Chapter 6. Stability

* Analysis with characteristic equation polynomial

1. All positive coefficients

$$s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3) \quad \text{stable}$$

2. A term is missing and all positive coefficients

$$s^2 + 1 = (s + j1)(s - j1) \quad \text{marginally stable}$$

3. A term is missing and some negative coefficients

$$s^3 - 7s - 6 = (s+1)(s+2)(s-3) \quad \text{unstable}$$

< Remark >

$$-s^3 - 6s^2 - 11s - 6 = 0 \;\rightarrow\; s^3 + 6s^2 + 11s + 6 = 0$$

stable

* Routh - Hurwitz Criterion

Apply to the characteristic equation

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad \text{..........} \quad (1)$$

Form the array (Routh array)

$$\begin{array}{lllll}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots \\
. & . & . & . & \\
. & . & . & . & \\
. & . & . & . & \\
s^2 & e_1 & e_2 & & \\
s^1 & f_1 & & & \\
s^0 & g_1 & & &
\end{array}$$

where

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} = \frac{-\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$$

etc.

•Routh-Stability Criterion

- The number of roots with positive real parts is equal to the number of sign change in the first column.

- A necessary and sufficient condition that all roots lie in the LHP is that all coefficients of Eq(1) be positive and all terms in the first column of the array have positive terms

Ex. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

The C.E is

$$s^4+3s^3+3s^2+2s+K=0$$

The Routh array is

$$\begin{array}{llll} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & \frac{7}{3} & K & \\ s^1 & 2-\frac{9}{7}K & & \\ s^0 & K & & \end{array}$$

$$* \quad \frac{7}{3} = \frac{3\times 3 - 1\times 2}{3}$$

$$K = \frac{3\times K - 1\times 0}{3}$$

$$2-\frac{9}{7}K = \frac{\frac{7}{3}\times 2 - 3K}{\frac{7}{3}}$$

so, K must be positive and

$$2-\frac{9}{7}K > 0 \;\Rightarrow\; 0 < K < \frac{14}{9} \;\Rightarrow\; stable$$

$$if\ \ K=\frac{14}{9}$$

$$A(s)=\frac{7}{3}s^2+K$$

$$\frac{dA(s)}{ds}=\frac{14}{3}s$$

s^4	1	3	K
s^3	3	2	0
s^2	$\frac{7}{3}$	$\frac{14}{9}$	
s^1	$\frac{14}{3}$		
s^0	$\frac{14}{9}$		

$$if\ \ K=0$$

s^2	$\frac{7}{3}$	0
s^1	2	
s^0	0	

Answer

$0<K<\frac{14}{9}$: *stable*

$K=0,\ or\ \ K=\frac{14}{9}$: marginally *stable*

$K<0,\ or\ \ K>\frac{14}{9}$: *unstable*

$Ex.\quad \Delta(s) = (s-1)(s+2)(s-4) = s^3 - 3s^2 - 6s + 8$

$$\begin{array}{lcc} s^3 & 1 & -6 \\ s^2 & -3 & 8 \\ s^1 & -\dfrac{10}{3} & 0 \\ s^0 & 8 & \end{array}$$

sign change are made twice, then 2 roots are in RHP

(2 unstable poles)

* Special Case

Case 1. In Routh's table, a zero only in the first column of a row

Case 2. In Routh's table, all the elements of a row are zero.

∘ Solution for case 1.

1. multiply by (s + a)   a > 0

2. replace by ε where ε is very small positive or negative number

Ex. $T(s) = \dfrac{N(s)}{D(s)}$

$$D(s) = s^3 - 3s + 2 = 0$$

guess at least one RHP pole.

$$\begin{array}{lll} s^3 & 1 & -3 \\ s^2 & 0 & 2 \\ s^1 & \dfrac{0-2}{0} = \infty & \\ s^0 & & \end{array}$$

1) *multiply by* $(s+3)$

$$D^{'}(s) = (s^3 - 3s + 2)(s+3) = s^4 + 3s^3 - 3s^2 - 7s + 6 = 0$$

$$
\begin{array}{llll}
s^4 & 1 & -3 & 6 \\
s^3 & 3 & -7 & 0 \\
s^2 & \dfrac{-9+7}{3} = -\dfrac{2}{3} & 6 & \\
s^1 & 2 & 0 & \\
s^0 & 6 & &
\end{array}
$$

2 RHP roots because of twice of sign changes.

2)
$$
\begin{array}{lll}
s^3 & 1 & -3 \\
s^2 & \varepsilon & 2 \\
s^1 & \dfrac{-3\varepsilon-2}{\varepsilon} & 0 \\
s^0 & 2 &
\end{array}
$$

