Chapter 4. Time Response

Things to know

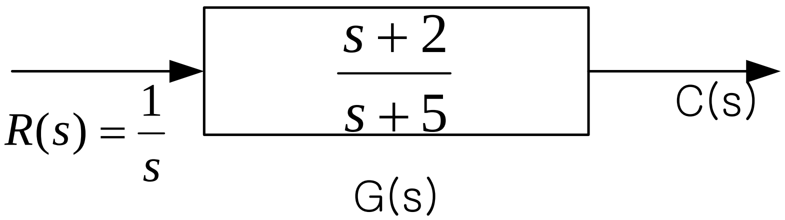
- **Poles and zeros**
- **Time constant, rise time, settling time**
- **Response of a second order system: overdamped, underdamped, critically damped**
- **Pole-zero cancellation**

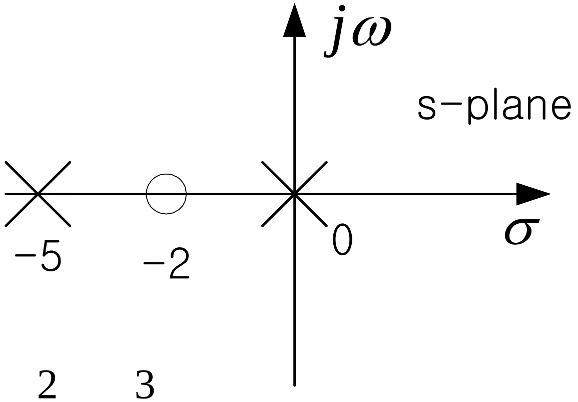
Time Response

* Poles and Zeros of First-order system

1. Pole of the input function generates the form of the forced response.
2. Pole of the transfer function generates the form of the natural response.
3. A pole on the real axis generates an exponential response of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis. The farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.
4. The zeros and poles generates the amplitudes for both the forced and natural responses.

Ex)





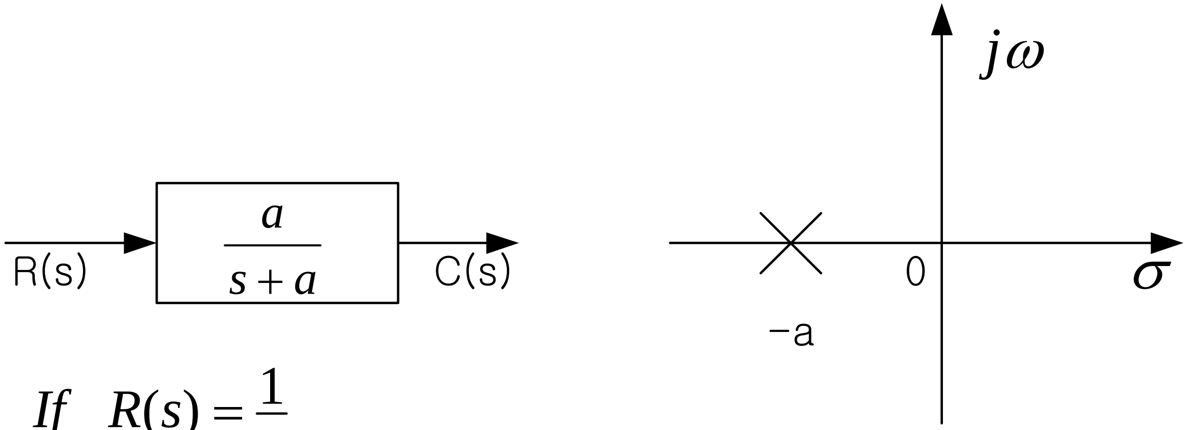
$$C(s)=\frac{s+2}{s(s+5)}=\frac{A}{s}+\frac{B}{s+5}=\frac{\frac{2}{5}}{s}+\frac{\frac{3}{5}}{s+5}$$

$$c(t)=\frac{2}{5}+\frac{3}{5}e^{-5t}, \qquad t\geq 0$$

Forced response

Natural response

* First-Order systems



$$If \quad R(s) = \frac{1}{s}$$

$$C(s) = G(s)R(s) = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

Taking ILT

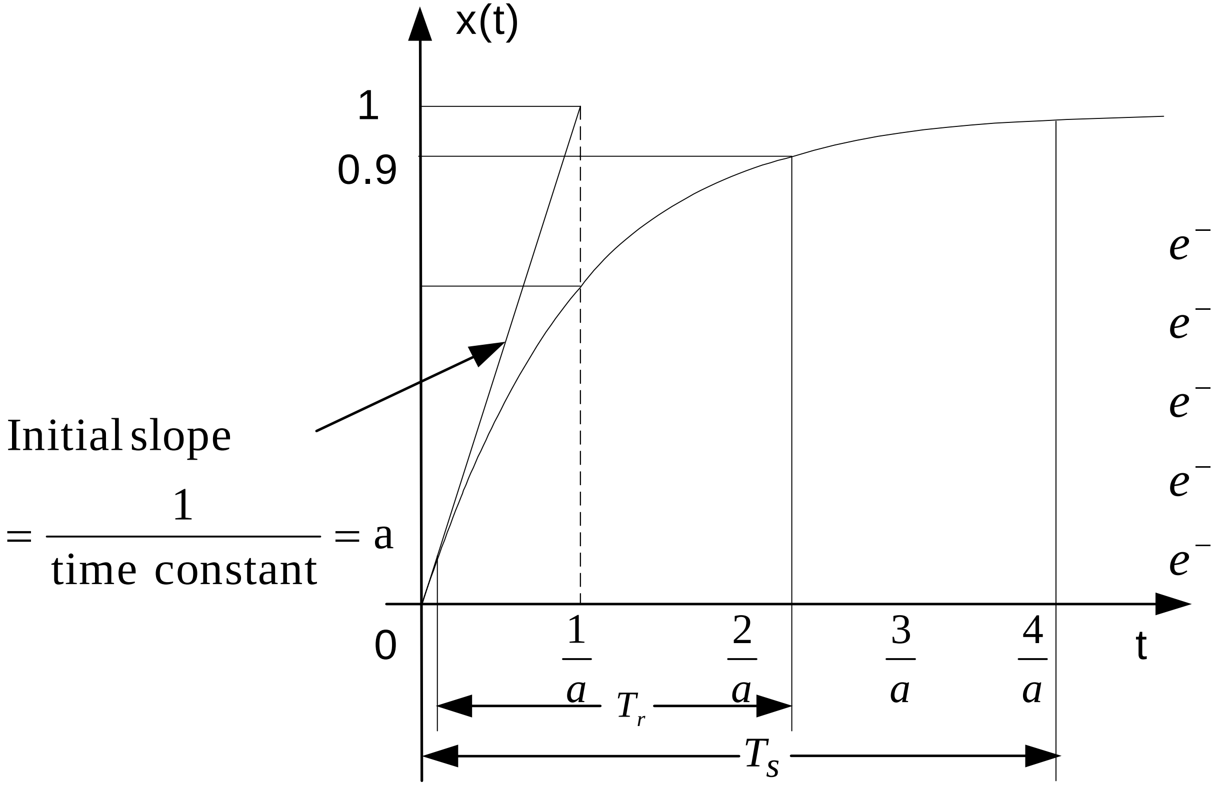
$$c(t) = 1 - e^{-at} \ , \ t \geq 0$$

$$= c_f(t) + c_n(t)$$

$$where \quad c_f(t) \ is \ forced \ response \quad (here, \ c_f(t) = 1)$$

$$c_n(t) \ is \ natural \ response \quad (here, \ c_n(t) = -e^{-at})$$

- Time Constant (τ) – reach 37% of its original for a decaying function
  – 63% of final value for a rising function



$$e^{-1} = 0.3679(\cong 0.37)$$

$$e^{-2} = 0.1353$$

$$e^{-3} = 0.0498$$

$$e^{-4} = 0.0163$$

$$e^{-5} = 0.0067 < 0.01 = 1\%$$

$$\tau = \frac{1}{a} \; or \; a = \frac{1}{\tau}$$

τ : *time* constant

a : exponential freq.

Further the pole from the imaginary axis, the faster the transient response.

- Rise Time (T_r): Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

$$1-e^{-at}=0.1 \quad 0.9=e^{-at} \quad at=0.11 \quad t_{0.1}=\frac{0.11}{a}$$

$$1-e^{-at}=0.9 \quad 0.1=e^{-at} \quad at=2.31 \quad t_{0.9}=\frac{2.31}{a}$$

$$T_r=t_{0.9}-t_{0.1}$$

$$\therefore T_r=\frac{2.2}{a}$$

- Settling time T_s ; Settling time is defined as the time for the response to reach 98% of its final value.

