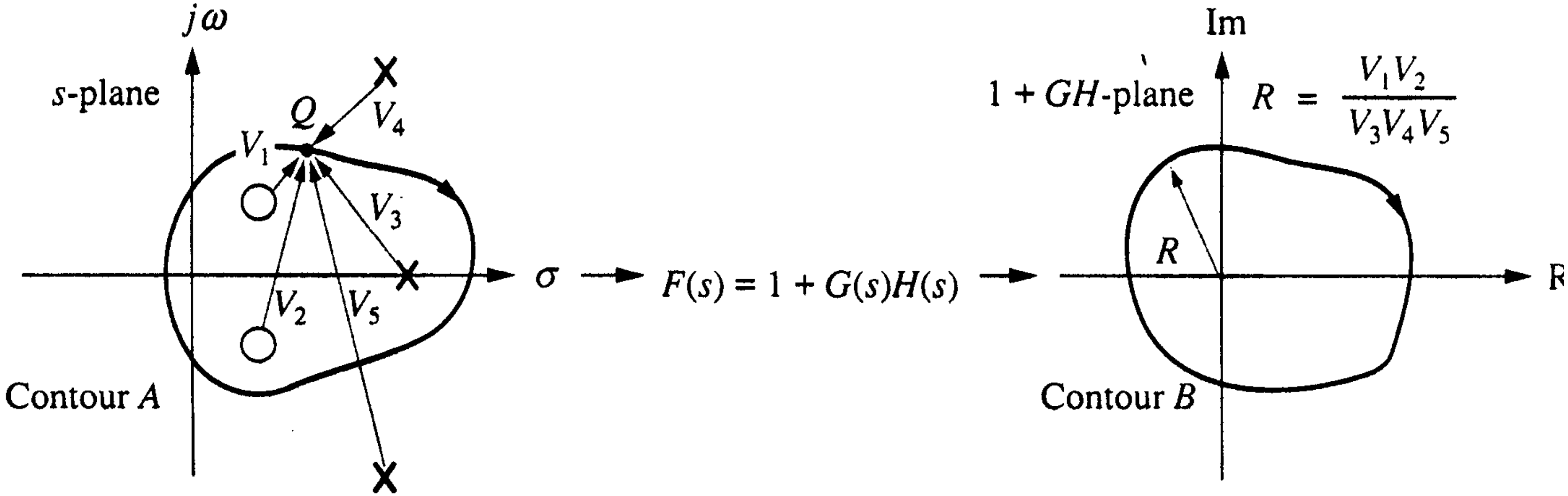


Figure 10.23 Vector representation of mapping

Each vector that lies inside contour A will make a change in angle of 360° .

Each vector that exist outside contour A will appear to oscillate and return to its previous position, undergoing a net angular change of 0° .

A mapping through G(s)H(s) is the same as a map through 1+G(s)H(s), except that it is translated one unit to the left. Thus we count rotations about –1 instead of rotations about the origin.

<Nyquist Stability Criterion>

If a contour, A, that encircles the entire right half-plane is mapping through G(s)H(s), then

$$Z=P-N$$

Z: the number of closed-loop poles in the RHP.

P: the number of open-loop poles in the RHP.

N: the number of counterclockwise revolutions around –1of the mapping .

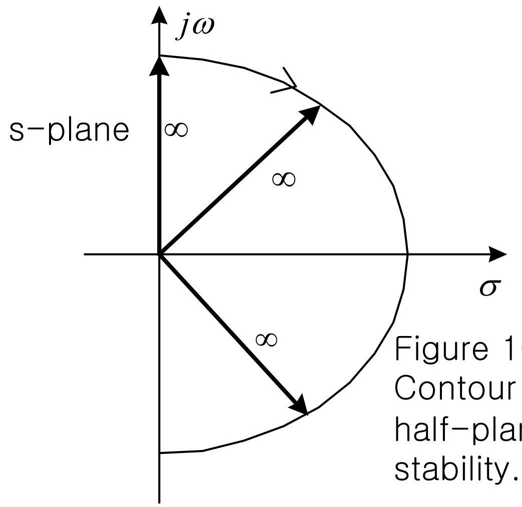


Figure 10.24
Contour enclosing right half-plane to determine stability.

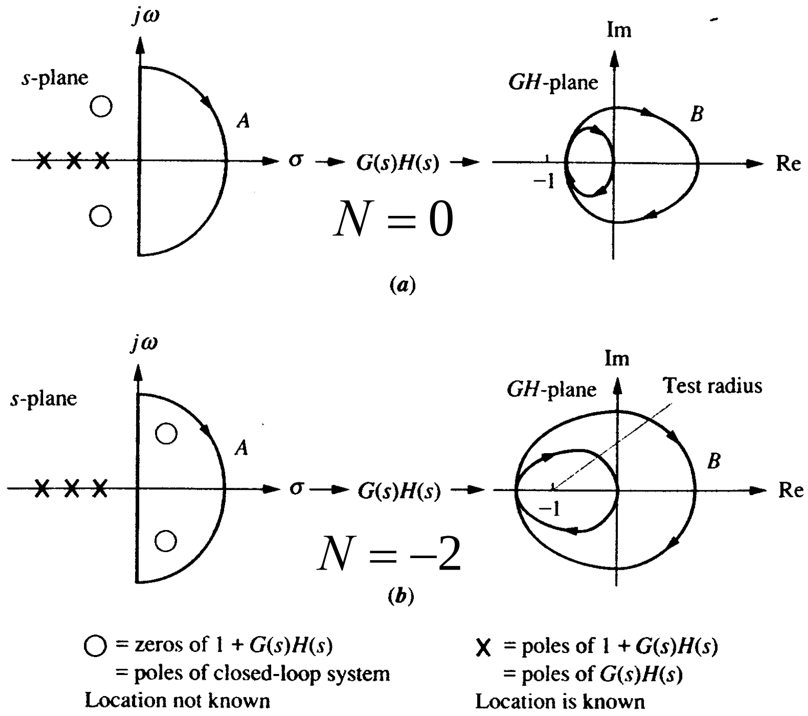


Figure 10.25(a) shows a contour A that does not enclose closed-loop poles, that is the zeros of 1+G(s)H(s). The contour thus maps through G(s)H(s) into a Nyquist diagram that does not encircle −1. Hence, P=0, N=0, and Z=P−N=0. Since Z is the number of closed-loop poles inside contour A, which encircles the right half-plane, this system has no right half-plane poles and is stable.

