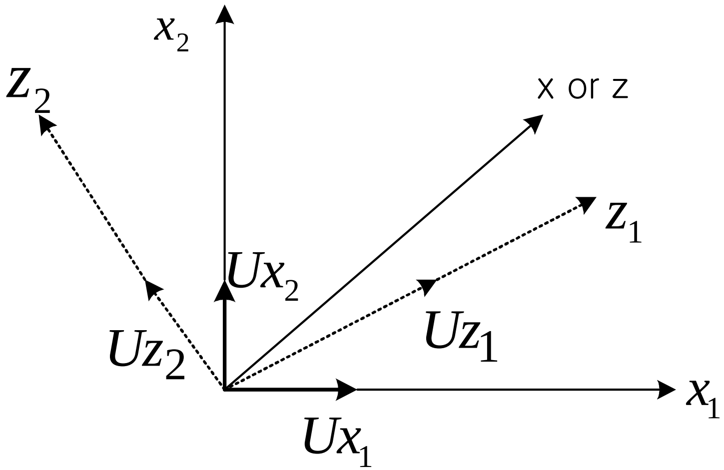
# * Similarity Transformations

Similar systems have the same transfer function and hence the same poles and eigenvalues




$x_1$ *and* $x_2$ *are orthogonal*

$;|Uz_1| = |Uz_2| = 1$

(linearly independent basis vector)

$$x = x_1 U_{x_1} + x_2 U_{x_2} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (1)$$

$$z = z_1 U_{z_1} + z_2 U_{z_2} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad (2)$$

$$U_{z_1} = p_{11}U_{x_1} + p_{21}U_{x_2} = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}$$

$$U_{z_2} = p_{12}U_{x_1} + p_{22}U_{x_2} = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} \qquad (3)$$

plugging (3) into (2) yields

$$x = (z_1 p_{11} + z_2 p_{12})\, U_{x_1} + (z_1 p_{21} + z_2 p_{22})\, U_{x_2}$$

$$= \begin{bmatrix} z_1 p_{11} + z_2 p_{12} \\ z_1 p_{21} + z_2 p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Pz$$

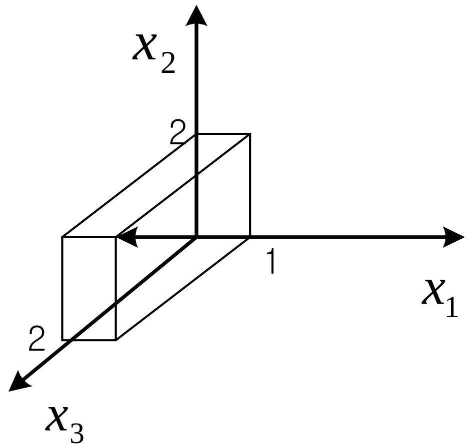
$$or \quad x = Pz \qquad (4)$$

$$z = P^{-1}x \qquad (5)$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} U_{z_1} & U_{z_2} \end{bmatrix} \qquad (6)$$

Ex. 5.9) Transfer the vector

$$x = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},\ U_{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ U_{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ U_{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$




$$U_{z_1} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ U_{z_2} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ U_{z_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$z = z_1 U_{z_1} + z_2 U_{z_2} + z_3 U_{z_3}$$

$$x = z_1 \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + z_2 \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + z_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} z_3 \\ \left(1/\sqrt{2}\right) z_1 - \left(1/\sqrt{2}\right) z_2 \\ \left(1/\sqrt{2}\right) z_1 + \left(1/\sqrt{2}\right) z_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = Pz$$

directly from (6)

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} U_{z_1} & U_{z_2} & U_{z_3} \end{bmatrix}$$

$$z = P^{-1}x = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.83 \\ 0 \\ 1 \end{bmatrix}$$

* Transforming the state equations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \qquad (7)$$

substituting (4) into (7) produces

$$P\dot{z} = APz + Bu$$

$$\dot{z} = P^{-1}APz + P^{-1}Bu$$

$$y = CPz + Du$$

$$A \Rightarrow P^{-1}AP \qquad B \Rightarrow P^{-1}B$$

$$C \Rightarrow CP \qquad D \Rightarrow D \tag{8}$$

*Review $T(s) = \dfrac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$ (9)

Substituting (8) into (9)

$$T(s) = CP(sI - P^{-1}AP)^{-1}P^{-1}B + D$$

using $(LM)^{-1} = M^{-1}L^{-1}$

$$T(s) = CP[P(sI - P^{-1}AP)]^{-1}B + D$$

$$= C[P(sI - P^{-1}AP)P^{-1}]^{-1}B + D$$

$$\text{since} \quad (sI - P^{-1}AP)P^{-1} = sP^{-1} - P^{-1}A$$

$$T(s) = C[P(sP^{-1} - P^{-1}A)]^{-1}B + D$$

$$= C[(sI - A)]^{-1}B + D \qquad (10)$$

$eq(9)$ *and* $eq(10)$ *are the same* .