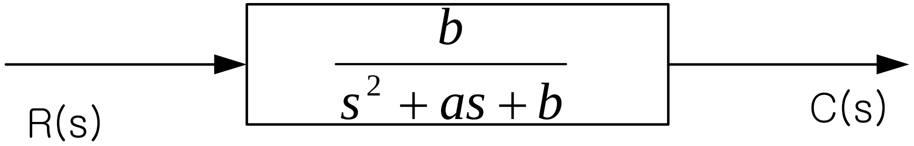
$$1-e^{-at}=0.98 \quad e^{-at}=0.02 \quad at\cong 3.9$$

$$T_s=\frac{3.9}{a}\cong\frac{4}{a}$$

* Second-order system Response

The parameters of a second-order system change the form of the response whereas a first-order system change the speed of the response

Basic System

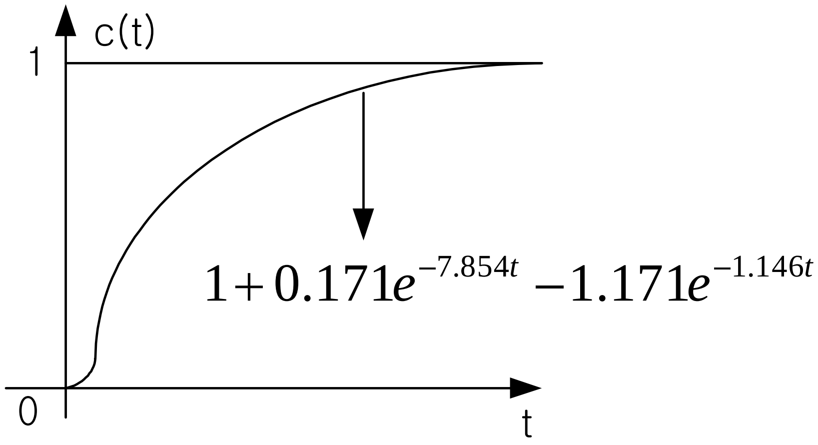


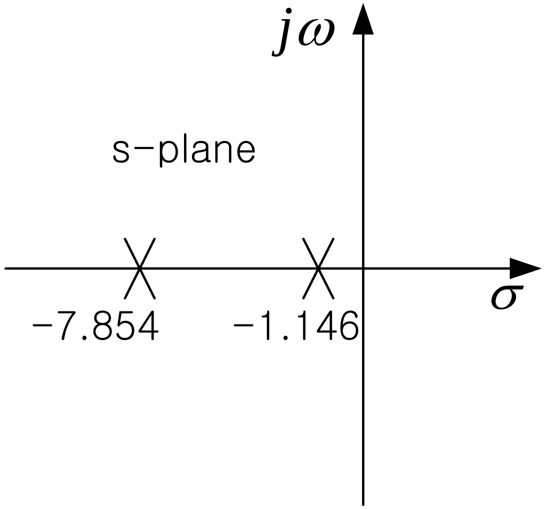
$$assume\ that\ R(s)=\frac{1}{s}$$

1. Overdamped response
(two distant real poles)

$$Transient\quad response\ : c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

$$Ex)\ G(s) = \frac{9}{s^2 + 9s + 9} = \frac{9}{(s + 7.854)(s + 1.146)}$$

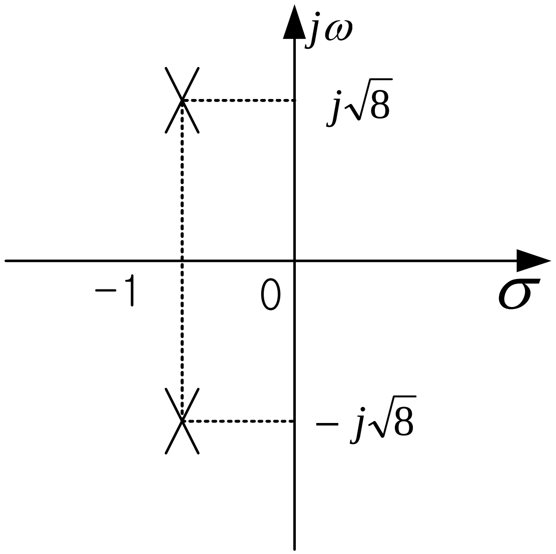


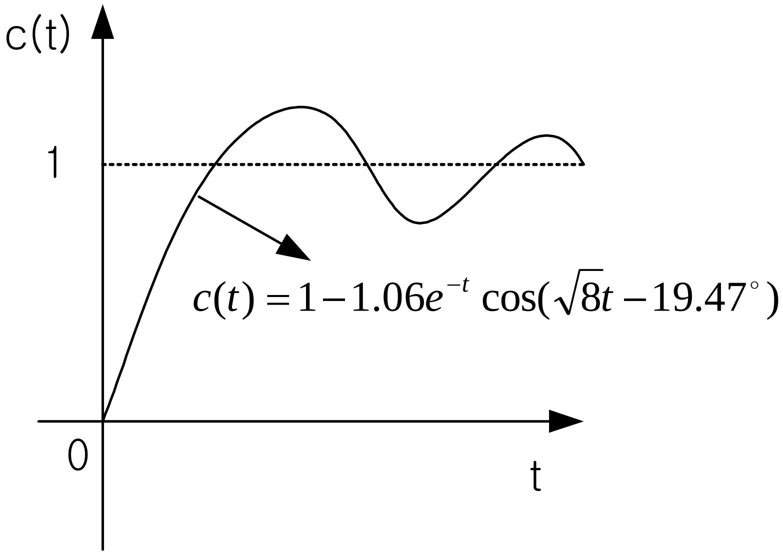


2. Underdamped response (complex poles)

$$\text{Transient response}: c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

$$Ex)\ G(s) = \frac{9}{s^2 + 2s + 9} = \frac{9}{(s+1+j\sqrt{8})(s+1-j\sqrt{8})}$$