Figure 10.25

Mapping examples:

a. contour does not enclose closed-loop poles;

b. contour does enclose closed-loop poles;

On the other hand, Figure 10.25(b) shows a contour A that, while it does not enclose open-loop poles, does generate two clockwise encirclements of –1. Thus, P=0, N=−2, and the system is unstable; it has two closed-loop poles in the right half-plane since Z=P−N=2.

Example 10.4

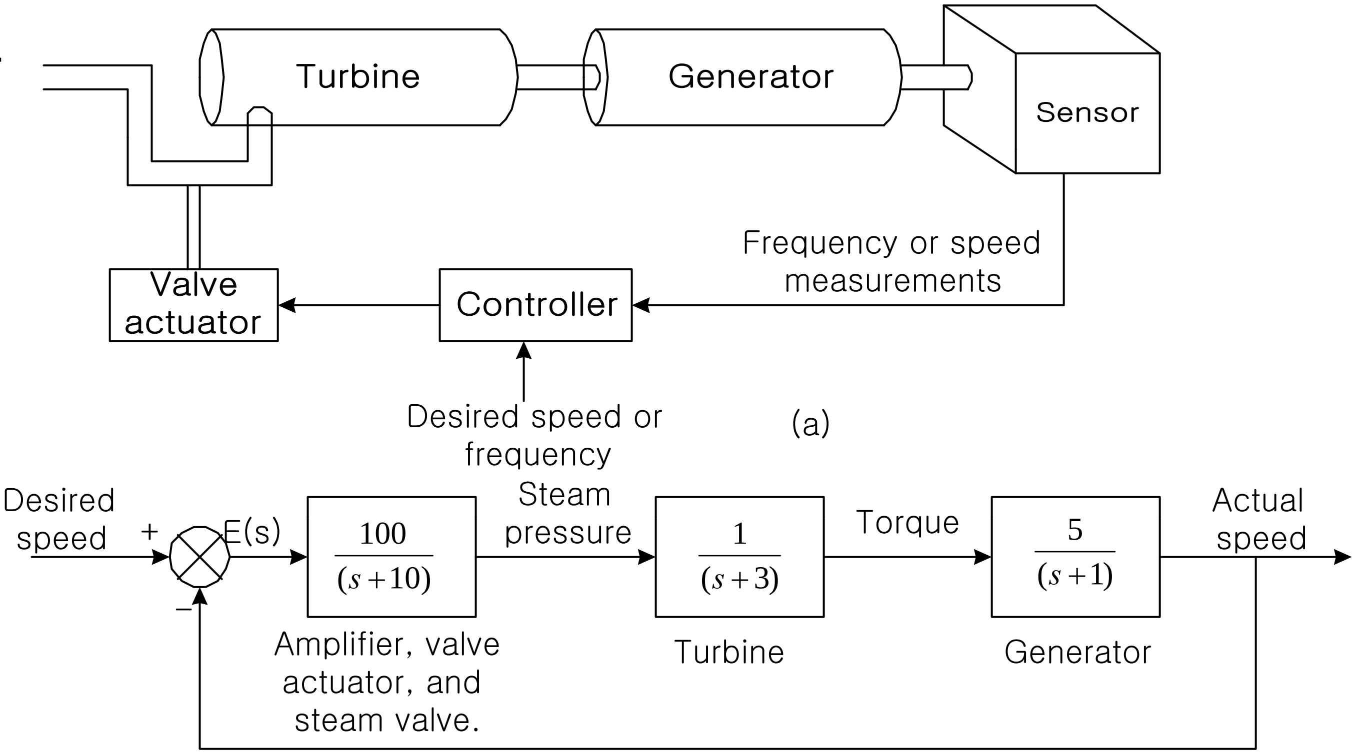


Figure 10.26
a. Turbine and generator
b. block diagram of speed control system for Example 10.4