$if\ \ \varepsilon > 0\ \ (positive)\ \ sign\ \ change\ \ twice.$

$if\ \ \varepsilon < 0\ (negative)\ \ sign\ \ change\ \ twice.$

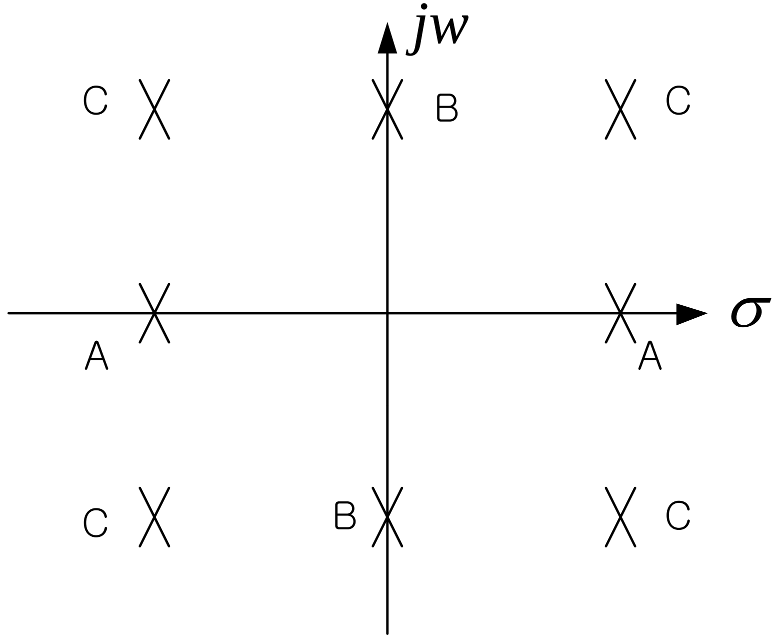
$\Rightarrow\ \ \therefore 2\ RHP\ \ Poles$

* Solution for case 2

A. a pair of real roots with different signs.

B. a pair of imaginary roots.

C. conjugate pair (Symmetric)



Answer : Use auxiliary equation A(s)

Ex. $s^4 + s^3 - 3s^2 - s + 2 = 0 \rightarrow (s+2)(s-1)^2(s+1) = 0$

$$\begin{array}{llll}
s^4 & 1 & -3 & 2 \\
s^3 & 1 & -1 & 0 \\
s^2 & -3+1=-2 & 2 & 0 \quad \rightarrow take \;\; previous \;\; row \\
s^1 & 0 & 0 & \\
s^0 & & &
\end{array}$$

$$A(s) = -2s^2 + 2 = 0$$

$$\frac{dA(s)}{ds} = -4s$$

$$\begin{array}{llll}
s^4 & 1 & -3 & 2 \\
s^3 & 1 & -1 & 0 \\
s^2 & -2 & 2 & 0 \\
s^1 & -4 & 0 & \\
s^0 & 2 & &
\end{array}$$

$sign \;\; change \;\; twice \;\; \rightarrow 2 \; RHP \;\; real \;\; poles$

$\rightarrow 2 \;\; LHP \;\; real \;\; poles$

$$Ex.\;\; D(s) = s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56$$

$$\begin{array}{llll}
s^5 & 1 & 6 & 8 \\
s^4 & 7 \to 1 & 42 \to 6 & 56 \to 8 \leftarrow take \\
s^3 & 0 \to 4 \to 1 & 0 \to 12 \to 3 & 0 \\
s^2 & 3 & 8 & \\
s^1 & \dfrac{1}{3} & & \\
s^0 & 8 & &
\end{array}$$

$$A(s) = 7s^4 + 42s^2 + 56$$

$$or\ A(s) = s^4 + 6s^2 + 8$$

$$\frac{dA}{ds} = 4s^3 + 12s$$

stable because of $\to$ *no RHP poles*

$\to$ *no sign changes.*

$$Ex.\ \ T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

$$\begin{array}{lll} s^3 & 1 & 77 \\ s^2 & 18 & K \\ s^1 & \dfrac{1386 - K}{18} & \\ s^0 & K & \end{array}$$

$$1386 - K > 0 \quad \rightarrow \quad 0 < K < 1386 : stable$$

$$K < 0 \ \ or \ \ K > 1386 : unstable$$

Chapter 7 Steady-State Errors

* Steady-State Errors

Steady-State Error is the difference between the input and the output for a prescribed test input as $t \rightarrow \infty$