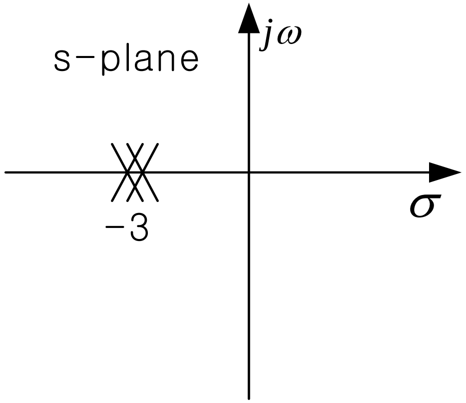


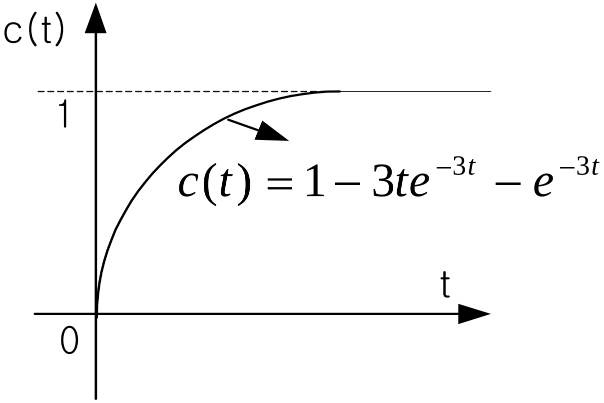


3. Critically damped (doubled real poles)
– fastest without overshoot

$$Transient\ \ response\ :c(t) = K_1 e^{-\sigma_1 t} + K_2 te^{-\sigma_2 t}$$

$$Ex)\ \ G(s) = \frac{9}{s^2 + 6s + 9} = \frac{9}{(s+3)^2}$$

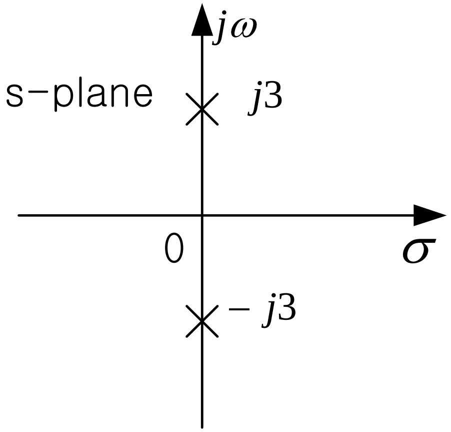


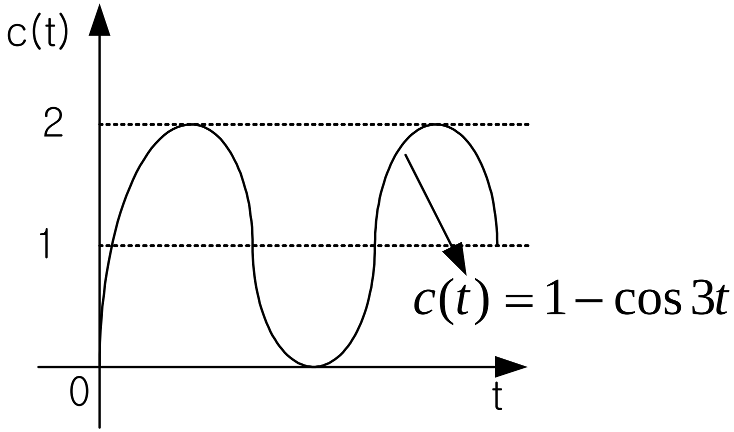


4. Oscillatory response (two imaginary poles)

$$Transient \quad response \quad : c(t) = A\cos\left(\omega t - \phi\right)$$

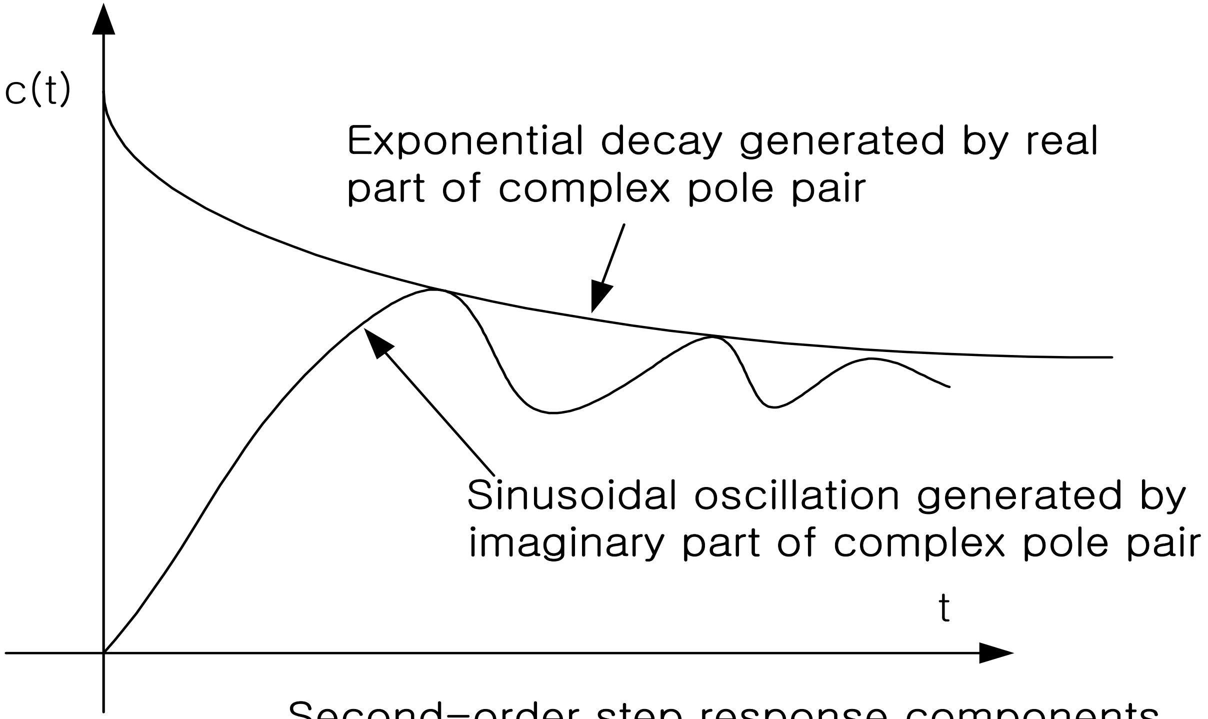
$$Ex)\ G(s) = \frac{9}{s^2+9} = \frac{9}{(s+j3)(s-j3)}$$



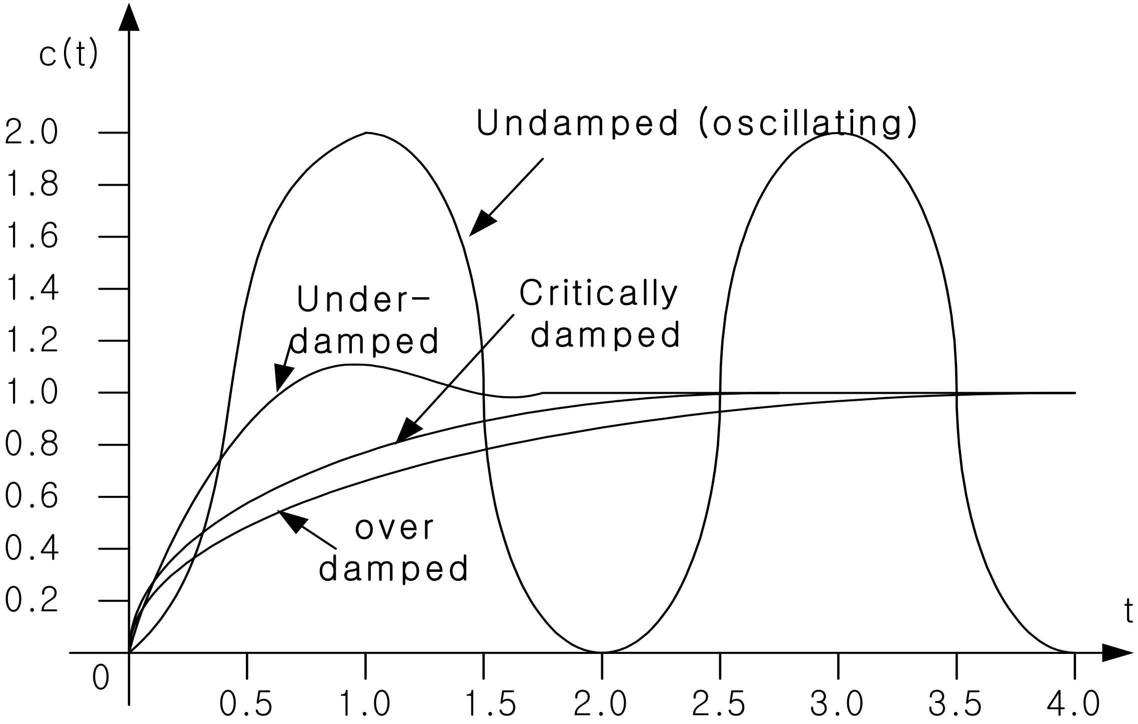


* Underdamped Response

$$C(s) = \frac{9}{s(s^2+2s+9)}$$



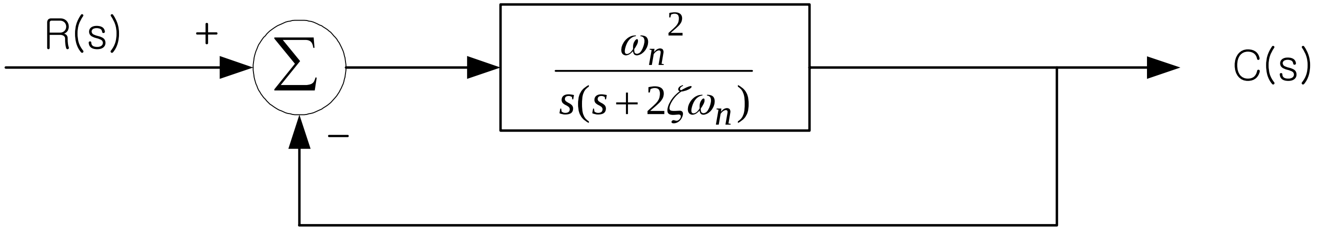
Second-order step response components generated by complex poles



Step responses for second-order system damping cases

* Second-order Systems

A prototype second-order system has the following diagram



The transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\frac{\omega_n^2}{s(s+2\zeta\omega_n)}}{1+\frac{\omega_n^2}{s(s+2\zeta\omega_n)}} \qquad \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad (1)$$

The response to a unit step input

$$\mathrm{r(t) = u(t) \ or \ R(s)} = \frac{1}{\mathrm{s}} \quad is$$

$$C(s) = \frac{{\omega_n}^2}{s(s^2 + 2\zeta\omega_n s + {\omega_n}^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{\mathrm{s}} - \frac{(s + \xi\omega) + \dfrac{\xi}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \xi\omega_n)^2 + \omega_n^2(1-\zeta^2)} \qquad (2)$$

Taking ILT yield

$$c(t) = 1 - e^{-\zeta\omega_n t}(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t)$$

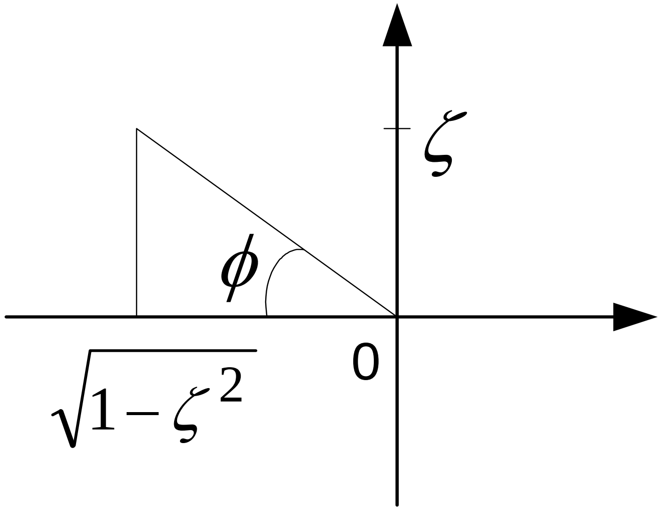
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi),\ t \geq 0$$

Where $\theta = \tan^{-1}(\frac{\zeta}{\sqrt{1-\zeta^2}})$ < As in the textbook >