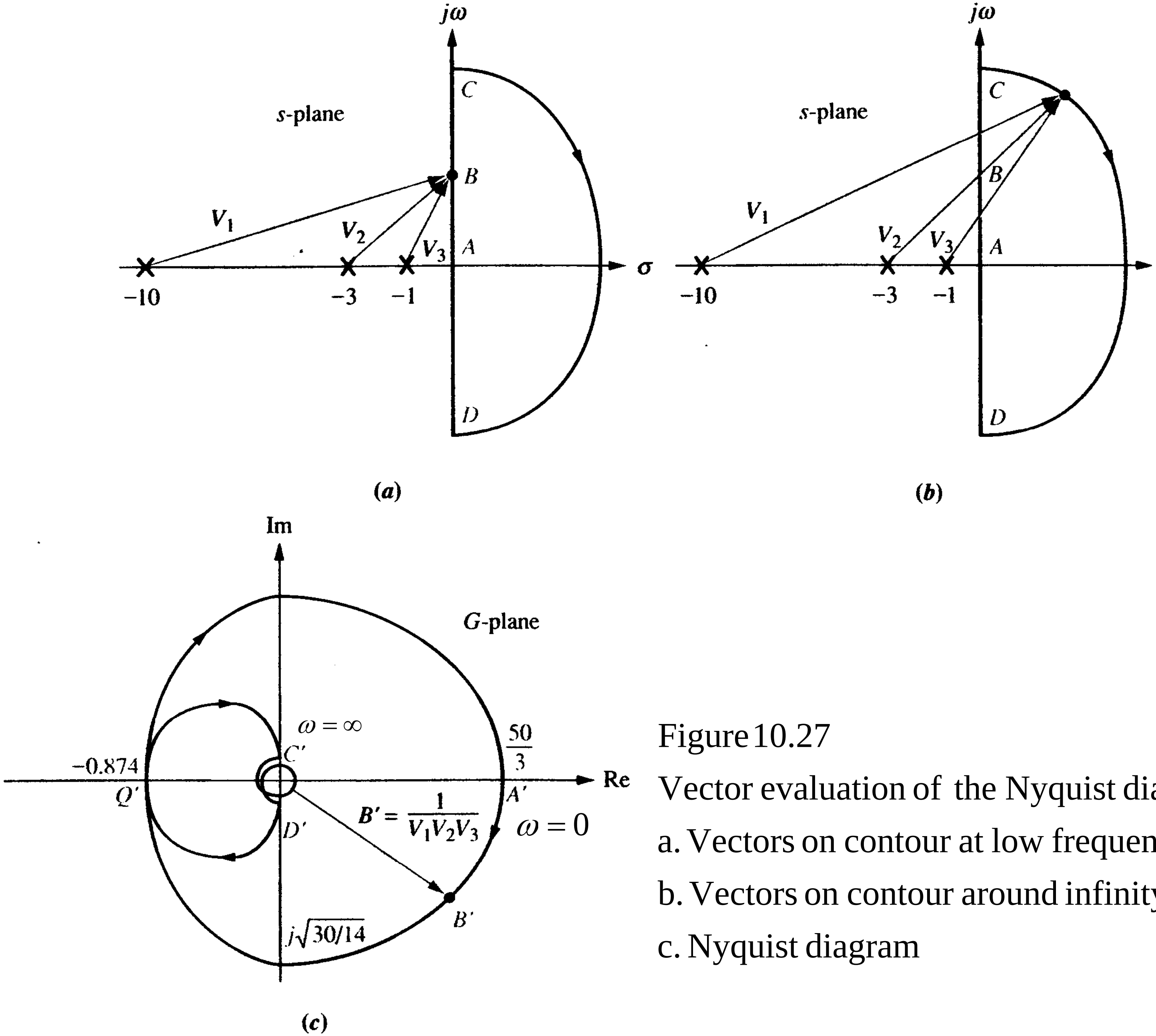


Figure 10.27

Vector evaluation of the Nyquist diagram for Example 10.4

a. Vectors on contour at low frequency;

b. Vectors on contour around infinity;

c. Nyquist diagram

Analysis >

$$G(j\omega) = \left.\frac{500}{(s+1)(s+3)(s+10)}\right]_{s\to j\omega} = \frac{500}{\left(-14\omega^2+30\right)+j(43\omega-\omega^3)}$$

$$or\ \ G(j\omega) = 500\frac{\left(-14\omega^2+30\right)-j(43\omega-\omega^3)}{\left(-14\omega^2+30\right)^2+(43\omega-\omega^3)^2}$$

$at\ \omega = \sqrt{\frac{30}{14}},$ The real part becomes negative.

$at\ \omega = \sqrt{43},$ The real axis crossing

$at\ \ \omega = \infty$ $G(j\omega) \cong \frac{500j}{\omega^3}$ about zero at 90°

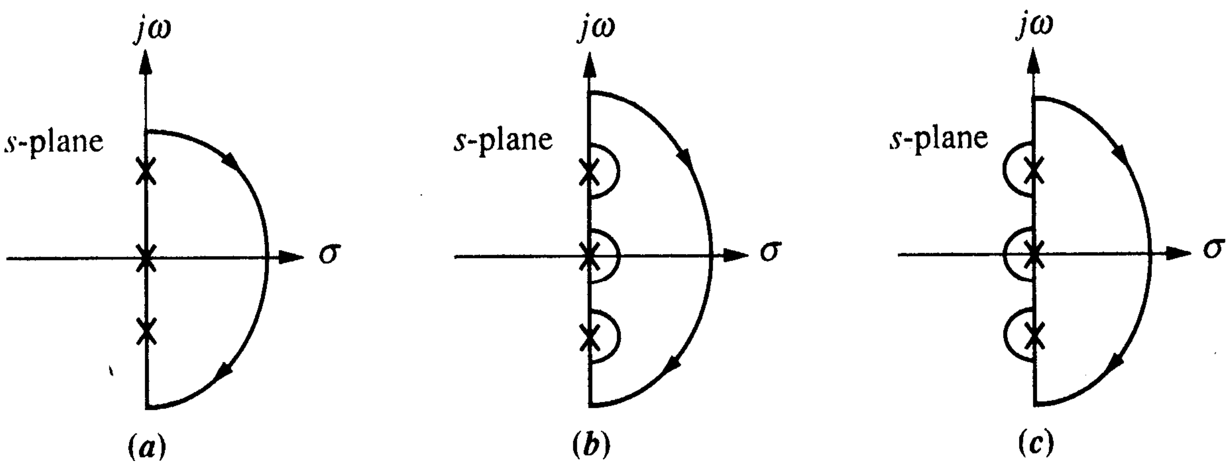
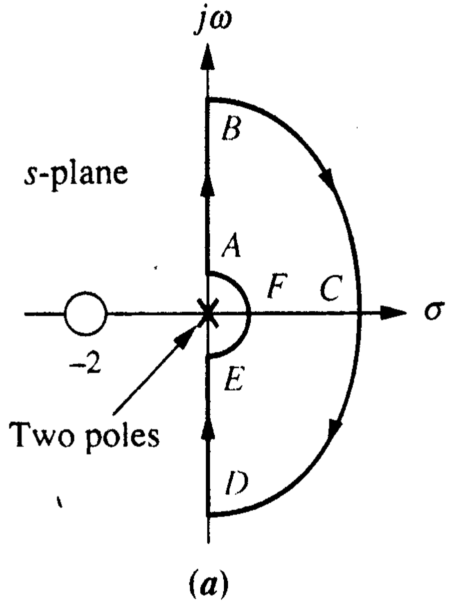


Figure 10.28
Detouring around open - loop poles :
a. poles on contour;
b. detour right;
c. detour left

