* Finding transfer function by block diagram reduction

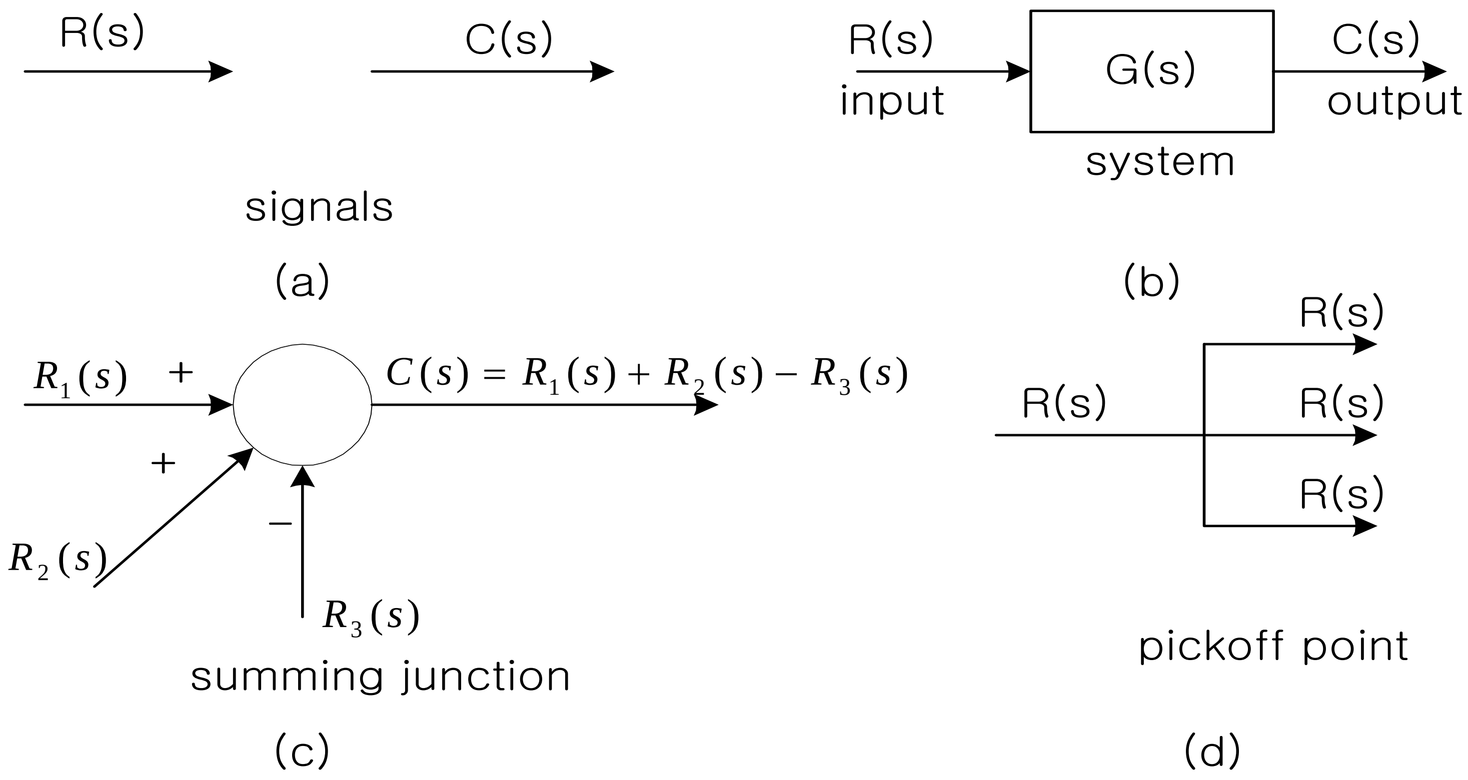
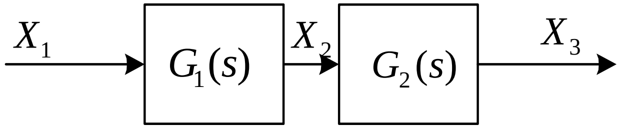
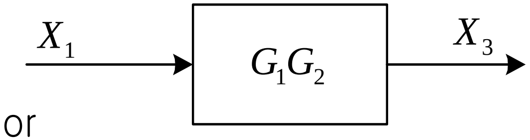
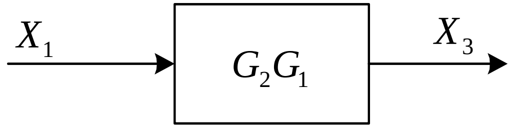
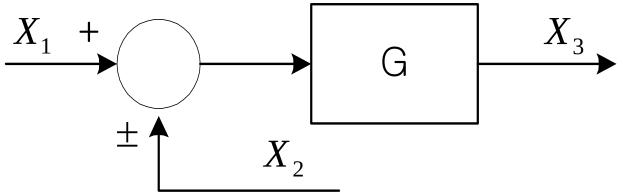
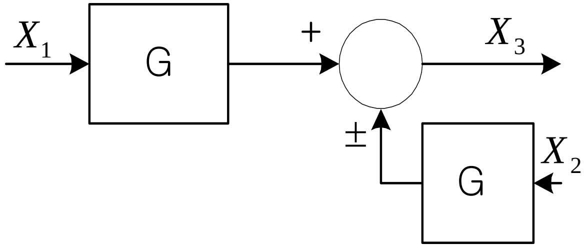
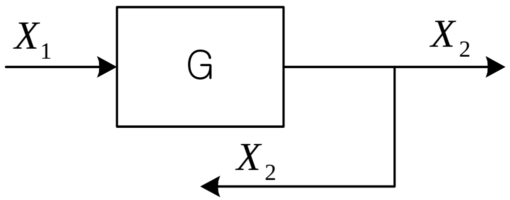
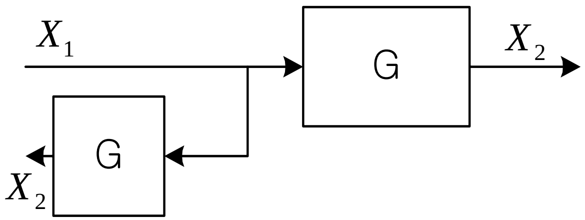
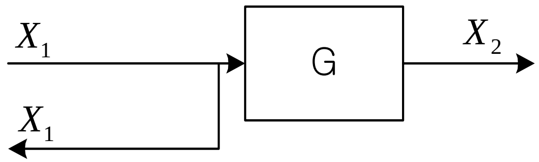
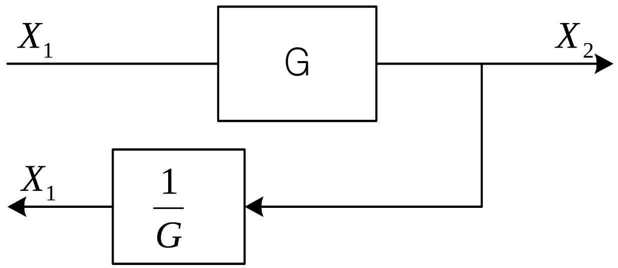
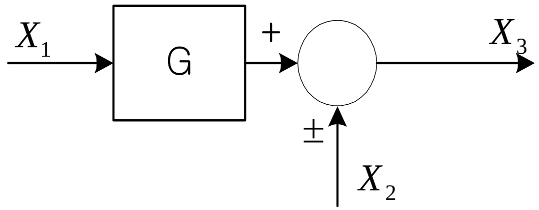
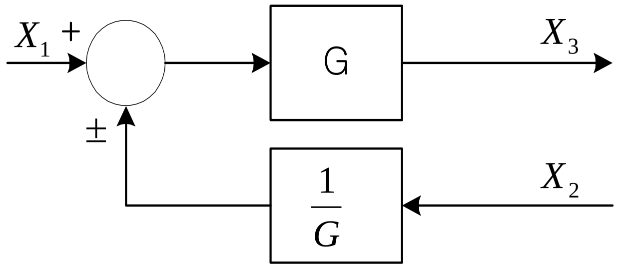
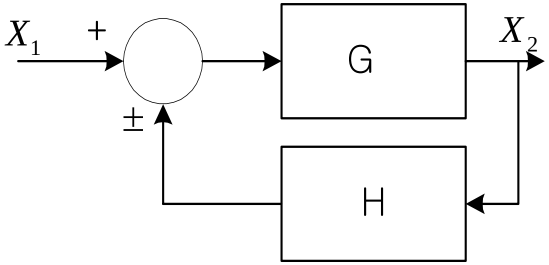
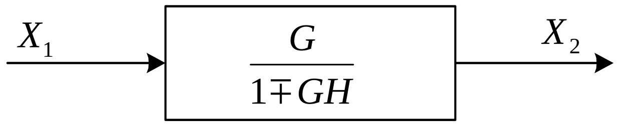