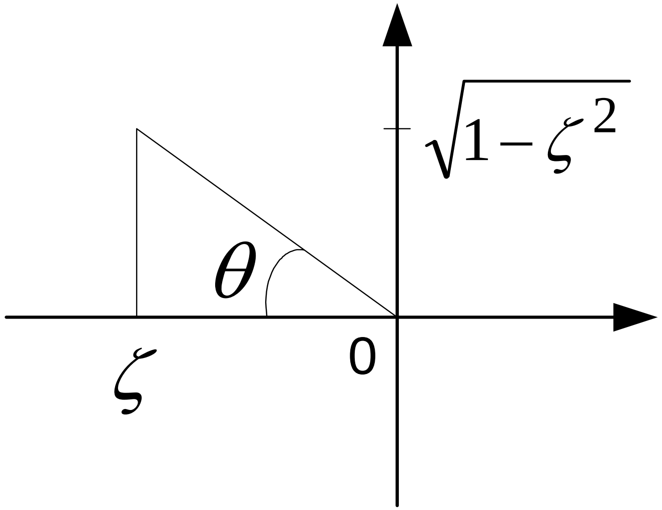
or

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \theta),\ t \geq 0$$

Where $\theta = \cos^{-1}\zeta$, $(0 < \zeta < 1)$ < Popular >



Textbook

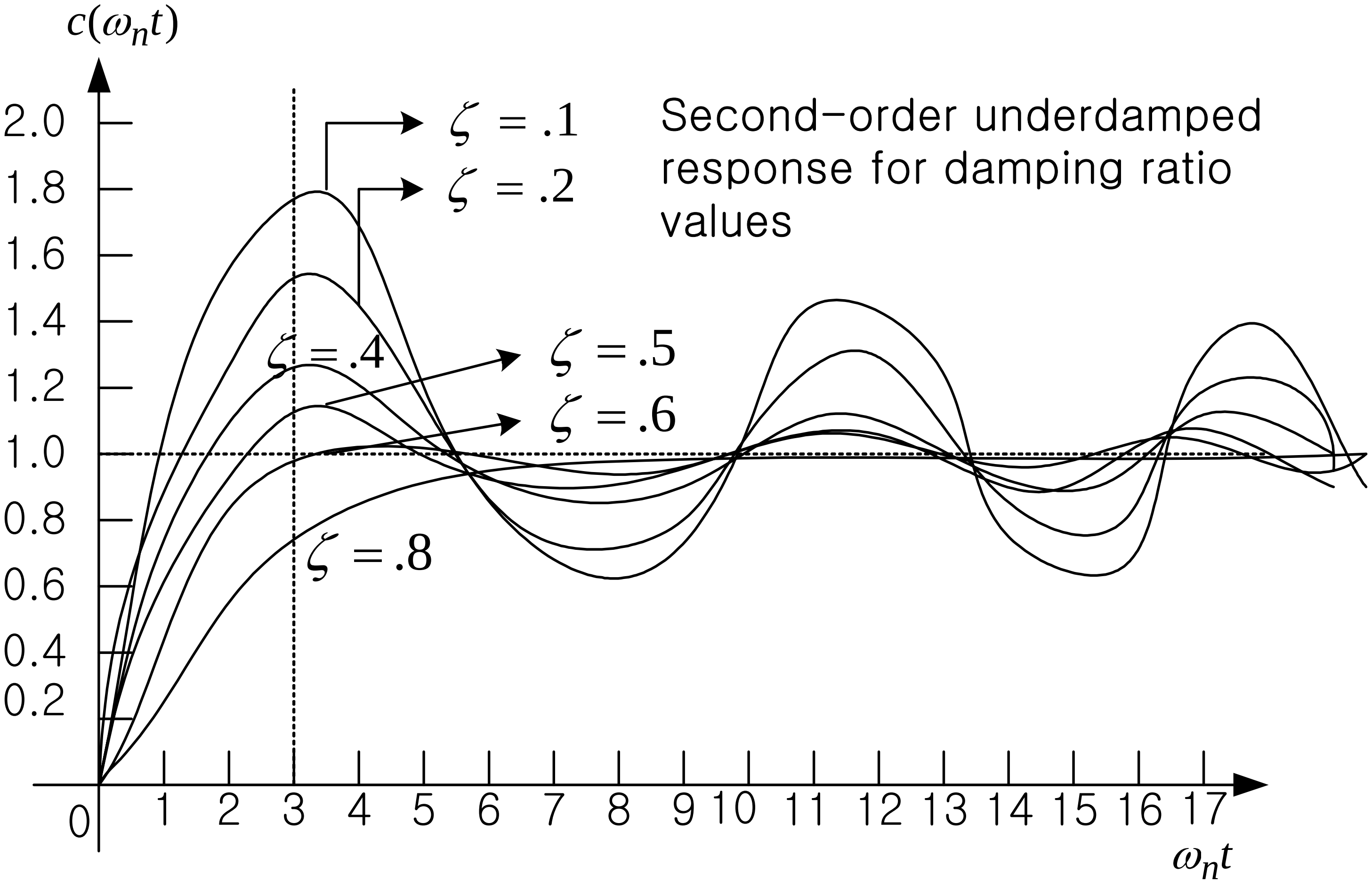


Popular

$\cos\phi = \sqrt{1-\zeta^2},\ \sin\phi = \zeta \quad \cos\theta = \zeta,\ \sin\theta = \sqrt{1-\zeta^2}$

$$\cos(A-\phi) = \cos A \cos\phi + \sin A \cos\phi$$

$$\sin(A+\theta) = \sin A \cos\theta + \cos A \sin\theta$$



The lower ζ, the more oscillatory the response.

Alternatively, we could write the differential equation which describe the system. Eq(1) is

$$\left(s^2 + 2\zeta\omega_n s + \omega_n{}^2\right)C(s) = \omega_n{}^2 R(s) \tag{4}$$

Taking the inverse LT of (4), we get

$$\frac{d^2c(t)}{dt^2} + 2\zeta\omega_n \frac{dc(t)}{dt} + \omega_n c(t) = \omega_n{}^2 r(t) \tag{5}$$

We can slove (5),

The homogeneous equation is

$$\frac{d^2c(t)}{dt^2} + 2\zeta\omega_n \frac{dc(t)}{dt} + \omega_n c(t) = 0 \tag{6}$$

The auxiliary equation is

$$r^2 + 2\zeta\omega_n r + \omega_n = 0 \qquad (7)$$

Solving for the roots of (7) gives

$$r_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n{}^2 - 4\omega_n{}^2}}{2}$$

$$r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad (8)$$

$$= -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

$$or \quad r_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}, \quad 0 < \zeta < 1$$

The homogeneous solution (the natural response) is

$$c(t) = a_1 e^{r_1 t} + a_2 e^{r_2 t} \tag{9}$$

There are four cases to consider :

1) If $\zeta > 1$, real and unequal roots

$$c(t) = a_1 e^{-\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right)t} + a_2 e^{-\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right)t} \tag{10}$$

(Note that $\sqrt{\zeta^2 - 1} < \zeta \quad \forall \zeta > 1$)

The response is said to be *overdamped*.

2) If $\zeta = 1$, real and equal roots

$$c(t) = a_1 t e^{-\omega_n t} + a_2 e^{-\omega_n t} \tag{11}$$

The response is said to be *critically damped*.

3) If $0 < \zeta < 1$, complex conjugates roots

$$c(t) = a_1 e^{-\omega_n\left(\zeta + j\sqrt{1-\zeta^2}\right)t} + a_2 e^{-\omega_n\left(\zeta - j\sqrt{1-\zeta^2}\right)t}$$

or

$$c(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_n\sqrt{1-\zeta^2}\,t + B\sin\omega_n\sqrt{1-\zeta^2}\,t\right) \quad (12)$$

The response is said to be *underdamped.*

4) If $\zeta = 0$, imaginary roots

$$c(t) = A\cos\omega_n t + B\sin\omega_n t \quad (13)$$

The response is said to be *undamped or oscillatory*.

< Definitions >

The system in (13) oscillates at frequency ω_n, so ω_n is called the natural frequency of the prototype second order system. ω_n is the frequency of oscillation of the system without damping.

The system, if underdamped, oscillates at frequency

$$\omega_d = \omega_n \sqrt{1-\zeta^2} \tag{14}$$

ω_d is called the *damped freqency*.

The damping of the underdamped system depends on the exponential term $e^{-\zeta\omega_n t}$

$\zeta\omega_n$ is called the damping factor.