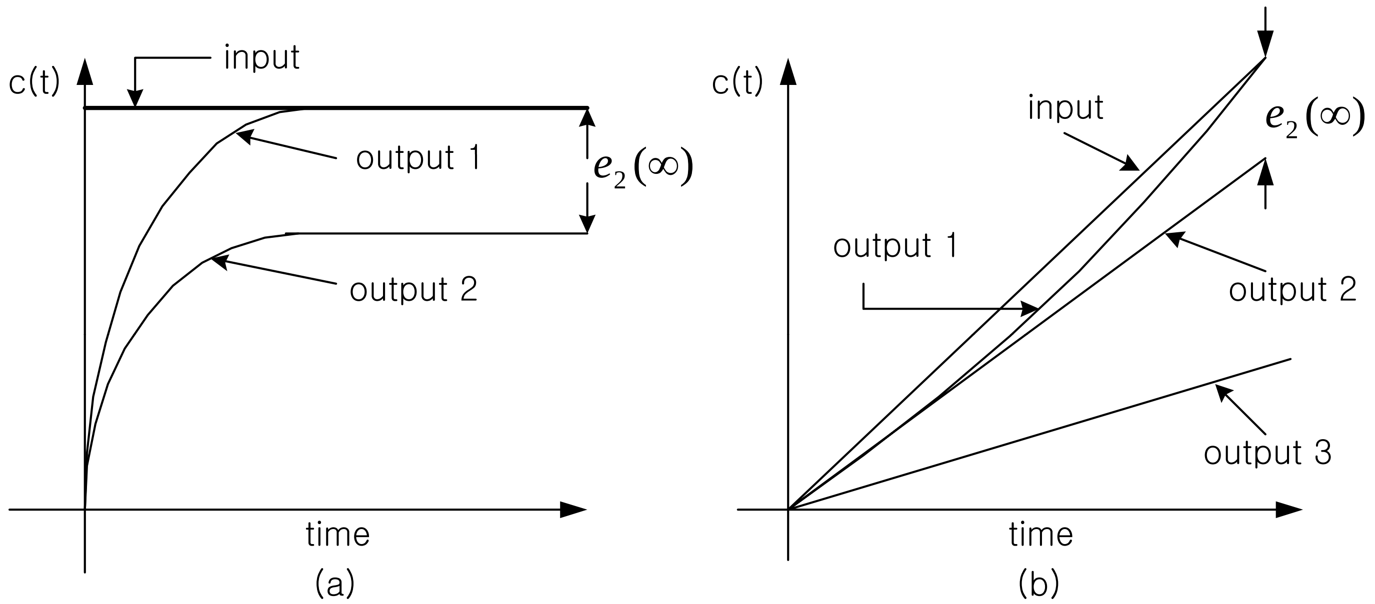
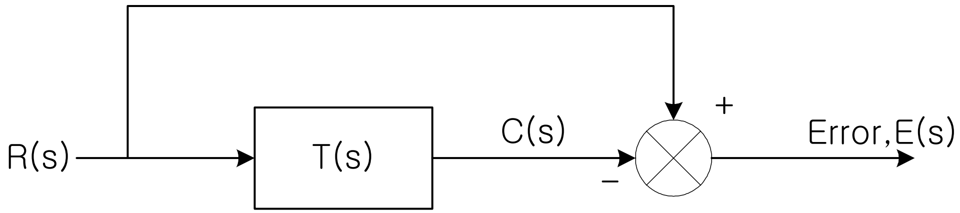
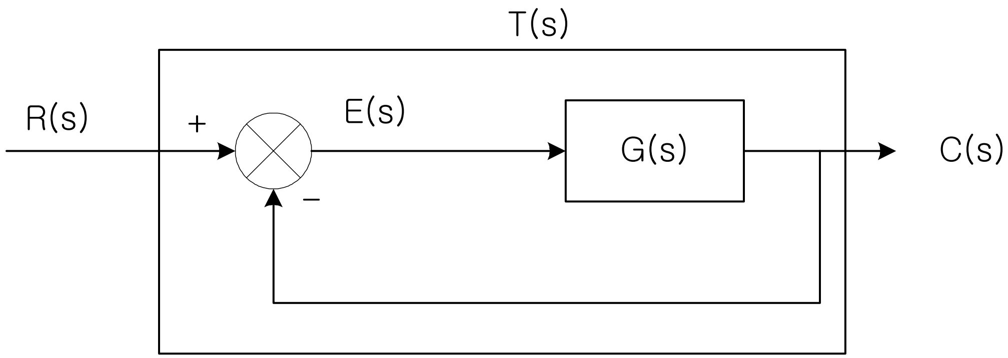