$$\text{Problem}: G(s)H(s) = \frac{s+2}{s^2}$$



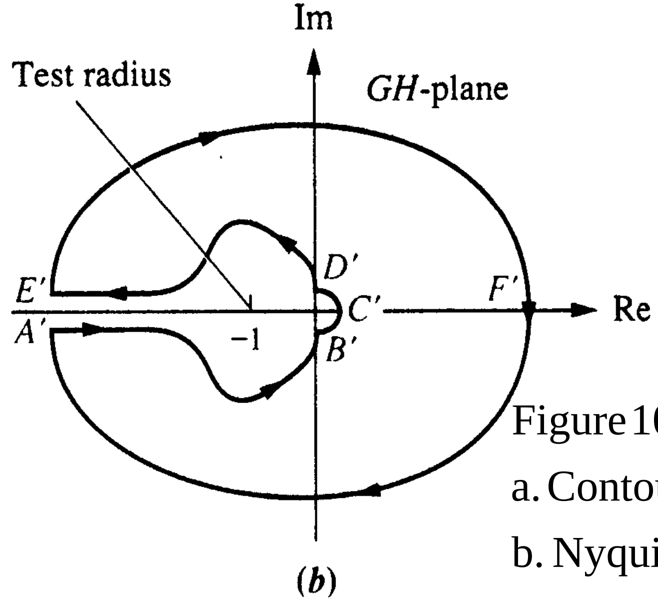


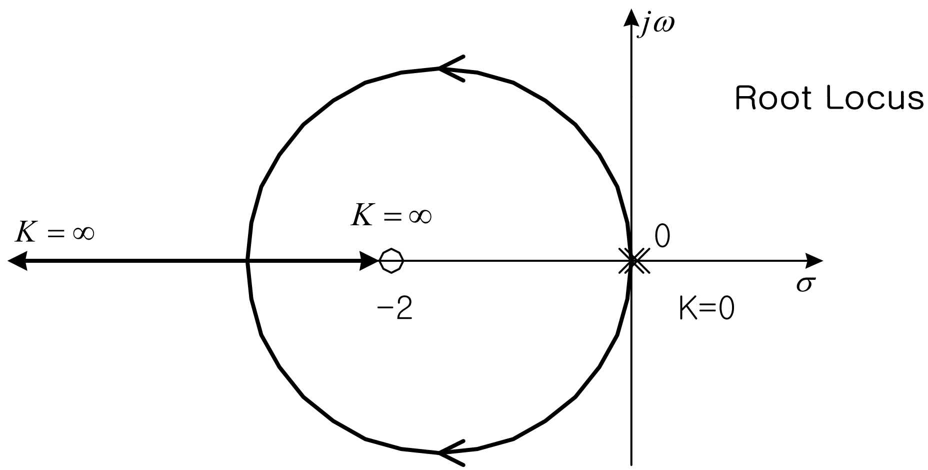
Figure 10.29

a. Contour for Example 10.5;

b. Nyquist diagram for Example 10.5

$$Z = P - N$$

$$0 = 0 - 0$$



Problem

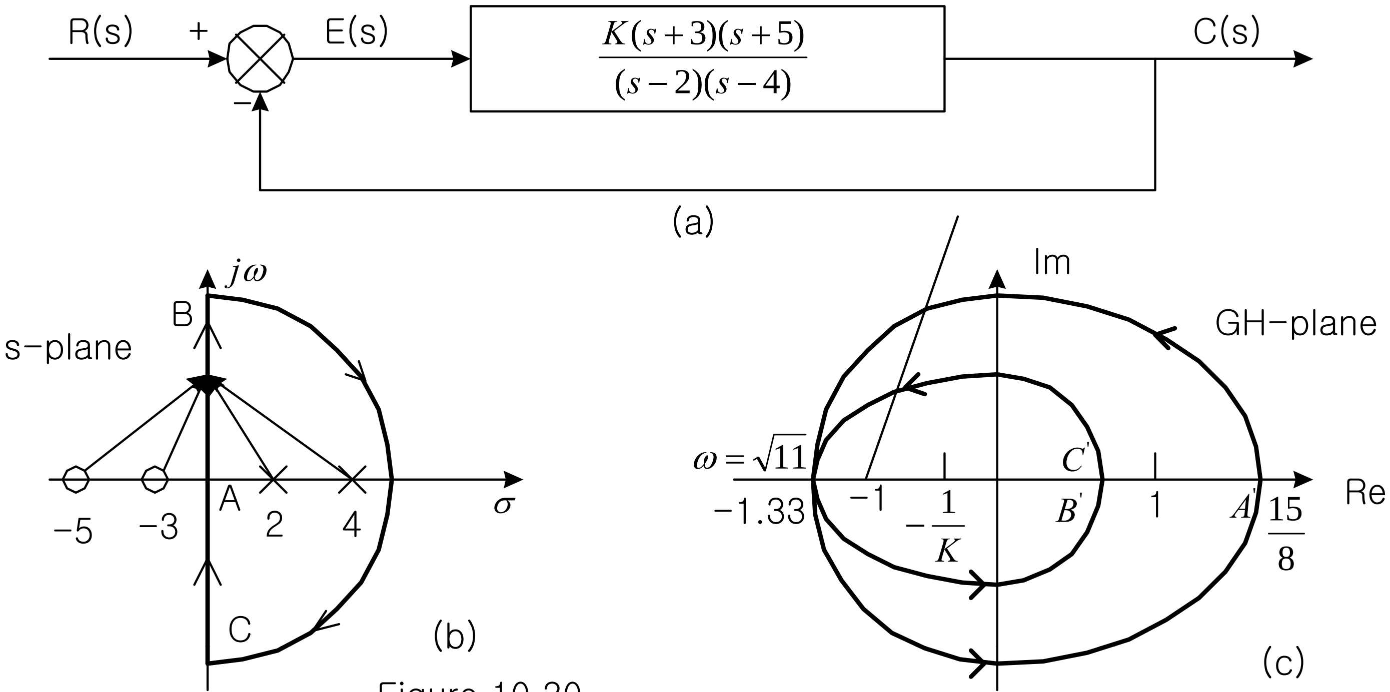
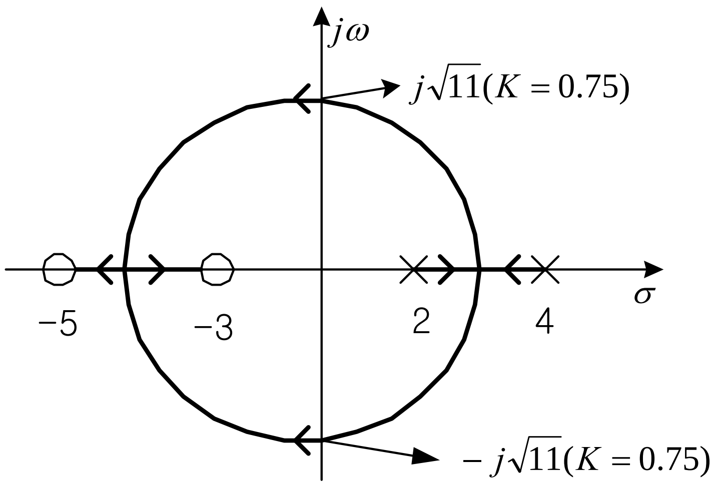