At critical damping, $\zeta = 1$ so $\zeta\omega_n = \omega_n$

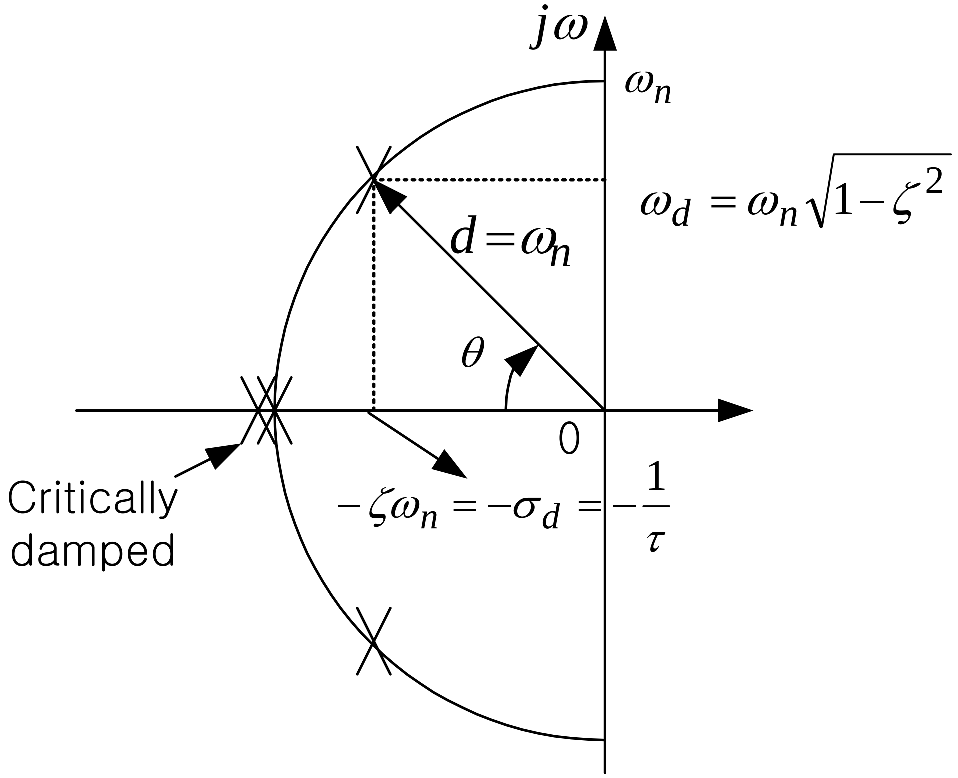
The ratio of the actual damping factor to the damping factor at critical damping is called the *damping ratio* and is given by

$$\text{damping ratio} = \frac{\zeta\omega_n}{\omega_n} = \zeta \quad \text{or}$$

$$\zeta = \frac{Exponential\ Decay\ Frequency}{Natural\ Frequency(rad/s)}$$

The Time constant is $\tau = \dfrac{1}{\zeta\omega_n}$ for $0 < \zeta < 1$

The placement in the s-plane of the roots(i.e. the poles of the closed-loop response) is as shown below :



Note that the distance of the roots from the origin is

$$d = \left[(\zeta\omega_n)^2 + \left(\omega_n \sqrt{1-\zeta^2} \right)^2 \right]^{\frac{1}{2}} = \omega_n \quad (15)$$

The angle θ is

$$\cos\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

or

$$\zeta = \cos\theta \qquad (16)$$

* Evaluation of the peak time or Maximum overshoots T_p

The response of the prototype second-order system to a unit-step input is again

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t + \cos^{-1} \zeta \right), \; t \geq 0$$

$$(\zeta < 1) \qquad (1)$$

The derivative of (1) : when set to zero, gives the necessary condition for the realtive minima and maxima. We get

$$\frac{dc(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} [\zeta \sin\left(\omega_n \sqrt{1-\zeta^2} t + \cos^{-1} \zeta \right)$$

$$- \sqrt{1-\zeta^2} \cos\left(\omega_n \sqrt{1-\zeta^2} t + \cos^{-1} \zeta \right)]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t \right) = 0 \quad , \qquad t \geq 0 \qquad (2)$$

For (2) to be satisfied, we have the possible solutions

$$\omega_n\sqrt{1-\zeta^2} = n\pi \ , \ n = 0,1,2,\cdots$$

and $\mathrm{t} = \infty$, *which gives* $t_0 = \dfrac{n\pi}{\omega_n\sqrt{1-\zeta^2}}, \ n = 0,1,2,\cdots$ (3)

The maxima can be found be taking the second derivative or by evaluating c(t) for various n. From (2) it should be easy to see that the first relative maximum occurs at

$$\mathrm{t}_0 = t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

The maximum overshoot, from (1), is

$$c(t_p)-1=-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}\sin(\pi+\cos^{-1}\zeta)$$

$$=e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \qquad (4)$$

For Eq (2) :

Let $\omega t=\omega_n\sqrt{1-\zeta^2}t$

$$\theta=\cos^{-1}\zeta \qquad (so\ \cos\theta=\zeta)$$

$$1-\cos^2\theta=1-\zeta^2=\sin^2\theta, \quad so\ \ \sin\theta=\sqrt{1-\zeta^2}$$

Use the trigonome tric identity

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

Let $\alpha=\omega t+\theta$ *and* $\beta=\theta$

Then

$$\sin\omega t = \sin(\omega t + \theta)\cos\theta - \cos(\omega t + \theta)\sin\theta$$

$$= \zeta\sin(\omega t + \theta) - \sqrt{1-\zeta^2}\cos(\omega t + \theta)$$

For Eqn, (4) :

$$-\frac{\sin(\pi + \cos^{-1}\zeta)}{\sqrt{1-\zeta^2}} = -\frac{\sin(\pi + \theta)}{\sin\theta}$$

$$= \frac{\sin\pi\cos\theta + \cos\pi\sin\theta}{-\sin\theta}$$

$$= 1$$

*Evaluation of % overshoot

$$\%OS = \frac{c_{\max} - c_{final}}{c_{final}} * 100$$

$$c_{\max} = c(T_p) = 1 - e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$= 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$For \quad unit \quad step \quad input \quad c_{final} = 1$$

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} * 100 \quad or \quad \zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

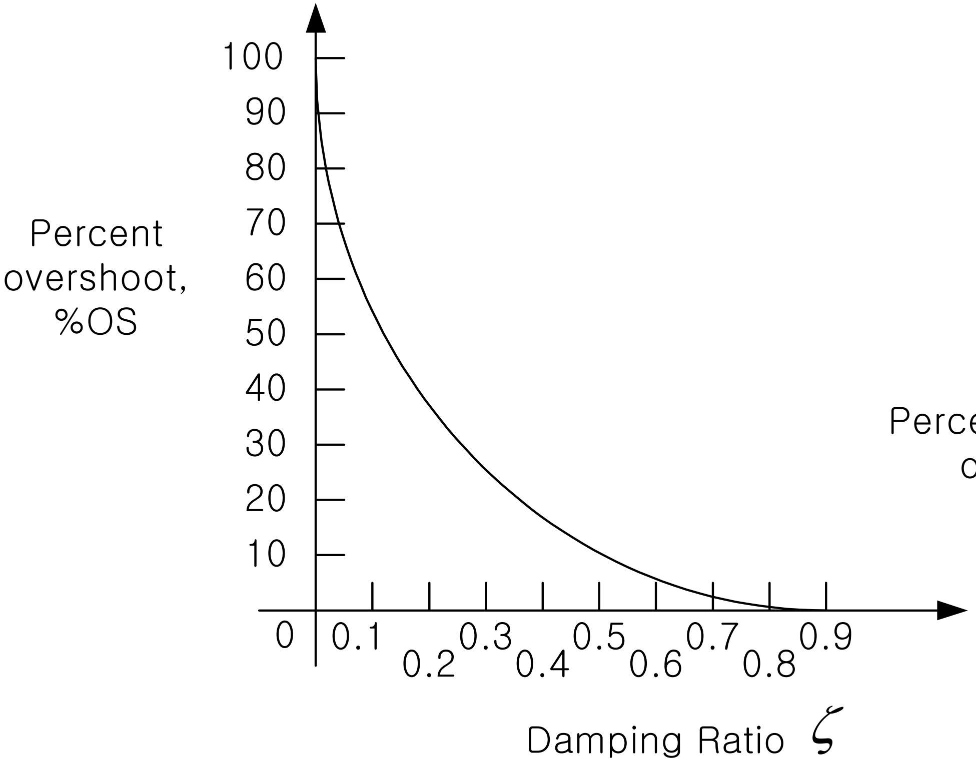
$Evaluation\ \ of\ \ T_s\ \ (overshoot\ \pm 2\%)$

$$e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}}=0.02$$

$$T_s=\frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

$$T_s=\frac{4}{\zeta\omega_n}$$

$$=\frac{4}{\sigma_d}$$



Percent overshoot vs. damping ratio

The relationship between $(T_p, \%OS, T_s)$ *and* (ω_n, ζ)

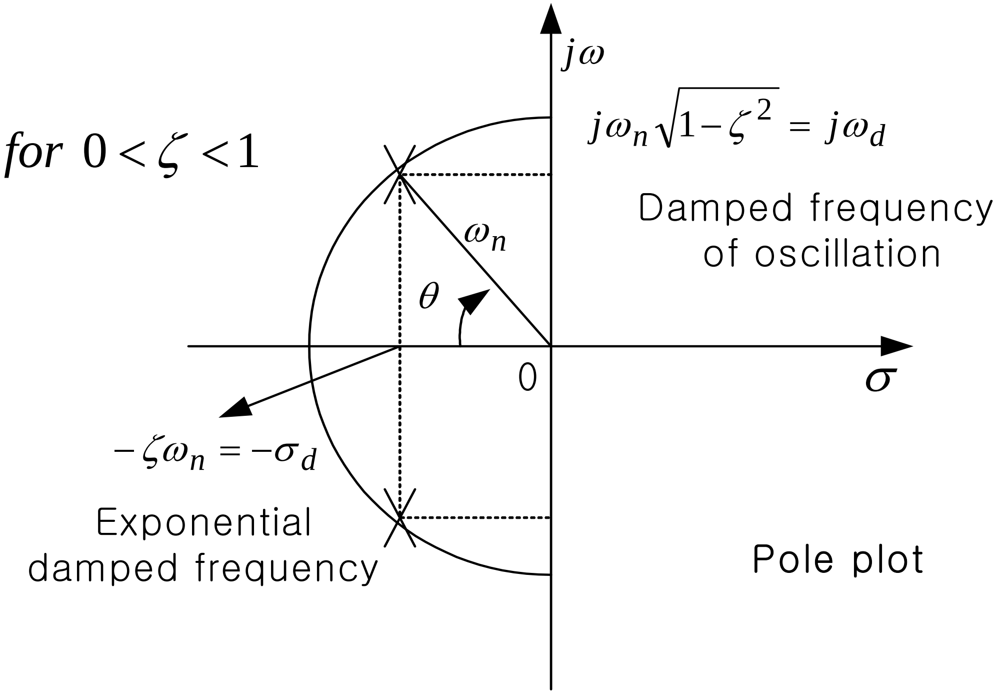
for underdamped second - order system.