$$\det(sI - A) = 0$$

$$\det(sI - P^{-1}AP) = 0$$

$$\det(sP^{-1}P - P^{-1}AP)$$

$$= \det[P^{-1}(sI - A)P] = \det[P^{-1}]\det(sI - A)\det[p]$$

$$\det[P^{-1}]\det[P] = \det[I] = 1$$

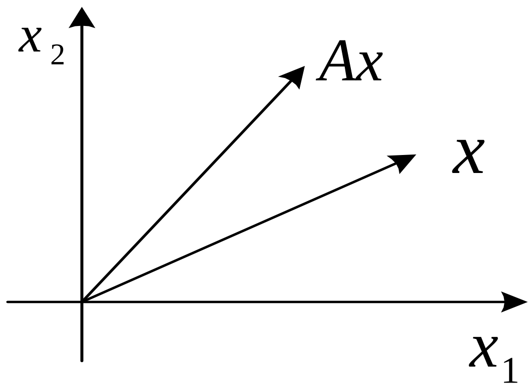
$$\therefore \det(sI - P^{-1}AP) = \det(sI - A) = 0$$

$$(\text{since} \quad \det[AB] = \det[A]\det[B])$$

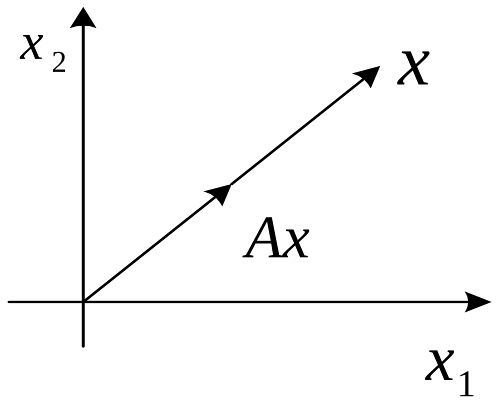
•Eigenvector : The eigenvector of the matrix A are all vectors, $x_i \neq 0$, which under the transformation A become multiples of themselves

$$Ax_i = \lambda_i x_i \qquad (1)$$

*where* $\lambda_i$*'s are* constant




x is not an eigenvector




An eigenvector

*Eigenvalue : The eigenvalues of the matrix A are the values of $\lambda_i$ that satisfy Eq. (1) for $x_i \neq 0$

$$From\ eq.(1)$$

$$0 = (\lambda_i I - A)x_i$$

$$(\lambda_i I - A)^{-1} 0 = x_i = \frac{adj(\lambda_i I - A)}{\det(\lambda_i I - A)} 0$$

$$\text{since } x_i \neq 0,\ a\ nonzero\ solution\ exists\ if$$

$$\det(\lambda_i I - A) = 0$$

* Diagonalizing a system matrix

If the eigenvectors of the matrix A are chosen as the basis vectors of a transformation, P, the resulting matrix will be diagonal.

since $x_i$ *are eigenvectors of* $Ax_i = \lambda_i x_i$

$$AP = PD$$

$$D = P^{-1}AP$$

*where* $P = \begin{bmatrix} x_1, & x_2, & x_3, & \cdot & \cdot & \cdot, & x_n \end{bmatrix}$ *and* $D$ *is a diagonal matrix consisting of* $\lambda_i$*'s*

*Ex.* 5.11) *Find the eigenvectors of the matrix*

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+3 & -1 \\ -1 & \lambda+3 \end{vmatrix} = \lambda^2 + 6\lambda + 8 = (\lambda+2)(\lambda+4)$$

*eigenvalues are* $\lambda = -2$ *and* $\lambda = -4$

$$Ax_i = \lambda x_i$$

*For* $\lambda = -2$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-3x_1 + \quad x_2 = -2x_1$$

$$x_1 - 3x_2 = -2x_2 \quad \Rightarrow x_1 = x_2 \rightarrow x = \begin{bmatrix} c \\ c \end{bmatrix}$$

$$For\ \lambda = -4$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-3x_1 + \quad x_2 = -4x_1$$

$$x_1 - 3x_2 = -4x_2 \quad \Rightarrow x_2 = -x_1 \rightarrow x = \begin{bmatrix} c \\ -c \end{bmatrix}$$

*one choice of eigenvectors*

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Ex.\ 5.12)\ \textit{Find the similar diagonal system, given}$$

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix} x$$

$$choose \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}, \text{ then } P^{-1} = \frac{\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}}{-2}$$

$$P^{-1}AP = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$$P^{-1}B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$CP = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} z + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & -1 \end{bmatrix} z$$

The system matrix is diagonal with eigenvalues along the diagonal.