Figure 5.2
Components of a block diagram for a linear,time-invariant system

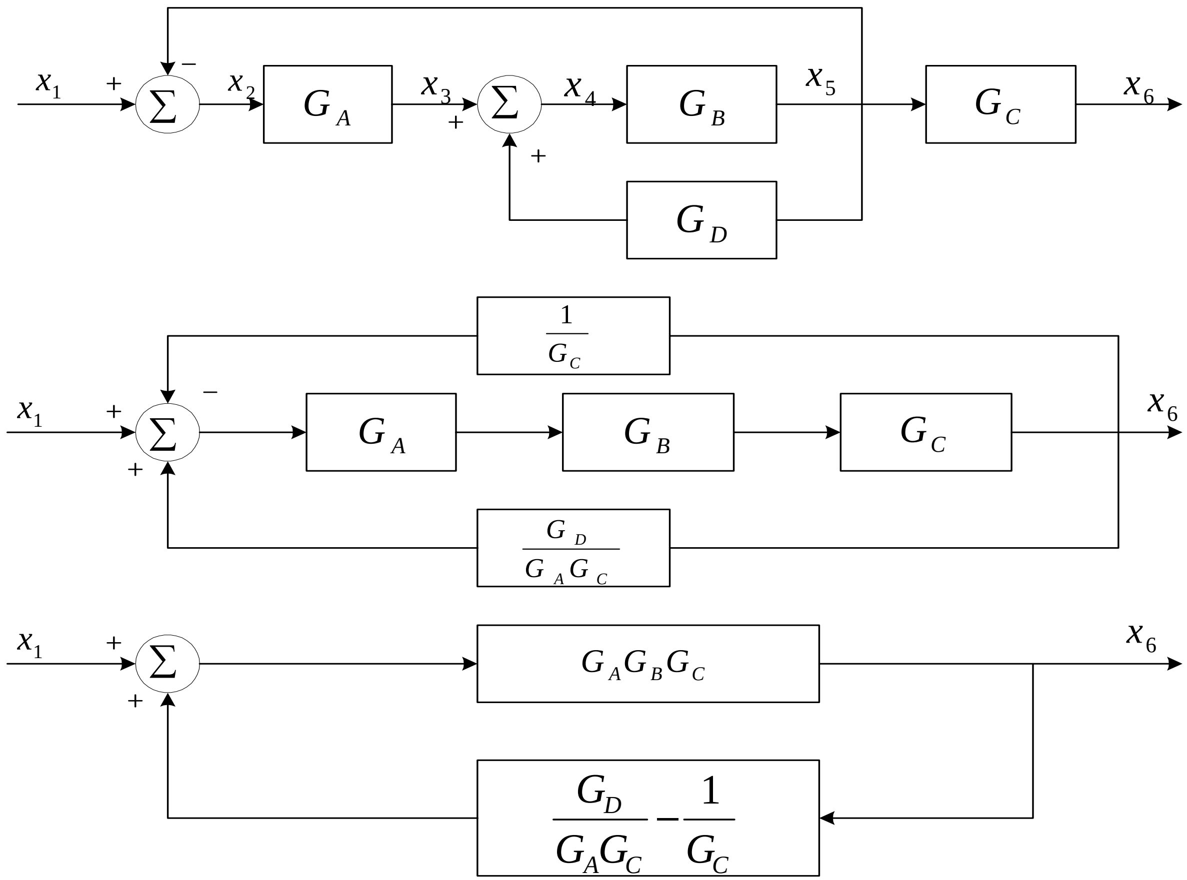
* Block Diagram Transformations

Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		 or 
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		

Transformation	Original Digram	Equivalent Diagram
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6.Eliminating a feedback loop		

Ex) Determine overall transfer Function

$$G_{61} = \frac{X_6}{X_1}$$



By standard feedback form

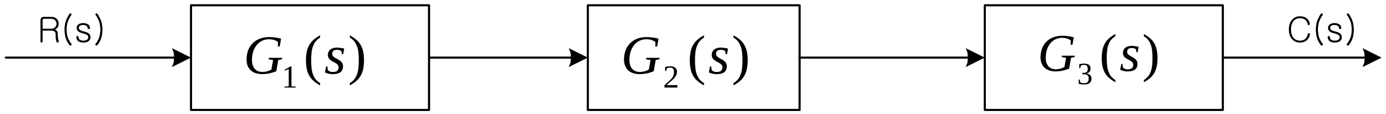
$$\therefore\ G_{61} = \frac{x_6}{x_1} = \frac{G_A G_B G_C}{1 - G_A G_B \left(\frac{G_D}{G_A} - 1 \right)}$$