$$G(s) = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Here,

$$poles \quad s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \quad for\ 0 < \zeta < 1$$



$$Let\ \ \zeta = \cos\theta \ \ \rightarrow \sin^2\theta + \cos^2\theta = 1 \rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\zeta^2}$$

$$\sigma_d = \omega_n \cos\theta = \zeta\omega_n \ \ (real): Exponential\ \ damping\ \ frequency$$

$$\omega_d = \omega_n \sin\theta = \omega_n\sqrt{1-\zeta^2}\ (imaginary)$$

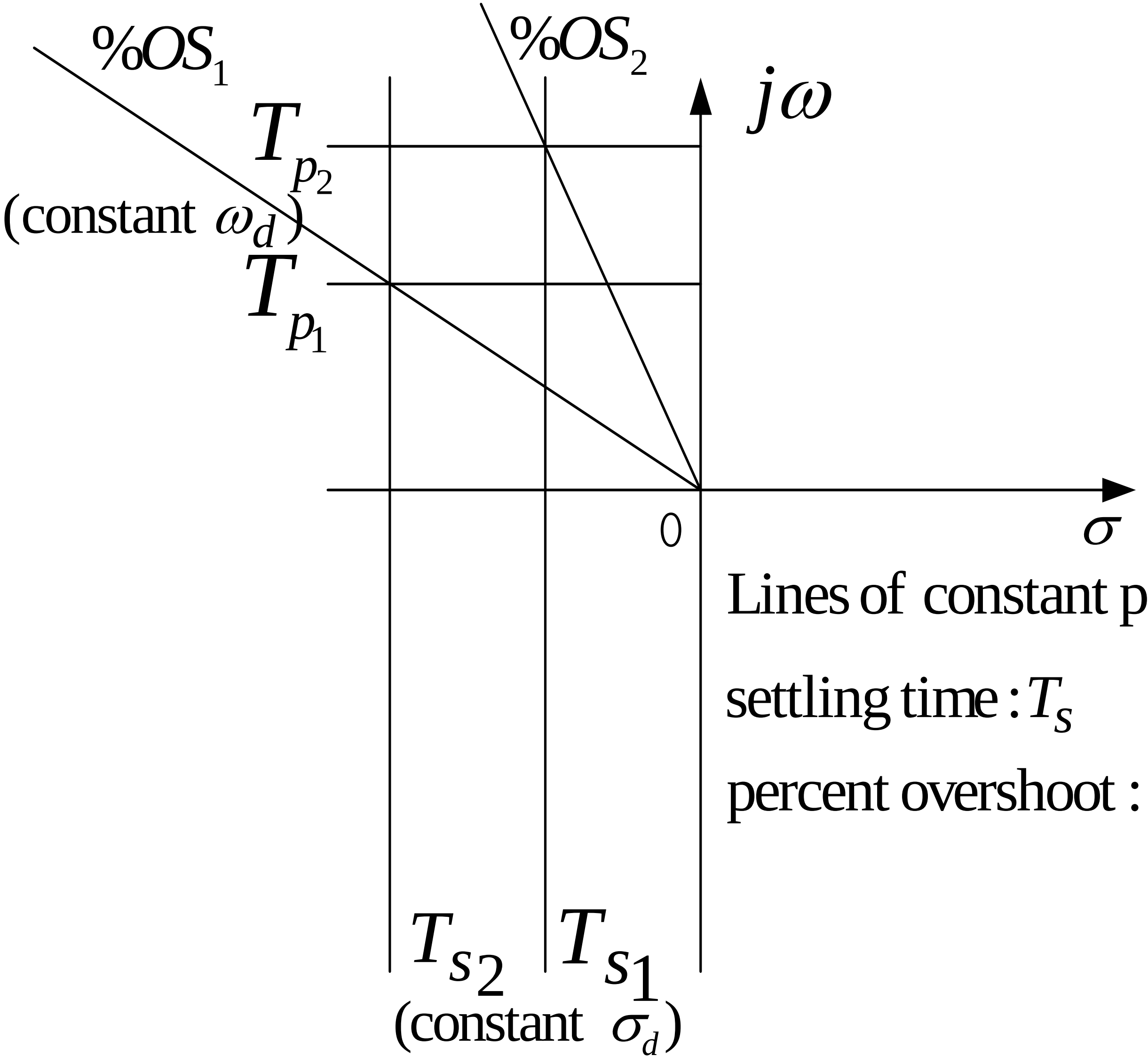
$$: Damped\ \ frequency\ \ of\ oscillation$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} : inversely\ proportional\ to\ \omega_d$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} : inversely\ proportional\ \ to\ \sigma_d$$

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} * 100 = e^{-\left(\frac{\pi\cos\theta}{\sin\theta}\right)} * 100 = e^{-\left(\frac{\pi}{\tan\theta}\right)} * 100$$