Figure 10.30
Demonstranting Nyquist stability:
a. System;
b. Contour;
c. Nyquist diagram

$$K = 1$$

$$P = 2,\ N = 2,\ Z = P - N = 0$$



Root Locus

$$1+\frac{K(s+3)(s+5)}{(s-2)(s-4)}=0$$

$$(s-2)(s-4)+K(s+3)(s+5)=0$$

$$(1+K)s^2+(8K-6)s+8+15K=0$$

s^2	$(1+K)$	$(8+15K)$
s^1	$(8K-6)$	0
s^0	$(8+15K)$	

$K>-1$

$K>\frac{3}{4}$

$K>-\frac{8}{15}$

$K>\frac{3}{4}$

$when\ K=0.75$

$$\frac{7}{4}s^2+\frac{77}{4}=0\rightarrow s^2+11=0\rightarrow s=\pm j\sqrt{11}$$

$when\ K=1$

$$2s^2+2s+23=0\rightarrow s=-1\pm j\sqrt{45}$$

the contour shown in Fig 10.31(a). For all points on the imaginary axis,

$$G(j\omega)H(j\omega) = \left.\frac{K}{s(s+3)(s+5)}\right|_{K=1, s=j\omega} = \frac{-8\omega^2 - j(15\omega - \omega^3)}{64\omega^4 + \omega^2(15-\omega^2)^2} \quad (10.45)$$

At $\omega = 0$, $G(j\omega)H(j\omega) = -0.0356 - j\infty$

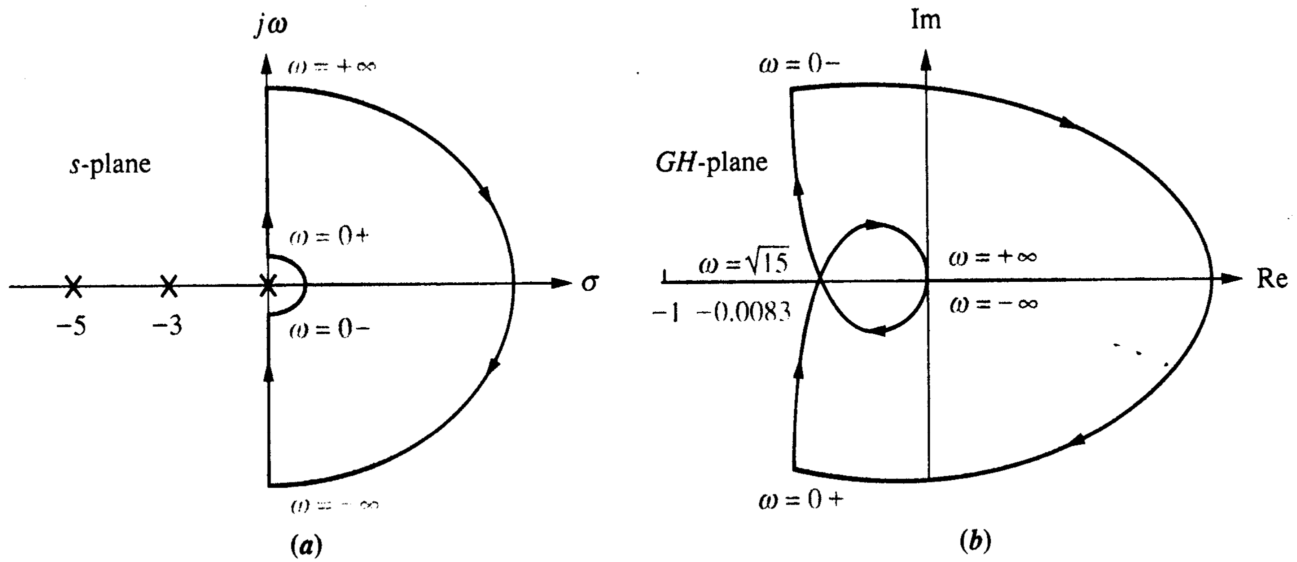


Figure 10.31

a. contour for Example 10.6;

b. Nyquist diagram

Next, find the point where the Nyquist diagram intersects the negative real axis. Setting the imaginary part of Eq. (10.45) equal to zero, we find $\omega = \sqrt{15}$. Substituting this value of ω back into Eq.(10.45) yields the real part of -0.0083.

Finally, at $\omega = \infty$, $G(j\omega)H(j\omega) = G(s)H(s)\big|_{s \to j\omega} = \frac{1}{(j\infty)^3} = 0\angle -270^\circ$.

From the contour of Figure 10.31(a), P = 0; for stability, N must then be equal to zero. Form Figure 10.31(b), the system is stable if the critical point lies outside the contour(N = 0), so that Z = P - N = 0. Thus, K can be increased by $\frac{1}{0.0083} = 120.48$, before the Nyquist diagram encircles -1.

Stability via Mapping only the positive $j\omega$-axis

Once the stability of a system is determined by the Nyquist criterion, continued evaluation of the system can be simplified by using just the mapping of the positive $j\omega$-axis. This concept plays a major role in the next two sections, where we discuss stability margin and the implementation of the Nyquist criterion with Bode plots.

Consider the system shown in Figure 10.32, which is stable at low values of gain and unstable at high values of gain. Since the contour does not encircle open-loop poles, the Nyquist criterion tell us that we must have no encirclements of –1 for the system to be stable.