* Representation of Mutiple Subsystem

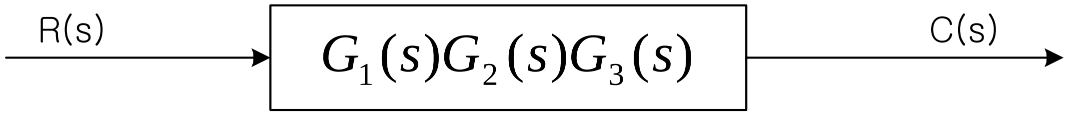
1. Block Diagram
2. Signal Flow Graphs

1) Cascade Form

a) Block diagram

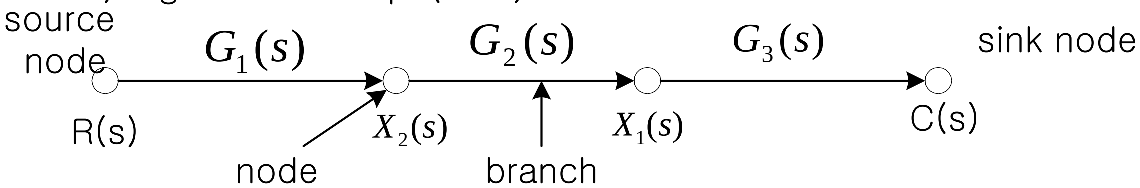


simplified or reduced



$$C(s) = G_1(s)G_2(s)G_3(s)R(s)$$

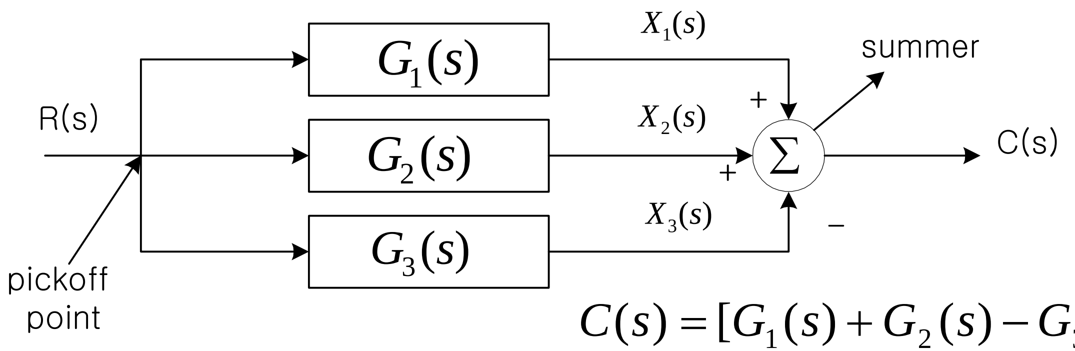
b) Signal Flow Graph(SFG)



$$path\ \ gain \Rightarrow G_e(s) = G_1(s)G_2(s)G_3(s)$$

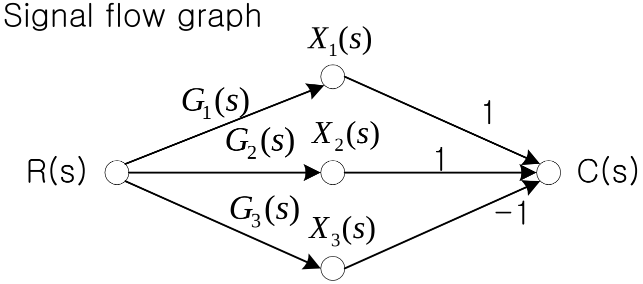
2. Parallel Form

a) Block diagram



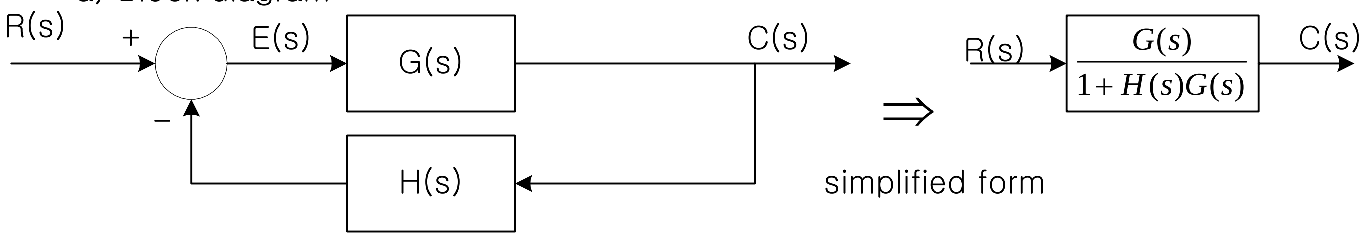
$$C(s) = [G_1(s) + G_2(s) - G_3(s)]R(s)$$