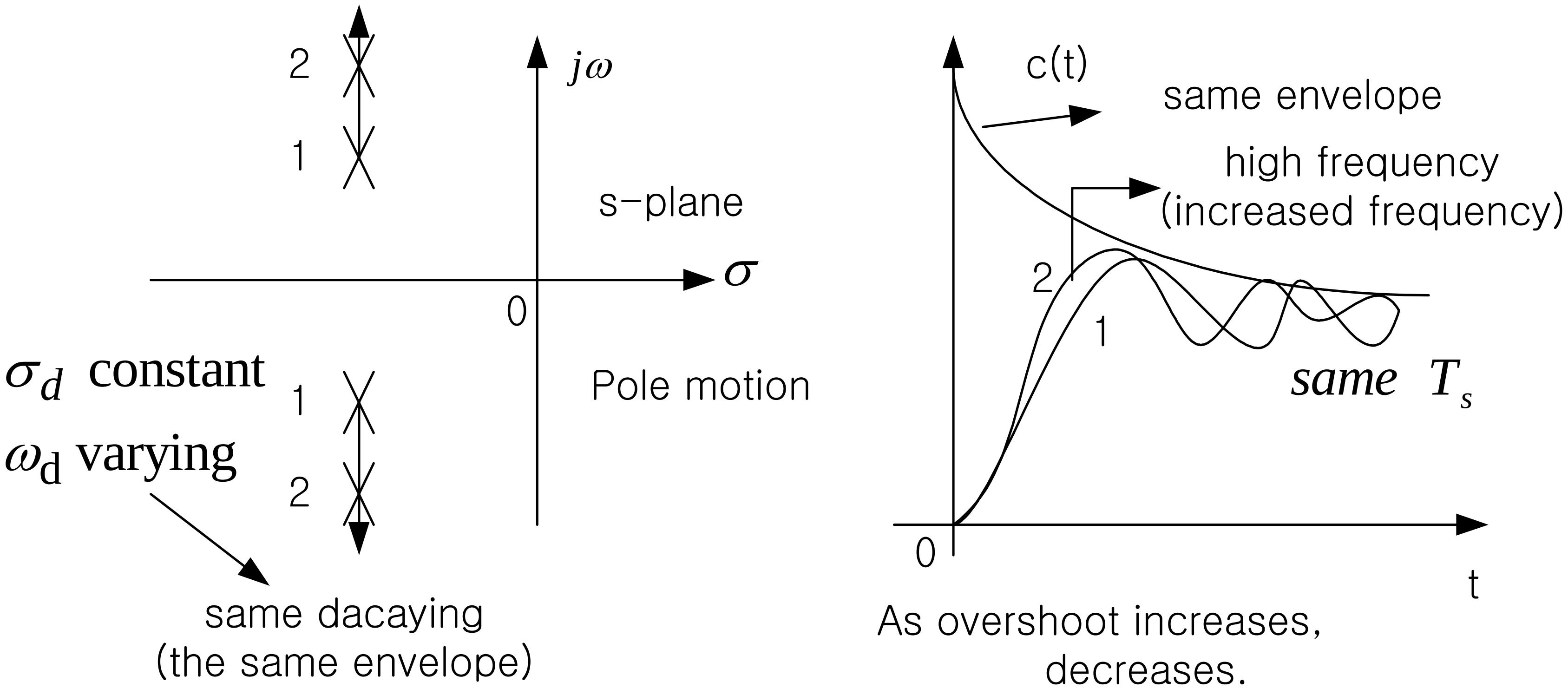
$$only\ a\ \ function\ \ of\ \ \zeta$$



Lines of constant peak time : T_p

settling time : T_s

percent overshoot : %OS



σ_d constant

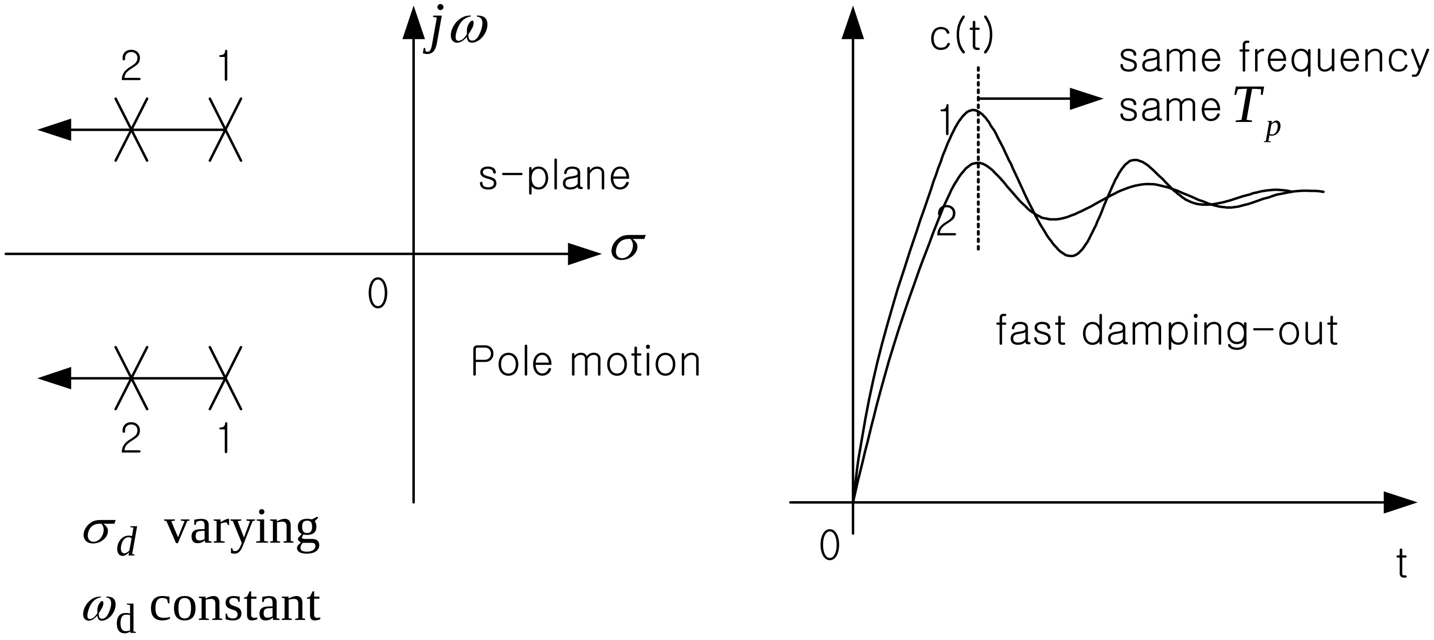
ω_d varying

same dacaying
(the same envelope)

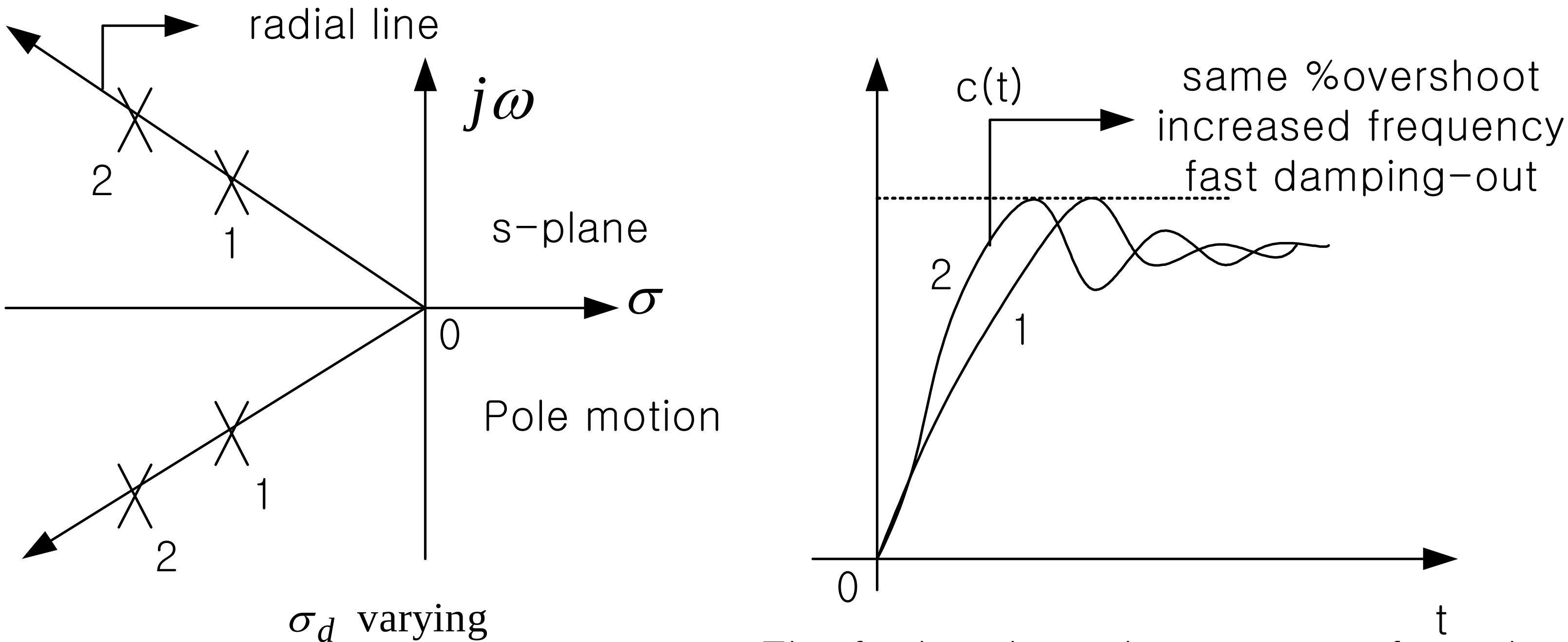
same T_s

As overshoot increases,
decreases.

a) constant real part



b) constant imaginary part



σ_d varying

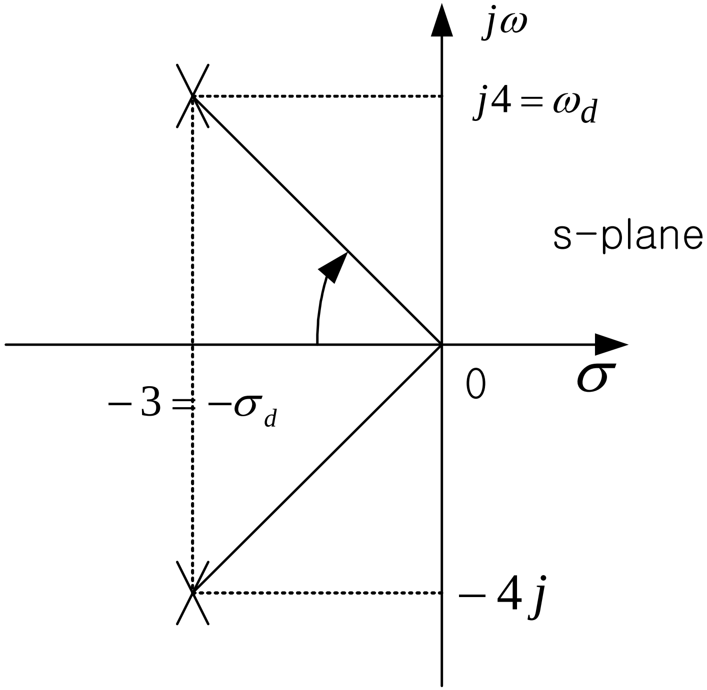
ω_d varying

The further the poles are away from the origin, the more rapid the response

c) constant damping ratio.

Ex) Given the following circuit

Find $\omega_n, \zeta, \theta, T_p, \%OS$, *and* T_s



$$\omega_n = \sqrt{3^2 + 4^2} = 5$$

$$\zeta = \cos\theta = \frac{\sigma_d}{\omega_n} = \frac{3}{5}$$

$$\theta = \cos^{-1}\frac{3}{5} = 53.13^\circ$$

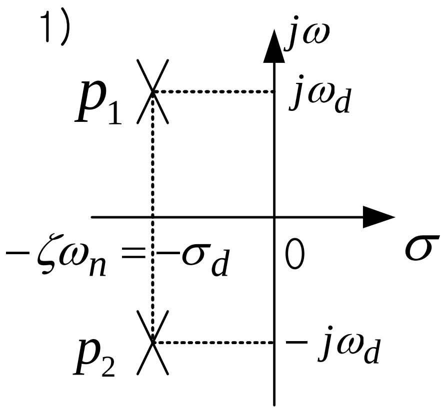
$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.78\,\text{sec}$$