Figure. 7.2 Steady-State Error
(a) Step input
(b) ramp input

Steady-State Errors for a unity feedback control system





$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)G(s)$$

$$E(s) = R(s) - E(s)G(s)$$

$$E(s) = \frac{R(s)}{1+G(s)}$$

By final theorem

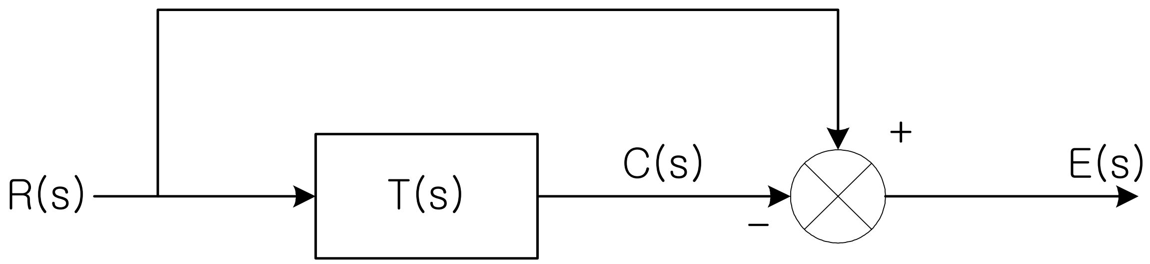
$$e_{ss} = \lim_{t\to\infty} e(t) = e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} \quad (1)$$

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}$$

$n: system\,type$

$n = 0 \;\; type\ 0$

$n = 1 \;\; type\ 1$



$$E(s) = R(s) - C(s) \qquad (1)$$

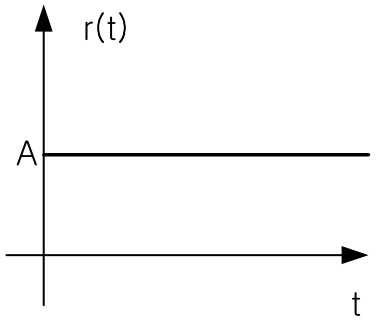
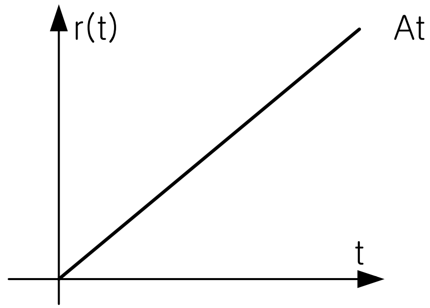
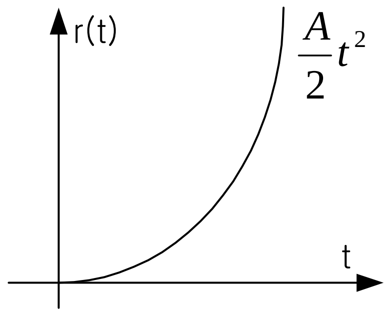
$$C(s) = R(s)T(s) \qquad (2)$$

$$E(s) = R(s)[1 - T(s)] \qquad (3)$$

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \qquad (4)$$

$$e(\infty) = \lim_{s \to 0} sR(s)[1 - T(s)] \qquad (5)$$

Test waveforms

waveform	Name	Time function	Laplace transform
	Step or constant position	A	$\frac{A}{s}$
	Ramp or constant velocity	At $\frac{dr(t)}{dt} = A$	$\frac{A}{s^2}$
	Parabola parabolic or constant acceleration	$\frac{A}{2}t^2$ $\frac{d^2r(t)}{d^2t} = A$	$\frac{A}{s^3}$

1. *Step input*

$$R(s)=\frac{A}{s}$$

$$Eq(1).\ becomes\ e_{ss}=e(\infty)=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{s\left(\dfrac{A}{s}\right)}{1+G(s)}=\frac{A}{1+\lim\limits_{s\to 0} G(s)}$$

$$\lim_{s\to 0} G(s)=\infty \quad\rightarrow\quad G(s)=\frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}$$

2. *Ramp input*

$$R(s)=\frac{A}{s^2} \qquad r(t)=Atu(t)$$

$$e(\infty)=\lim_{s\to 0}\frac{s\left(\dfrac{A}{s^2}\right)}{1+G(s)}=\lim_{s\to 0}\frac{A}{s+sG(s)}=\frac{A}{\lim\limits_{s\to 0} sG(s)}$$

3. *Parabolic input*

$$R(s) = \frac{A}{s^3}$$

$$e(\infty) = \lim_{s \to 0} \frac{s\left(\frac{A}{s^3}\right)}{1 + G(s)} = \lim_{s \to 0} \frac{A}{s^2 + s^2 G(s)} = \frac{A}{\lim_{s \to 0} s^2 G(s)}$$