b)Signal flow graph

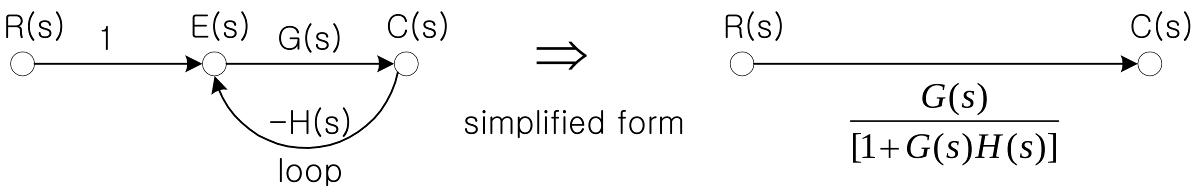


3. Feedback Form

a) Block diagram



b) Signal flow graph



* Mason's rule for reduction of SFG

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{k} T_i \Delta_i}{\Delta}$$

where

k = number of forward paths

T_i = the ith forward path gain

$\Delta = 1 - \sum$ individual loop gains

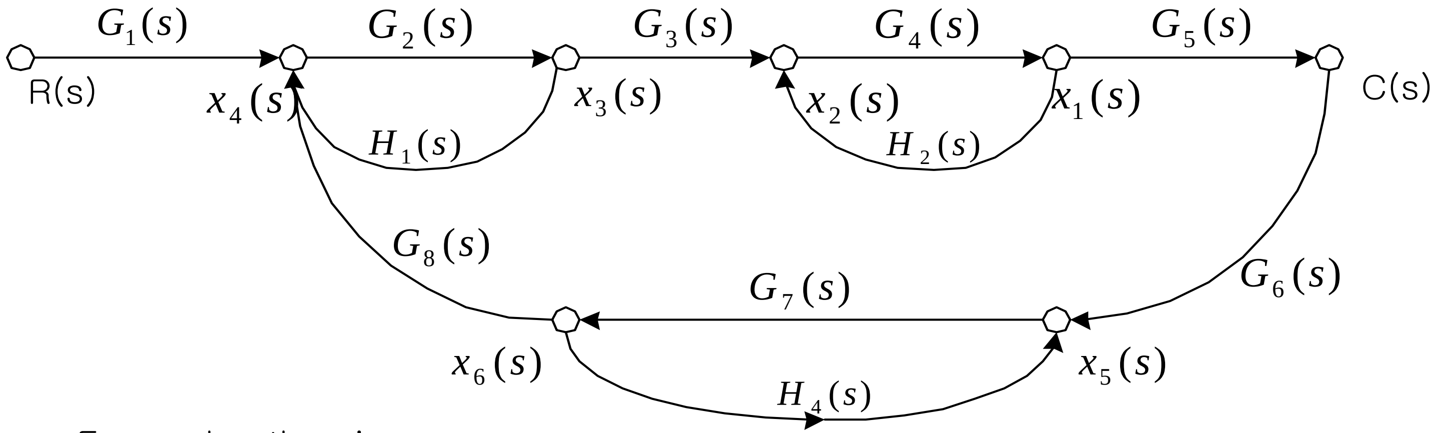
$+ \sum$ nontouching – loop gains taken two at a time

$- \sum$ nontouching – loop gains taken three at a time

$+ \sum$ nontouching – loop gains taken four at a time

$\Delta_i = 1 - \sum$ loop gains not touching the ith forward path

Ex) Using Mason's rule find $\dfrac{C(s)}{R(s)}$



* Forward path gain

$$G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)$$

* Closed-loop gains

(1) $G_2(s)H_1(s)$

(2) $G_4(s)H_2(s)$

(3) $G_7(s)H_4(s)$

(4) $G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)$

* Nontouching loops taken two at a time

(5) *loop* (1) *and* *loop* (2) $G_2(s)H_1(s)G_4(s)H_2(s)$

(6) *loop* (1) *and* *loop* (3) $G_2(s)H_1(s)G_7(s)H_4(s)$

(7) *loop* (2) *and* *loop* (3) $G_4(s)H_2(s)G_7(s)H_4(s)$

$* \textit{Nontouching loops taken three at a time}$

8) $loops (1),(2),(3)\ ;\ G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)$

$\Delta = 1-[(1)+(2)+(3)+(4)]+[(5)+(6)+(7)]-(8)$

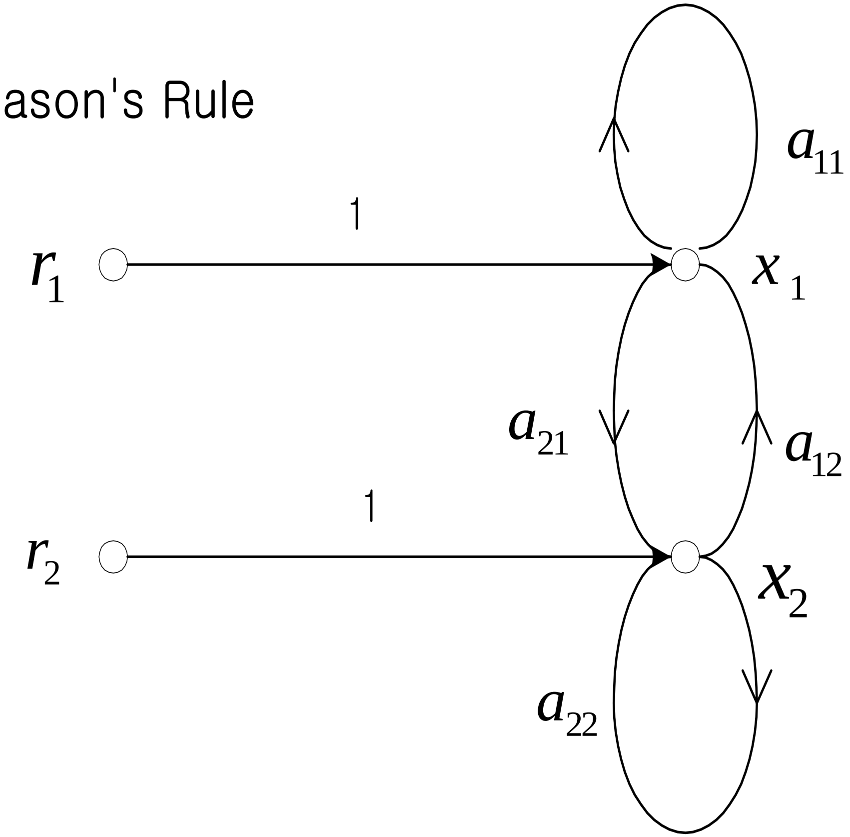
portion of Δ *not touching the forward path*

$\Delta_1 = 1 - G_7(s)H_4(s)$

$$G(s) = \frac{C(s)}{R(s)} = \frac{T_1\Delta_1}{\Delta}$$

$$= \frac{G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)[1-G_7(s)H_4(s)]}{\Delta}$$

* Idea for Mason's Rule



$$a_{11}x_1 + a_{12}x_2 + r_1 = x_1 \rightarrow (1 - a_{11})x_1 + (-a_{12})x_2 = r_1$$

$$a_{21}x_1 + a_{22}x_2 + r_2 = x_2 \rightarrow (-a_{21})x_1 + (1 - a_{22})x_2 = r_2$$