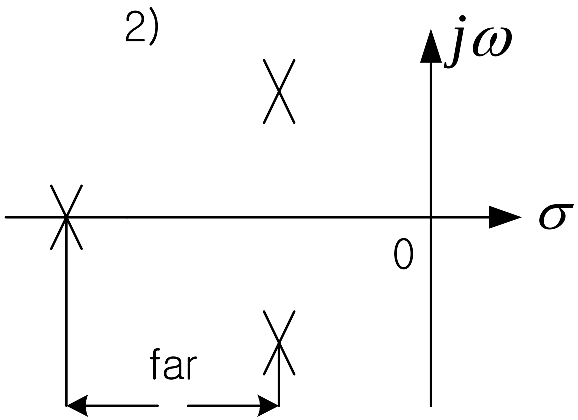
$$\%OS = e^{-\left(\zeta\pi / \sqrt{1-\zeta^2}\right)} * 100 = 9.478\%$$

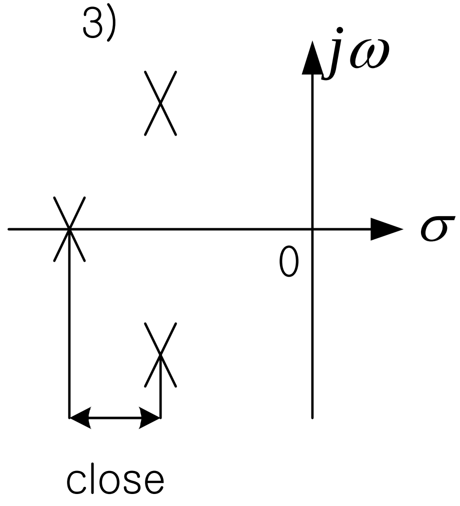
$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds.}$$

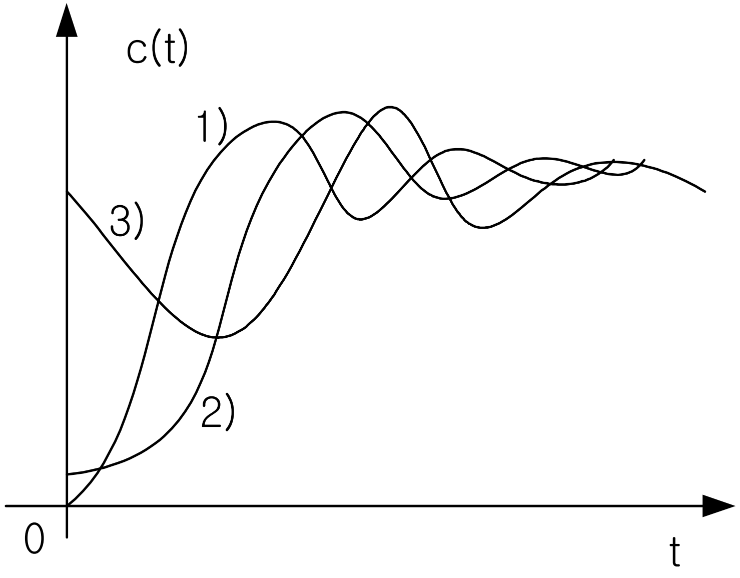
Study Example 4.7

* System Response with additional poles









If the real pole is five times further to the left than the dominant poles, (we assume) that the system is represented by its dominant second-order pair of poles.

Study Example 4.8

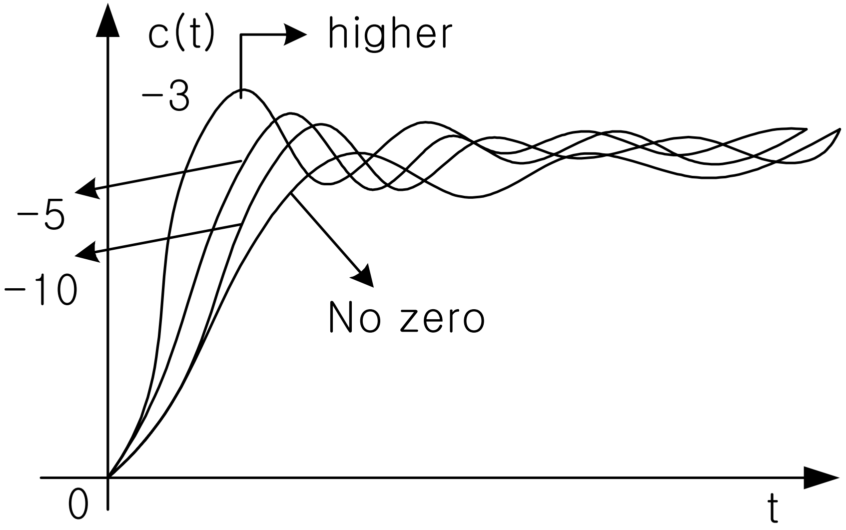
System Response with zeros

Generally, the zeros of a response affect (the residue or), amplitude but do not affect the nature of response – exponential, damped sinusoid.

$$G(s) = \frac{9}{(s+1+j2.828)(s+1-j2.828)} = \frac{9}{s^2+2s+9}$$

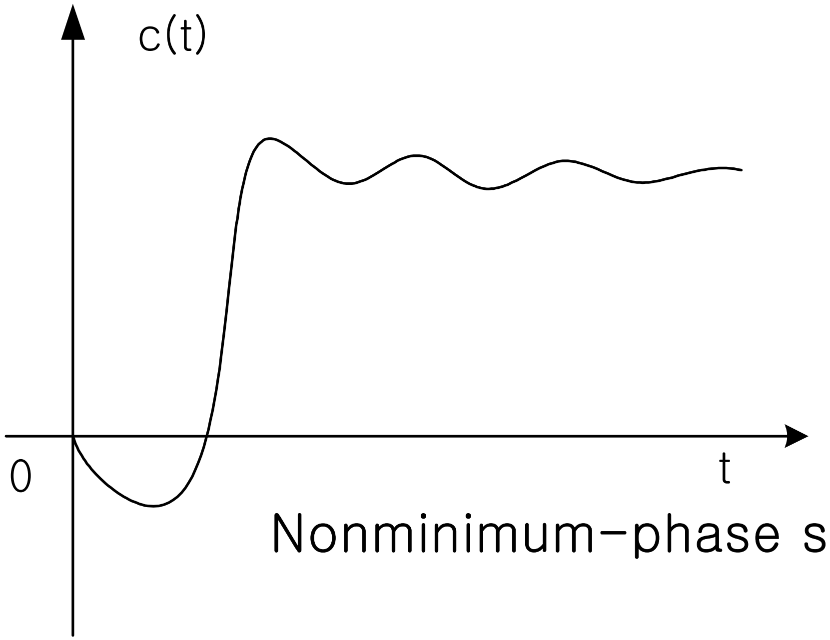
Add a real axis zero to a two pole system, −3,−5,−10

$$G(s) = \frac{9(s+a)}{s^2+2s+9}, \quad a = 3,5,10 \;\; left\ half \;\; a > 0$$



The closer the zero is to the dominant poles the greater its effect on the trasient response. As the zero moves away from the dominant poles; the response approaches that of the two-pole system.

If a is negative, placing the zero in the right-half plane.



Nonminimum-phase system

If a motorcycle is a nonminimum-phase system, it would initially veer left when commanded to steer right

Study Example 4.9

< Analysis I >

$$T(s)=\frac{(s+a)}{(s+b)(s+c)}=\frac{A}{s+b}+\frac{B}{s+c}=\frac{(-b+a)/(-b+c)}{s+b}+\frac{(-c+a)/(-c+b)}{s+c}$$

If the zero is far from the poles, then a is large compared to b and c

$$T(s)=a\left[\frac{1/(-b+c)}{s+b}+\frac{1/(-c+b)}{s+c}\right]=\frac{a}{(s+b)(s+c)}$$

The zero look like a simple gain factor and does not change the relative amplitudes of the components of the response.

< Analysis 2>

Another way to look at the effect of zero

$$T(s) = \frac{C(s)}{R(s)}$$

$$T(s) \rightarrow (s+a)T(s) : add\ a\ zero\ to\ a\ transfer\ \ function$$

$$C_a(s) = (s+a)C(s) = \underbrace{sC(s)}_{\text{derivative version}} + \underbrace{aC(s)}_{\text{scaled version}}$$

$If\ a > 0$, and $|a|$ is very large

$\mathrm{C_a}(s) = aC(s)$ scaled version of original response

$If\ a > 0$, and $|a|$ is not very large, both response important

$If\ a > 0$, and $|a|$ is very small,

derivative term dominate more overshoot

in the second-order system

Pole-Zero cancellation

$$T(s) = \frac{K\cancel{(s+z)}}{\cancel{(s+p_3)}(s^2+as+b)} \quad when \;\; z = p_3$$

Example 4.10

$$G_1(s)=\frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$

$$G_2(s)=\frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$

PFE,

$$G_1(s)=\frac{1}{s}-\frac{3.5}{s+5}+\frac{3.5}{s+6}-\frac{1}{s+3.5}$$

$$g_1(t)=1-3.5e^{-5t}+3.5e^{-6t}-e^{-3.5t}$$

$$G_2(s)=\frac{0.87}{s}-\frac{5.3}{s+5}+\frac{4.4}{s+6}+\frac{0.033}{s+4.01}$$

$$0.033\ is\ very\ small\rightarrow neglect$$

$$G_2(s)\cong\frac{0.87}{s}-\frac{5.3}{s+5}+\frac{4.4}{s+6}$$

$$g_2(t)\cong 0.87-5.3e^{-5t}+4.4e^{-6t}$$