Static error constant

$$Au(t) \qquad e_{ss} = e(\infty) = \frac{A}{1 + \boxed{\lim_{s \to 0} G(s)}} \qquad K_p = \lim_{s \to 0} G(s)$$

$$\boxed{\lim_{s \to 0} G(s)} \rightarrow K_p$$

$$e_{ss} = \frac{A}{1 + K_p}$$

K_p : *position* constant , step - error constant

$$Atu(t) \qquad e_{ss} = e(\infty) = \frac{A}{\boxed{\lim_{s \to 0} sG(s)}} \qquad K_v = \lim_{s \to 0} sG(s)$$

$$\boxed{\lim_{s \to 0} sG(s)} \rightarrow K_v$$

$$e_{ss} = \frac{A}{K_v}$$

K_v : velocity constant, ramp – error constant

$$\frac{A}{2} t^2 u(t) \quad e_{ss} = e(\infty) = \frac{A}{\boxed{\lim_{s \to 0} s^2 G(s)}} \qquad K_a = \lim_{s \to 0} s^2 G(s)$$

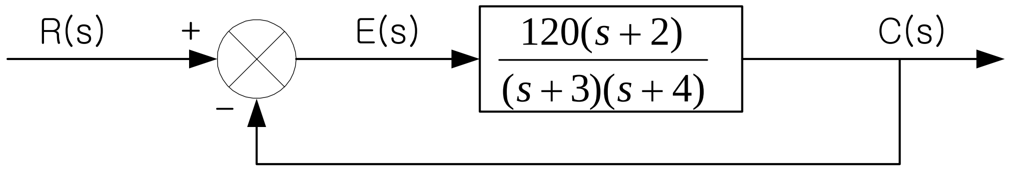
$$\boxed{\lim_{s \to 0} s^2 G(s)} \rightarrow K_a$$

$$e_{ss} = \frac{A}{K_a}$$

K_a : Acceleration constant

Example7.2 Steady-state errors for systems with no integrations

problem: Find the steady-state errors for input $5u(t)$, $5tu(t)$ and $5t^2u(t)$ to the system shown in Figure. The function u(t) is the unit step



Figure

solution

1) $system\ input: 5u(t) \rightarrow \frac{5}{s}(LT)$

$$e(\infty) = e_{step}(\infty) = \frac{5}{1+\lim_{s\to 0} G(s)} = \frac{5}{1+20} = \frac{5}{21}$$

2) $system\ input: 5tu(t) \rightarrow \frac{5}{s^2}(LT)$

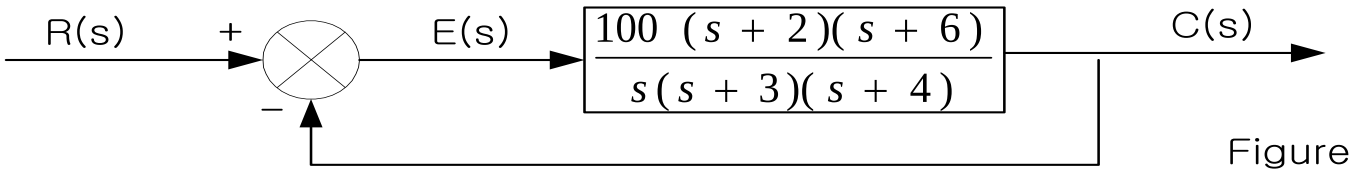
$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{0} = \infty$$

3) $system\ input: 5t^2u(t) \rightarrow \frac{10}{s^3}(LT)$

$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim_{s\to 0} s^2G(s)} = \frac{10}{0} = \infty$$

Example 7.3 Steady-state errors for systems with one integrations

problem : Find the steady-state errors for input $5u(t)$, $5tu(t)$ *and* $5t^2u(t)$ to the system shown in Figure. The function u(t) is the unit step



Figure

solution

1) $system \quad input \quad : 5u(t) \;\rightarrow\; \dfrac{5}{s}(LT)$

$$e(\infty) = e_{step}(\infty) = \frac{5}{1 + \lim_{s\to 0} G(s)} = \frac{5}{\infty} = 0$$

2) $system \quad input \quad : 5tu(t) \;\rightarrow\; \dfrac{5}{s^2}(LT)$

$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{100} = \frac{1}{20}$$

3) $system \quad input \quad : 5t^2u(t) \;\rightarrow\; \dfrac{10}{s^3}(LT)$

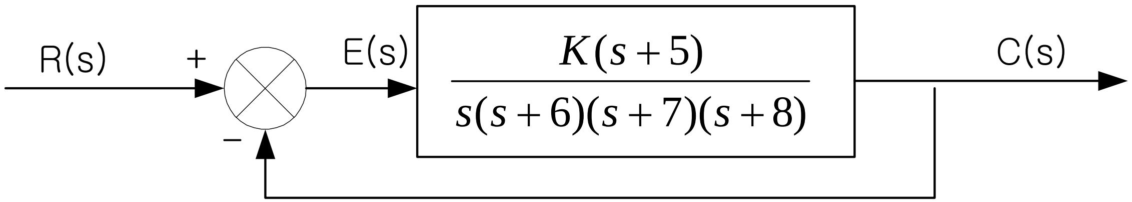
$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim_{s\to 0} s^2G(s)} = \frac{10}{0} = \infty$$