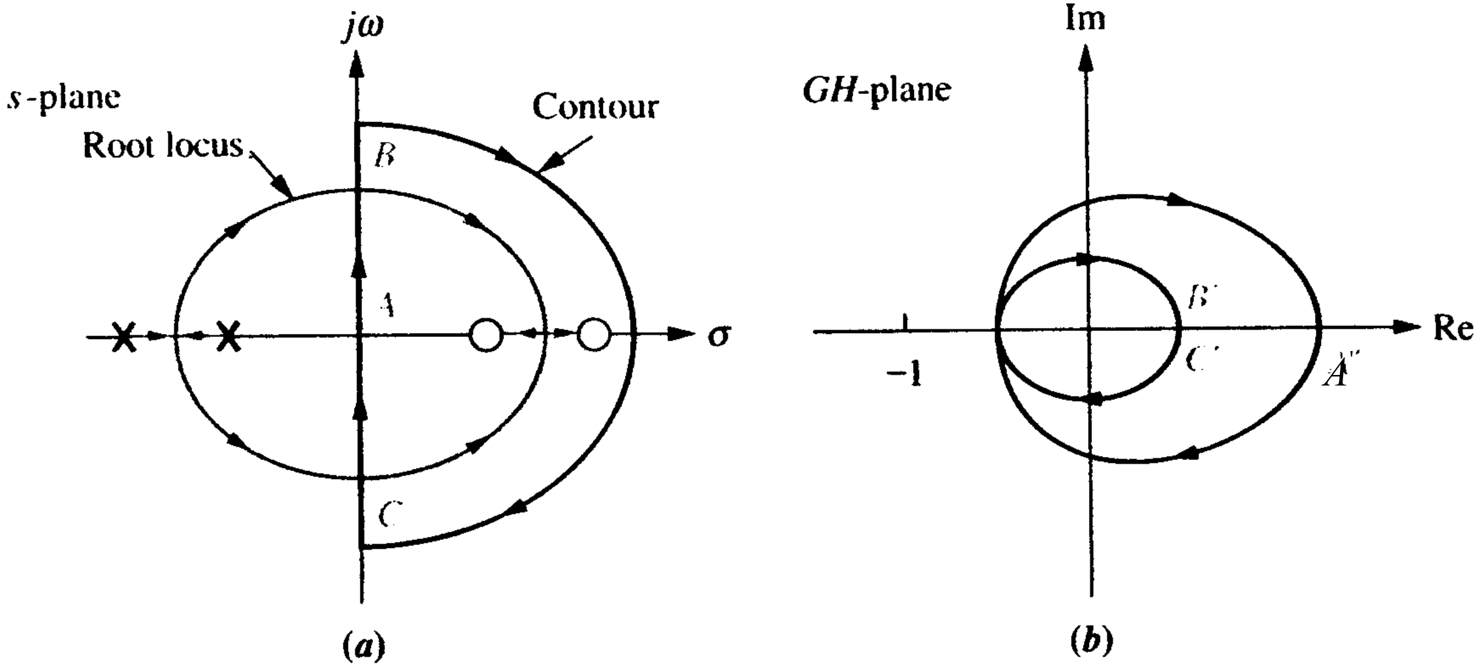


Figure 10.32

a. Contour and root locus of system that is stable for small gain and unstable for large gain;

b. Nyquist diagram

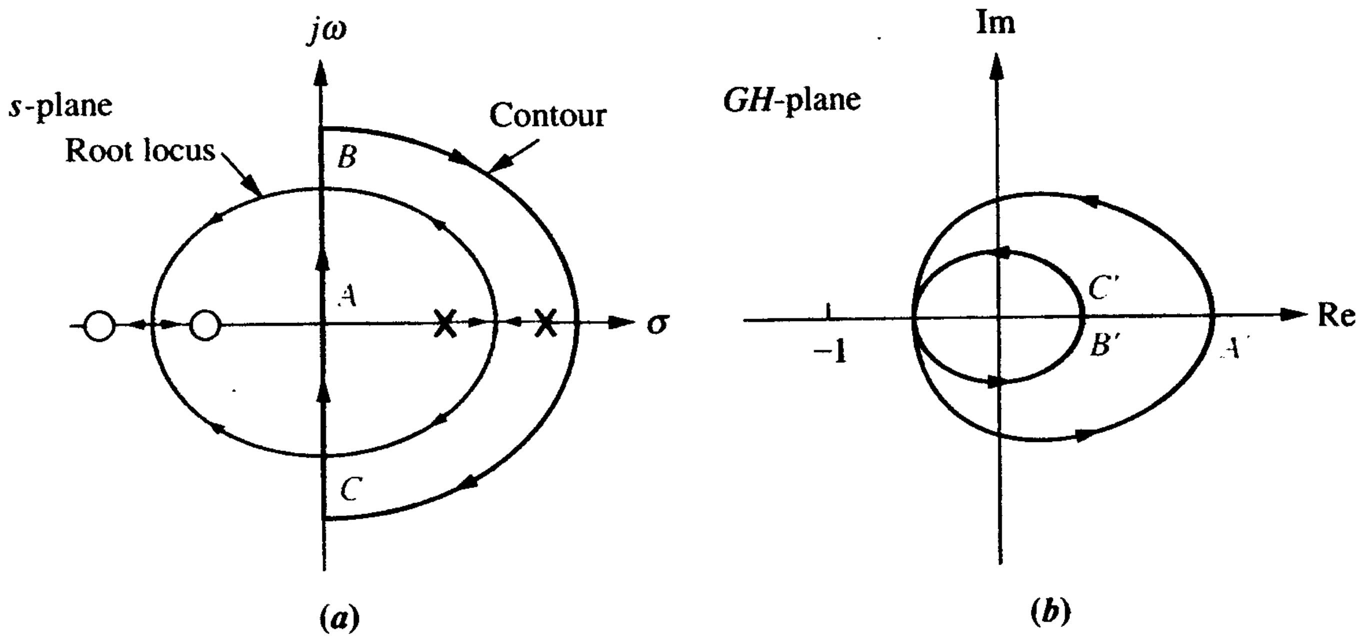


Figure 10.33

a. Contour and root locus of system that is unstable for small gain and stable for large gain

b. Nyquist diagram

Stability design via mapping positive $j\omega$-axis

Problem) Find the range of gain for stability and instability, and the gain for marginal stability for the unity feedback system shown in Figure 10.10, $G(s) = \frac{K}{[(s^2+2s+2)(s+2)]}$. For marginal stability, find the radian frequency of oscillation. Use the Nyquist criterion and mapping of only the positive imaginary axis.

Solution) Since the open-loop poles are only in the left half-plane, the Nyquist criterion tells us that we want no encirclements of -1 for stability. Hence, a gain less than unity at $\pm 180^\circ$ is required. Begin by letting K = 1 and draw the portion of the contour along the positive imaginary axis as shown in Figure 10.34(a). In Figure 10.34(b), the intersection with the negative real axis is found by letting $s = j\omega$ in G(s)H(s), setting the imaginary part equal to zero to find the frequency, and then substituting the frequency into real part of $G(j\omega)H(j\omega)$.