By Cramer's Rule

$$x_1 = \frac{\begin{vmatrix} r_1 & -a_{12} \\ r_2 & 1-a_{22} \end{vmatrix}}{\begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix}} = \frac{(1-a_{22})r_1 + a_{12}r_2}{(1-a_{11})(1-a_{22}) - a_{12}a_{21}}$$

$$= \frac{(1-a_{22})}{\Delta} r_1 + \frac{a_{12}}{\Delta} r_2$$

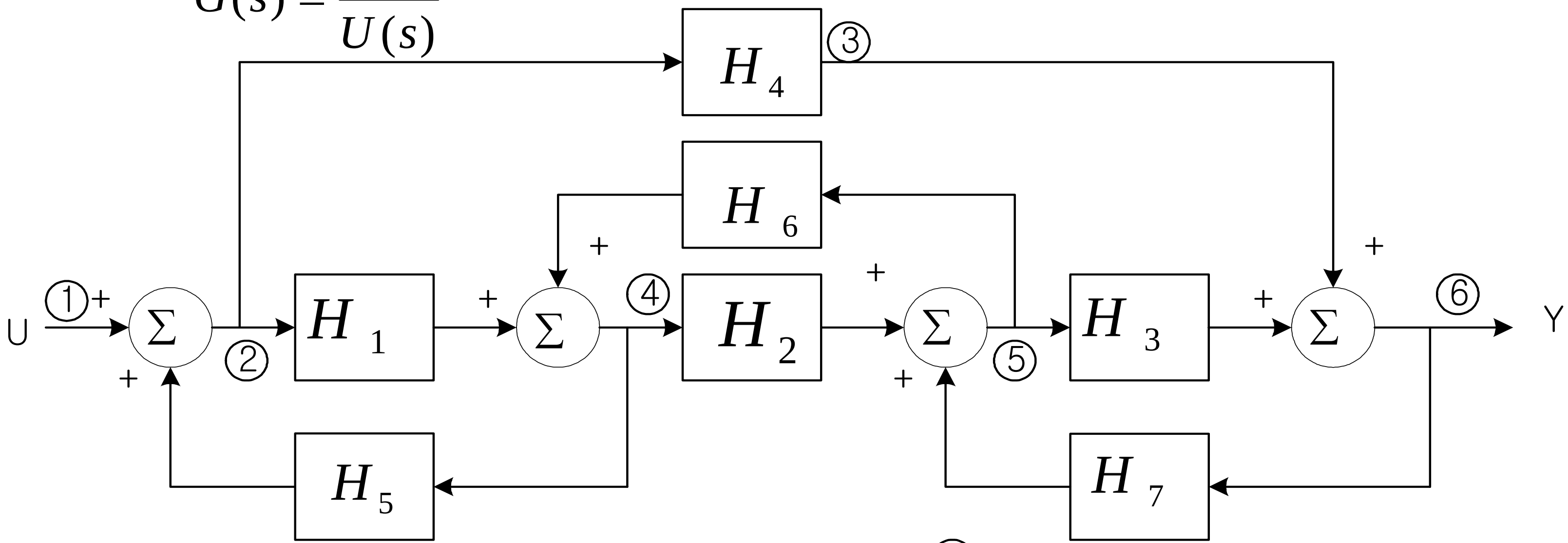
$$x_2 = \frac{\begin{vmatrix} 1-a_{11} & r_1 \\ -a_{21} & r_2 \end{vmatrix}}{\begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix}} = \frac{a_{21}r_1 + (1-a_{11})r_2}{\Delta}$$

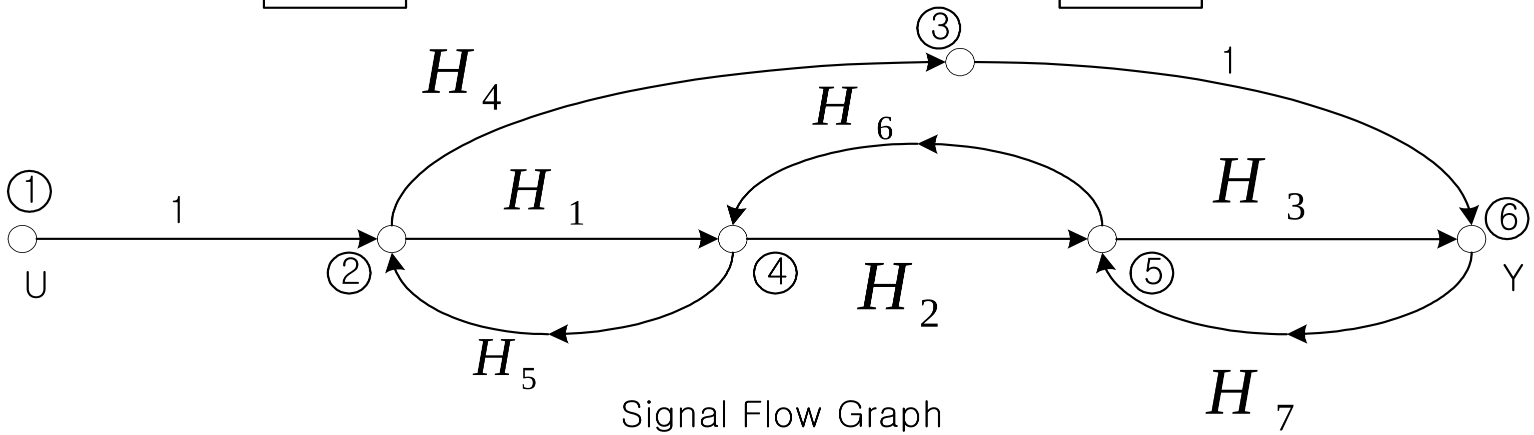
$$= \frac{a_{21}}{\Delta} r_1 + \frac{(1-a_{11})}{\Delta} r_2$$

$$where\ \Delta = (1-a_{11})(1-a_{22}) - a_{12}a_{21} = 1 - a_{11} - a_{22} - a_{12}a_{21} + a_{11}a_{22}$$

Ex) Find the transfer Function

$$G(s) = \frac{Y(s)}{U(s)}$$





Signal Flow Graph

solution.

forward path	*path gain*
12456	$G_1 = H_1 H_2 H_3$
1236	$G_2 = H_4$
	loop path gain
242	$l_1 = H_1 H_5 (\textit{does not touch } l_3)$
454	$l_2 = H_2 H_6$
565	$l_3 = H_3 H_7 (\textit{does not touch } l_1)$
236542	$l_4 = H_4 H_7 H_6 H_5$

and the determinants are

$$\Delta = 1 - \left(H_1 H_5 + H_2 H_6 + H_3 H_7 + H_4 H_7 H_6 H_5\right) + \left(H_1 H_5 H_3 H_7\right)$$