System Type

Input	Steady-state error formula	Type 0		Type 1		Type 2	
		Static error constant	error	Static error constant	error	Static error constant	error
step Au(t)	$\frac{A}{1+K_p}$	$K_p =$ constant	$\frac{A}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp Atu(t)	$\frac{A}{K_v}$	$K_v = 0$	∞	$K_v =$ constant	$\frac{A}{K_v}$	$K_v = \infty$	0
$parabola$ $\frac{A}{2}t^2u(t)$	$\frac{A}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ constant	$\frac{A}{K_a}$

7.4 Steady-State Error Specifications and Examples

Example 7.6) Given the following control system, find K so that there is 10% error in the steady state



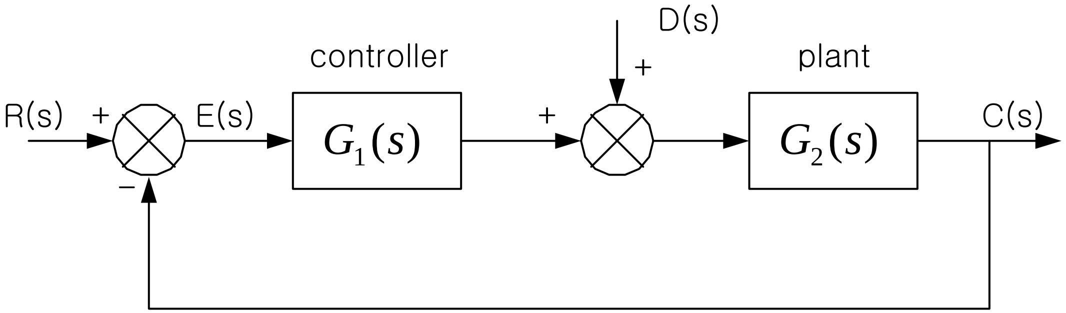
solution) since the system is Type1, a ramp input should be applied for yielding a finite error.

$$e(\infty) = \frac{1}{K_v} = 0.1$$

$$K_v = 10 = \lim_{s \to 0} sG(s) = \frac{K \times 5}{6 \times 7 \times 8}$$

$$\therefore K = 672$$

Steady-State Error for Disturbance



$$\text{The output } C(s) = R(s) - E(s)$$

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

$$E(s) = \boxed{\frac{1}{1+G_1(s)G_2(s)}}R(s)\boxed{-\frac{G_2(s)}{1+G_1(s)G_2(s)}}D(s)$$

Kind of transfer function

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s}{1+G_1(s)G_2(s)}R(s) - \lim_{s\to 0}\frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s) \qquad (1)$$

$$= \boxed{e_R(\infty)} + \boxed{e_D(\infty)}$$

due to the input

due to disturbance

$$If \quad D(s) = \frac{1}{s}$$

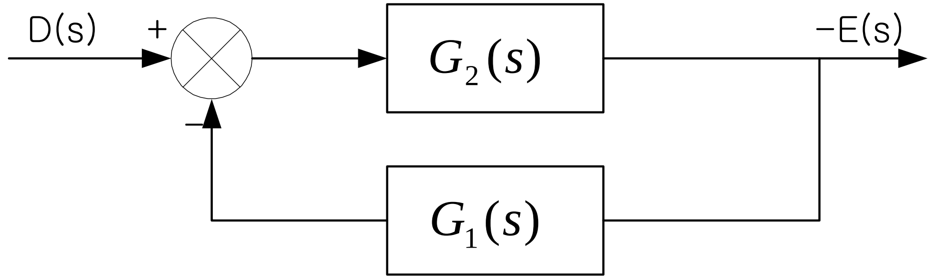
$$e_D(\infty) = -\frac{1}{\lim_{s\to 0}\frac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)}$$