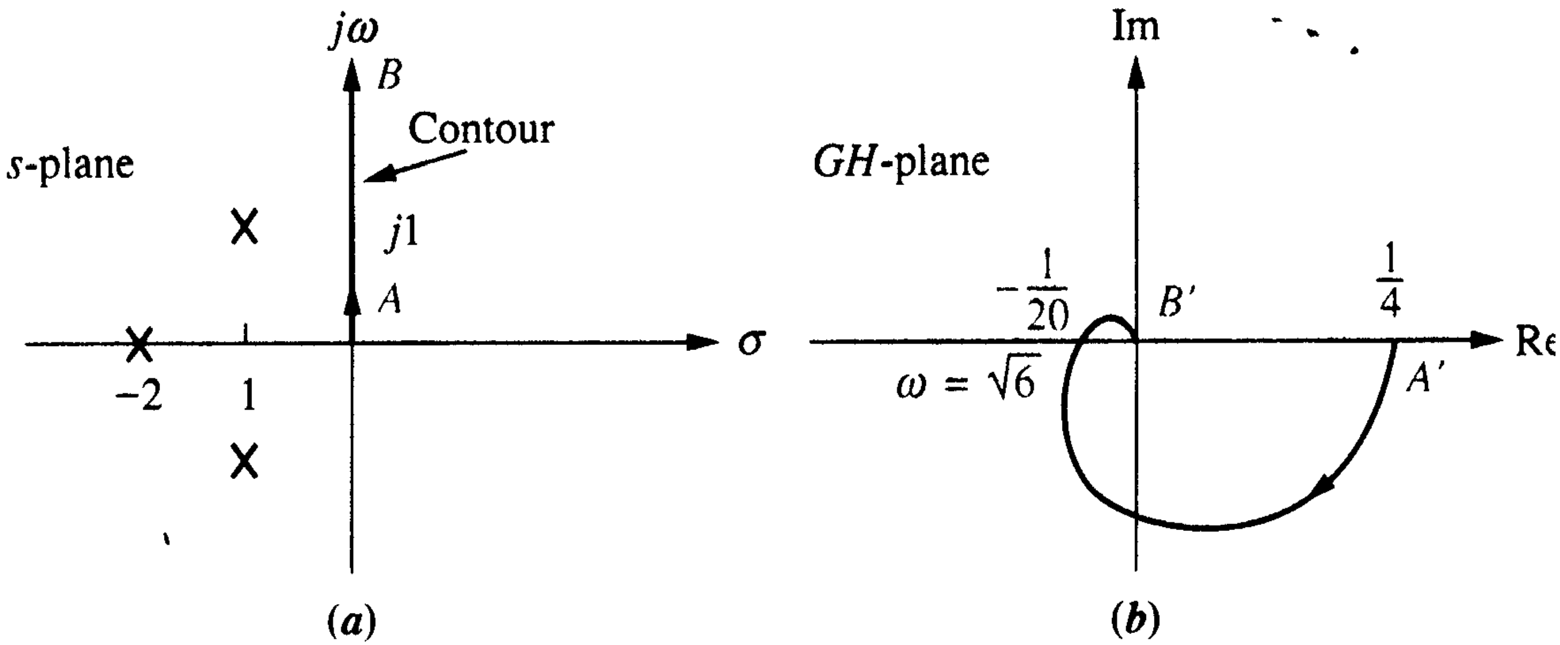


Figure 10.34

a. Portion of contour to be mapped for Example 10.7;

b. Nyquist diagram of mapping of positive imaginary axis.

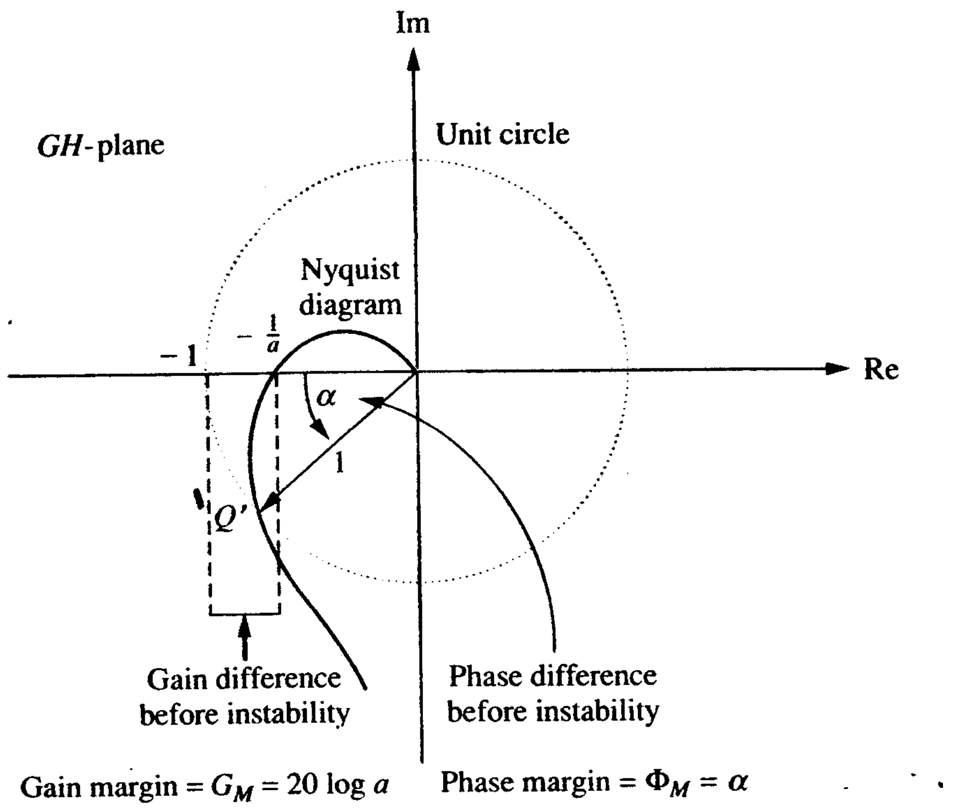
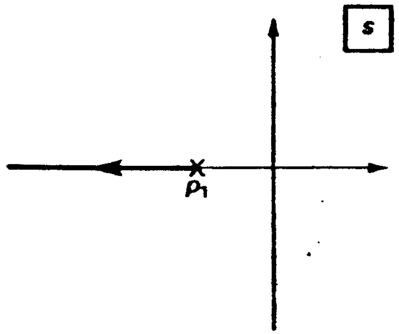