$$\Delta_1 = 1 - 0$$

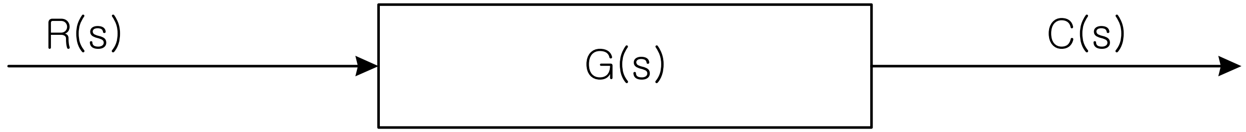
$$\Delta_2 = 1 - H_2 H_6$$

Therefore,

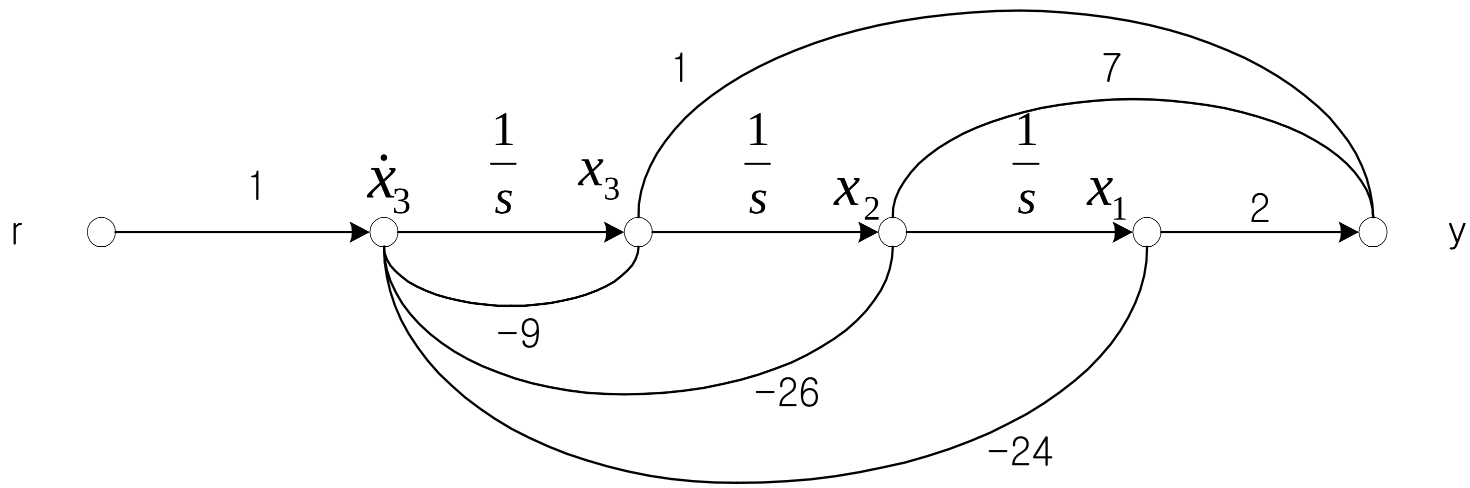
$$\frac{Y(s)}{U(s)} = \frac{H_1 H_2 H_3 + H_4 - H_4 H_2 H_6}{1 - H_1 H_5 - H_2 H_6 - H_3 H_7 - H_4 H_7 H_6 H_5 + H_1 H_5 H_3 H_7}$$

* Decompositions of transfer Function
  * Types
    1. Phase-variable(canonical) form
    2. Controller canonical form
    3. Cascade form
    4. Parallel form
       1) First-order pole
       2) multi-order pole
    5. Dual Phase-variable form

1. Phase-variable form



$$G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$



$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} x$$

The system matrix A has the coefficients of the system's characteristic polynomial along the last row.

2. Controller canonical form

This form is obtained from the phase-variable form simply by ordering the phase-variable the reverse order

$$\dot{x} = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

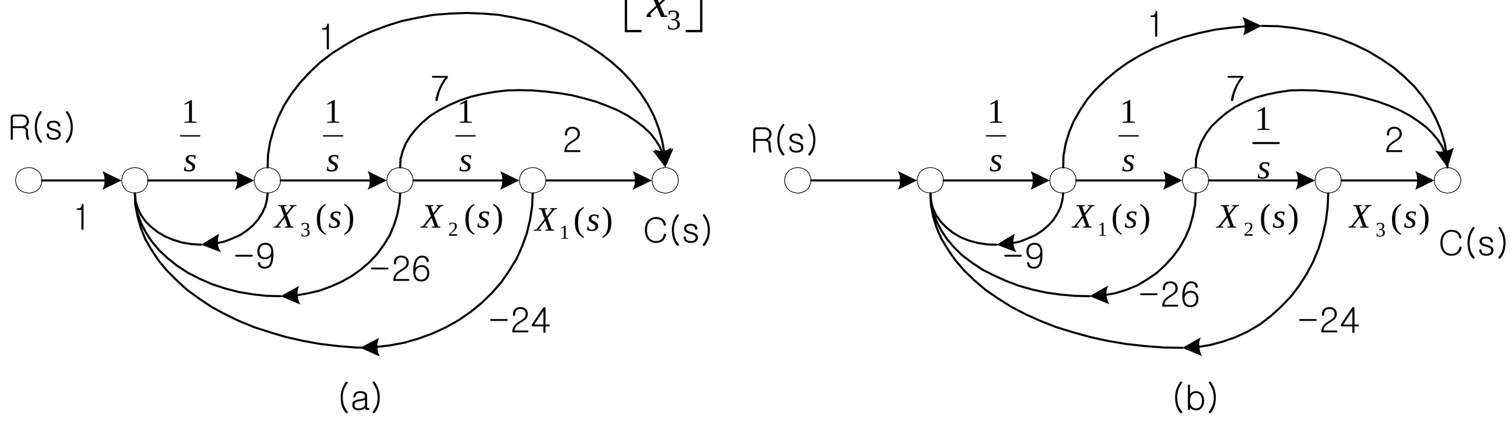
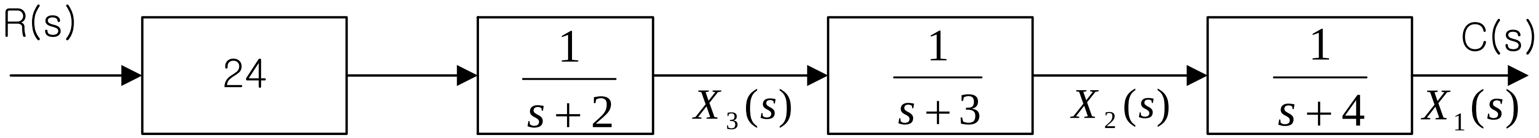


Figure 5.27
a. Phase-variable form
b. controller canonical form

3. Cascade form

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24} = \frac{24}{(s+2)(s+3)(s+4)}$$