$$e_D(\infty) \to \; small$$

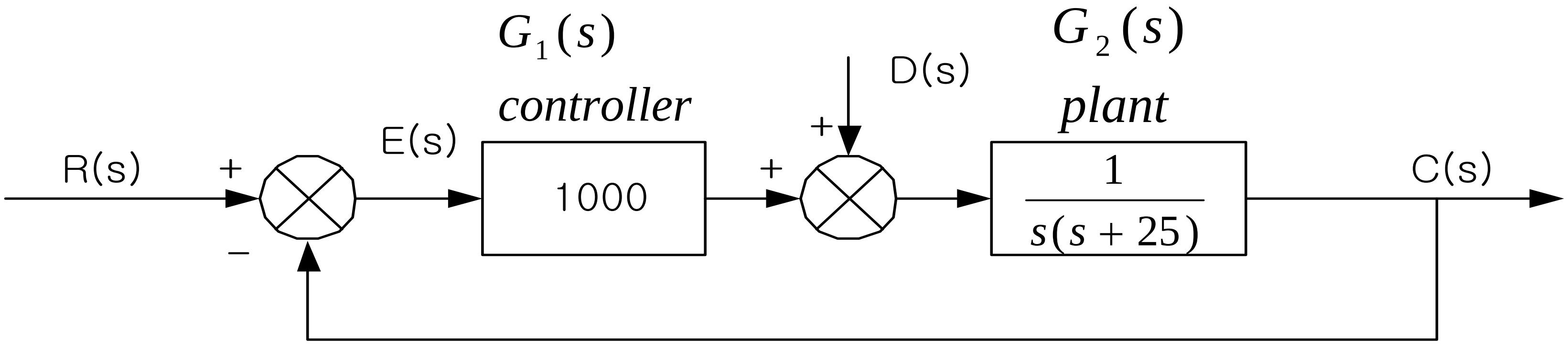
$$if \;\; G_1(s) \;\to\; increases$$

$$G_2(s) \to decreases$$

$$If \quad R(s) = 0$$



Example 7.7 Find the steady-state error component due to a step disturbance for the system of Figure

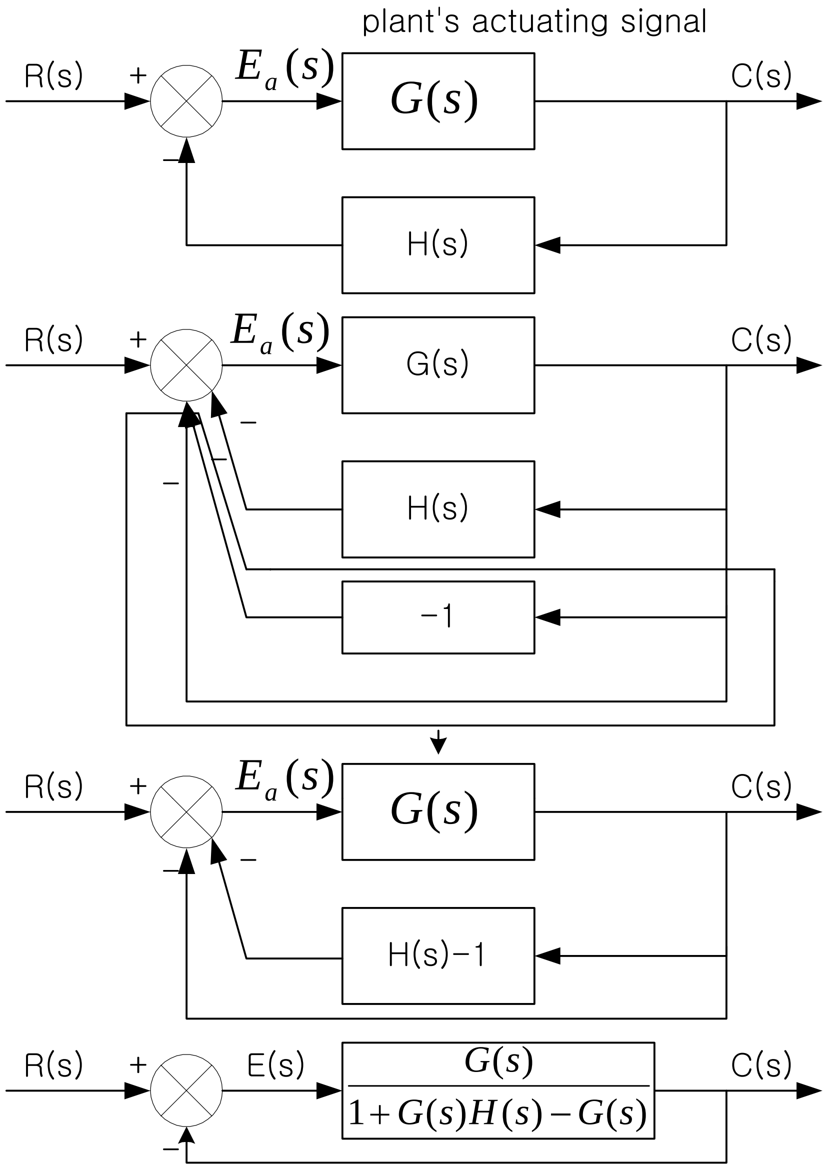


solution) The system is stable

$$e_D(\infty) = -\frac{1}{\lim_{s\to 0}\frac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)} = -\frac{1}{0+1000} = -\frac{1}{1000}$$

The result shows that the steady-state error produced by the step disturbance is inversely proportional to the dc gain of $G_1(s)$. The dc gain of $G_2(s)$ is infinite in this example.