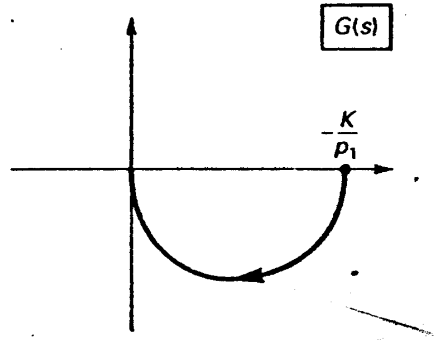
Figure 10.35

Nyquist diagram showing gain and phase margins

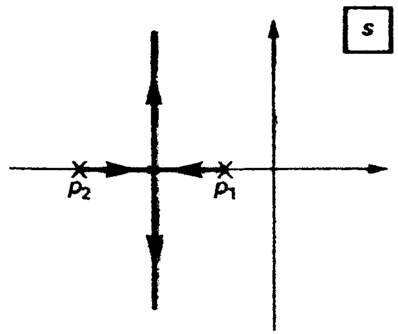
TABLE 8.4 COMPARISON OF ROOT LOCI AND NYQUIST DIAGRAMS



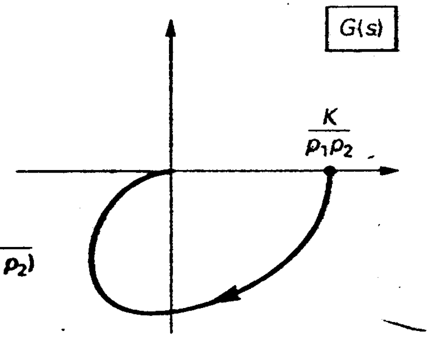
$$G(s) = \frac{K}{s - p_1}$$



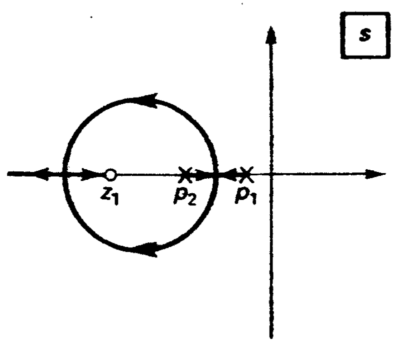
(a)



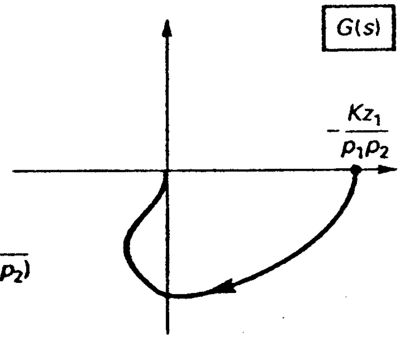
$$G(s) = \frac{K}{(s - p_1)(s - p_2)}$$



(b)

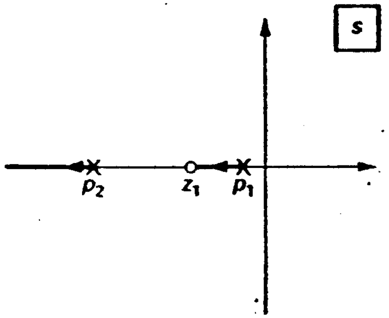


$$G(s) = \frac{K(s - z_1)}{(s - p_1)(s - p_2)}$$

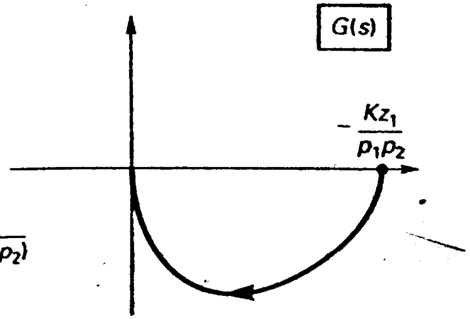


(c)

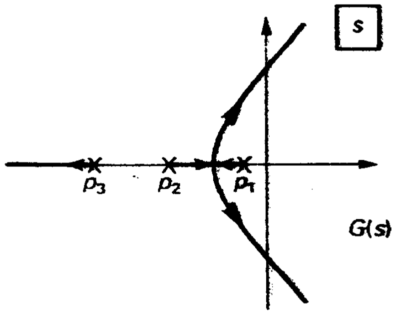
TABLE 8.4 (CONTINUED)



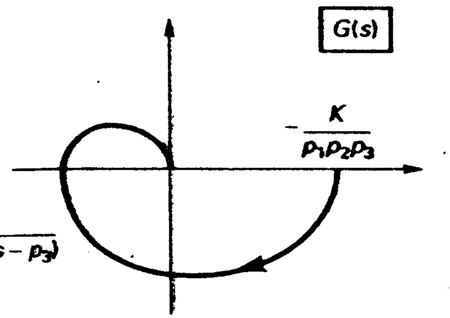
$$G(s) = \frac{K(s - z_1)}{(s - p_1)(s - p_2)}$$



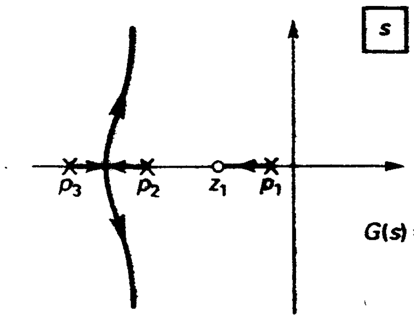
(d)



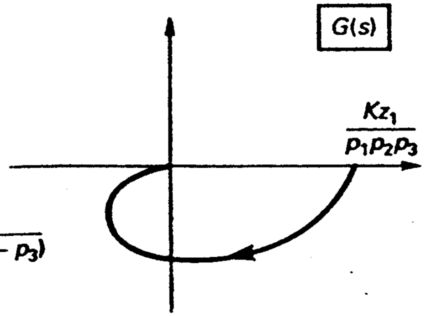
$$G(s) = \frac{K}{(s - p_1)(s - p_2)(s - p_3)}$$



(e)



$$G(s) = \frac{K(s - z_1)}{(s - p_1)(s - p_2)(s - p_3)}$$



(f)