For each stage

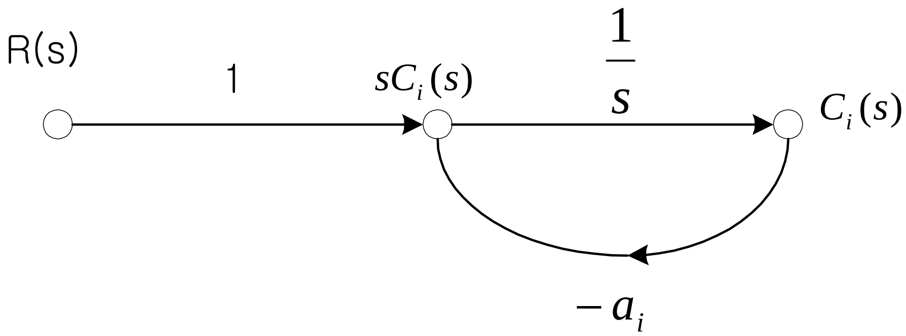
$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s+a_i)}$$

$$(s+a_i)C_i(s) = R_i(s)$$

$By\ \ ILT,$

$$\frac{dc_i}{dt} + a_i c_i = r_i(t)$$

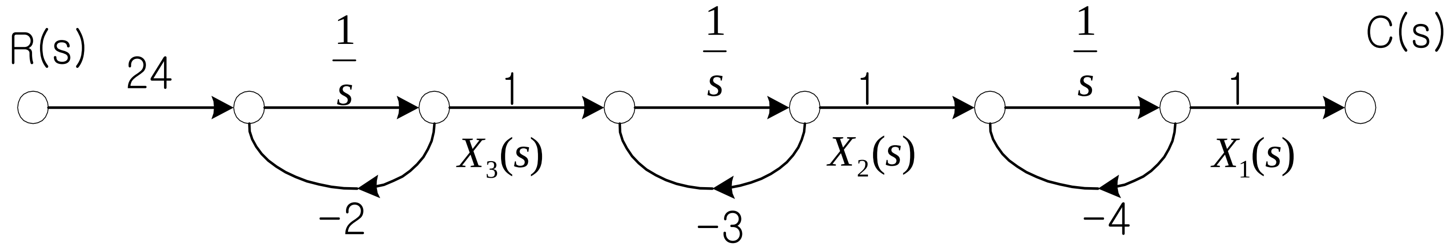
$$\frac{dc_i}{dt} = -a_i c_i + r_i(t)$$



$$\dot{x}_1 = -4x_1 + x_2$$

$$\dot{x}_2 = -3x_2 + x_3$$

$$\dot{x}_3 = -2x_3 + 24r$$



$$\dot{x} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = [1\ 0\ 0]x$$

The system matrix A has the poles along the diagonal and the terms relative to the internal system itself

4. Parallel form

(1) With first-order pole

If no system pole is a repeated root, 'A' matrix becomes purly diagonal

By PFE,

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$

$$= \frac{12}{s+2} - \frac{24}{s+3} + \frac{12}{s+4}$$

$$C(s) = \frac{12}{s+2}R(s) - \frac{24}{s+3}R(s) + \frac{12}{s+4}R(s)$$

$$\dot{x}_1 = -2x_1 \qquad\qquad\qquad +12r(t)$$

$$\dot{x}_2 = \qquad -3x_2 \qquad\quad -24r(t)$$

$$\dot{x}_3 = \qquad\qquad -4x_3 + 12r(t)$$

each equation has only one state-variable

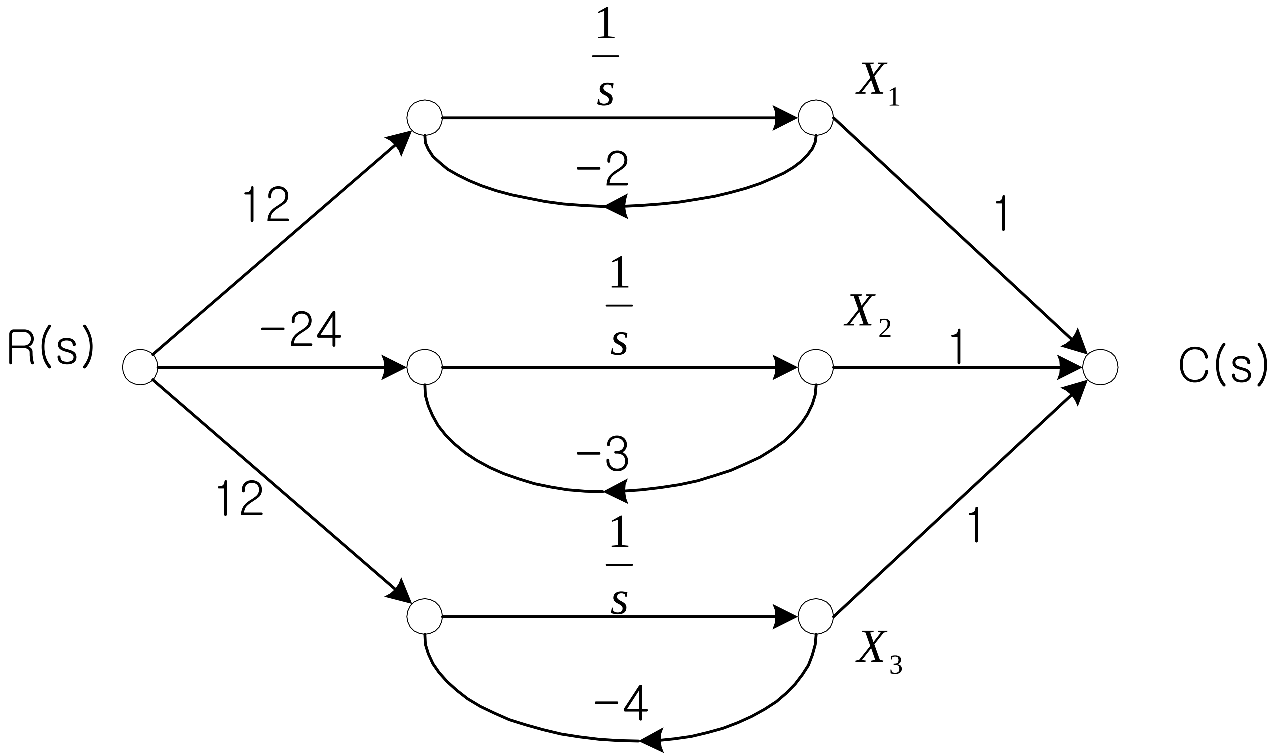
$\Rightarrow$ independent

$\Rightarrow$ decoupled(purely diagonal)

$$y = c(t) = x_1 + x_2 + x_3$$

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} x + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