Steady-State Error for Nonunity Feedback

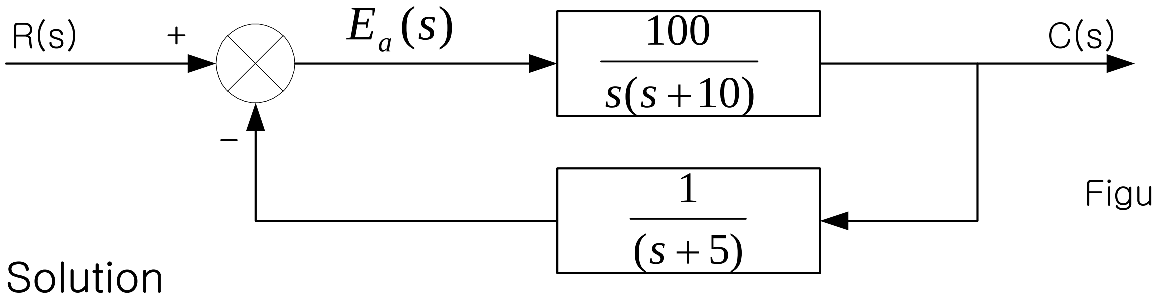


$$\frac{G(s)}{1+G(s)[H(s)-1]}$$

$$=\frac{G(s)}{1+G(s)H(s)-G(s)}$$

Example 7.8

Problem: For the system shown in Figure, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input. Assume input and output units are the same



Figure

Solution

The system is type 1. This may not be the case, since there is a nonunity feedback element, and the plant's actuating siganl is not the difference between the input and the output. The first step in solving the problem is to convert the system of Figure into an equivalent unity feedback system.

$$G(s) = \frac{100}{s(s+10)}, \quad H(s) = \frac{1}{(s+5)}$$

$$G_e(s) = \frac{G(s)}{1+G(s)H(s)-G(s)} = \frac{100(s+5)}{s^3+15s^2-50s-400}$$

The system is type 0

$$K_p = \lim_{s\to 0} G_e(s) = -\frac{100\times 5}{400} = -\frac{5}{4}$$

$$e(\infty) = \frac{1}{1+K_p} = \frac{1}{1-(5/4)} = -4$$

* Sensitivity

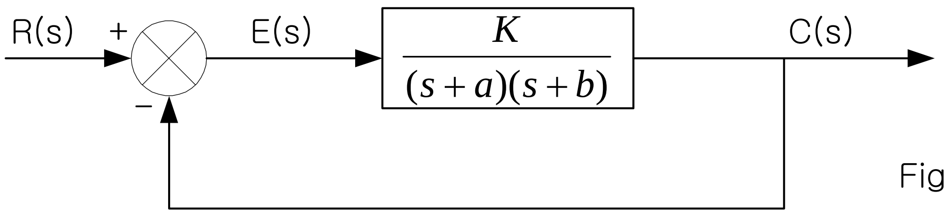
$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\text{Fractional change in the function, F}}{\text{Fractional change in the parameter, P}}$$

$$= \lim_{\Delta P \to 0} \frac{\dfrac{\Delta F}{F}}{\dfrac{\Delta P}{P}} = \lim_{\Delta P \to 0} \frac{P}{F} \frac{\Delta F}{\Delta P}$$

$$S_{F:P} = \frac{P}{F} \frac{\delta F}{\delta P}$$

Example 7.12

Problem : Find the sensitivity of the steady-state error to changes in parameter K and parameter a for the system shown in Figure with a step input



Figure

Solution

The steady-state error for this type 0 system is

$$e(\infty)=\frac{1}{1+K_p}=\frac{1}{1+\frac{K}{ab}}=\frac{ab}{ab+K}$$

The sensitivity of $e(\infty)$ *to changes in parameter a is*

$$S_{e:a}=\frac{a}{e}\frac{\delta e}{\delta a}=\frac{a}{\left(\frac{ab}{ab+K}\right)}\frac{(ab+K)b-ab^2}{(ab+K)^2}=\frac{K}{ab+K}$$

The sensitivity of $e(\infty)$ *to changes in parameter k is*

$$S_{e:K}=\frac{K}{e}\frac{\delta e}{\delta K}=\frac{K}{\left(\frac{ab}{ab+K}\right)}\frac{-ab}{(ab+K)^2}=\frac{-K}{ab+K}$$

Home Work #4 (Due date: one week from today)

<Chapter 6 >

1. Do the review questions #1, #5, #12, #15
2. Solve problem #5 on p. 355
3. Solve problem #6 on p. 355
4. Solve problem #35 on p. 359
5. Solve problem #45 on p. 361

<Chapter 7 >

6.Do the review question #1, #3, #6, #11
7. Solve problem #4 on p. 406
8. Solve problem #20 on p. 409
9. Solve problem #32 on p. 411