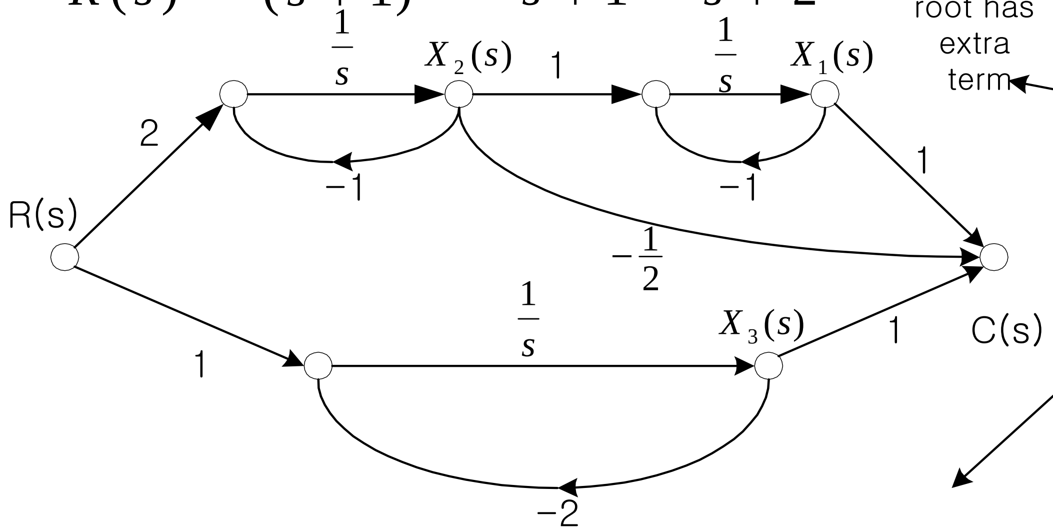

R(s)
C(s)
12
-24
12
1/s
1/s
1/s
-2
-3
-4
X1
X2
X3
1
1
1

(2) With multiple-order poles (repeated roots)

$$\frac{C(s)}{R(s)} = \frac{s+3}{(s+1)^2(s+2)}$$

By PFE

$$\frac{C(s)}{R(s)} = \frac{2}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2}$$



Repeated root has extra term

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = \quad - x_2 \quad + 2r(t)$$
$$\dot{x}_3 = \quad - 2x_3 + r(t)$$

$$y(t) = c(t) = x_1 - \frac{1}{2}x_2 + x_3$$

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} 1 & -\frac{1}{2} & 1 \end{bmatrix} x$$

"A" matrix is called the "Jordan Canonical Form"

Not purely diagonal but the system poles along the diagonal.

5. Observer canonical Form (or Dual phase-variable form)

- Useful for systems with finite zeros.

$$\frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

divide by s^3

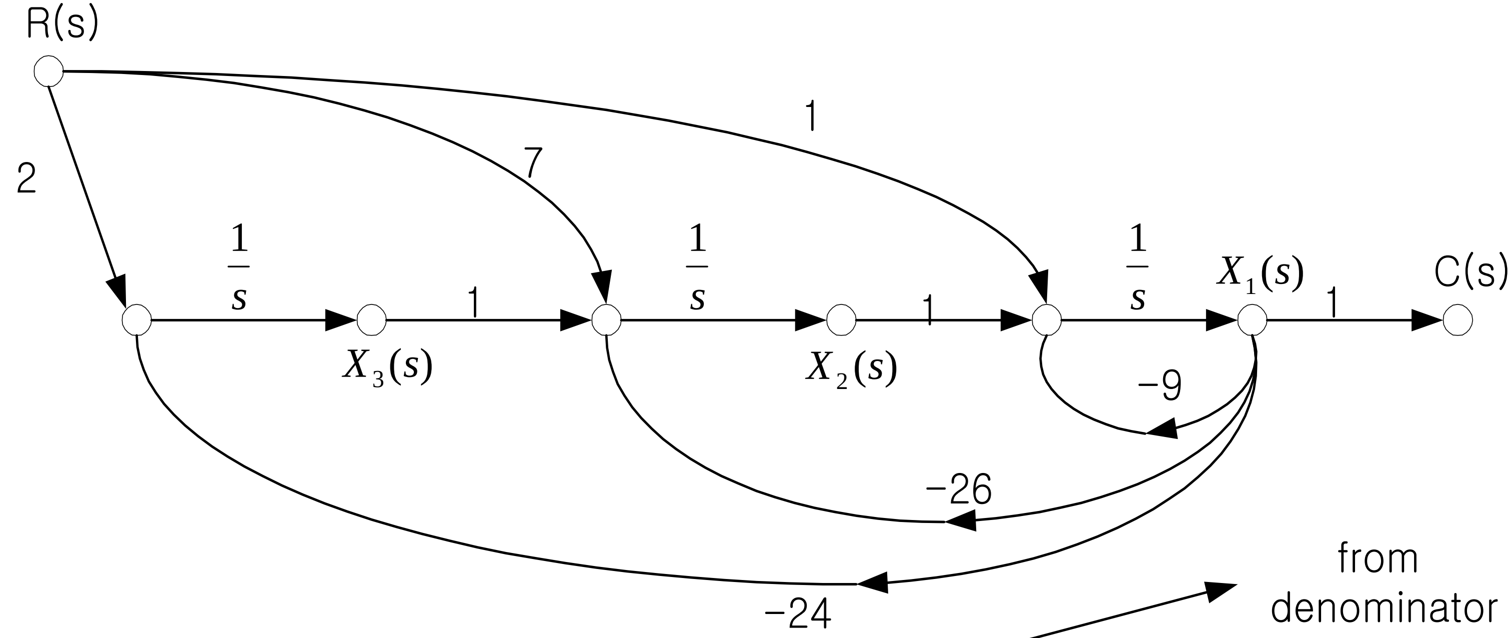
$$\frac{C(s)}{R(s)} = \frac{\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3}}{1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3}}$$

crossmultiplying yields

$$\left(\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3}\right)R(s) = \left(1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3}\right)C(s)$$

$$C(s) = \frac{1}{s}[R(s) - 9C(s)] + \frac{1}{s^2}[7R(s) - 26C(s)] + \frac{1}{s^3}[2R(s) - 24C(s)]$$

$$or\ C(s) = \frac{1}{s}\left\{[R(s) - 9C(s)] + \frac{1}{s}\left\{[7R(s) - 26C(s)] + \frac{1}{s}[2R(s) - 24C(s)]\right\}\right\}$$



$$\dot{x}_1 = -9x_1 + x_2 + r(t)$$
$$\dot{x}_2 = -26x_1 + x_3 + 7r(t)$$
$$\dot{x}_3 = -24x_1 + 2r(t)$$

$$\dot{x} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

from denominator

From numerator

This form is dual with the phase-variable form.