Chapter 3. Modeling in the Time Domain

Things to know

- State-space representation
- Conversion from a transfer function and a state-space representation and vice versa
- Getting the solution using a state equation

Frequency Domain vs. Time Domain

1. Frequency domain (classical approach)

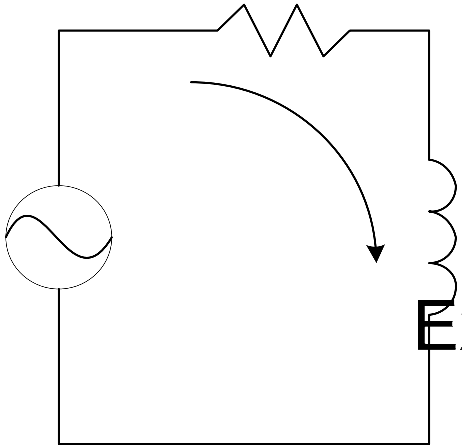
- Only for linear time-invariant (LTI) SISO systems
- Easy understanding (graphical analysis possible)
- Both zeros and poles obtainable
- Cannot predict the behavior of higher orders poles (usually for up to second-order poles)

2. Time domain or state-space domain (Modern approach)

- Unified method
- Can be extended to nonlinear, time-varying systems
- Multi-input, multi-output systems (MIMO)
- Easy computer installation
- May be sensitive to parameter changes (no specification of closed-loop zeros)

Examples: State Equation

Ex.1)



$$L\frac{di(t)}{dt}+Ri(t)=v(t) \qquad (1)$$

$$then,\ V(s)=\frac{1}{s}$$

$$I(s)=\frac{\frac{1}{s}}{Ls+R}+\frac{Li(0)}{Ls+R}$$

$$=\frac{\frac{1}{L}}{s(s+\frac{R}{L})}+\frac{i(0)}{(s+\frac{R}{L})}$$

+ *By PFE*

$$I(s)=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+\frac{R}{L}}\right)+\frac{i(0)}{s+\frac{R}{L}}$$

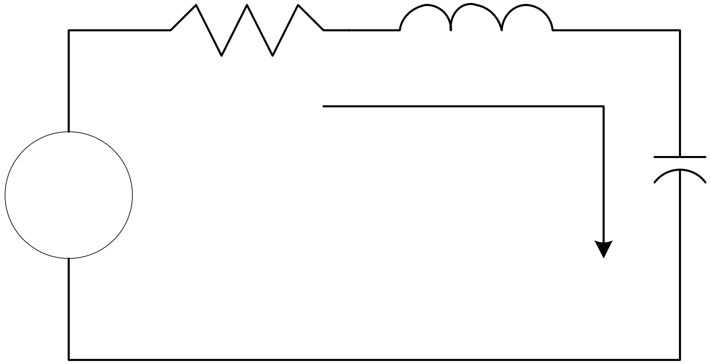
$$L^{-1}\{I(s)\}=i(t)=\frac{1}{R}\left(1-e^{-\frac{R}{L}t}\right)+i(0)e^{-\frac{R}{L}t}$$

$$\frac{di(t)}{dt} = \dot{i}(t) = -\frac{R}{L} i(t) + \frac{v(t)}{L} \Rightarrow state \;\; eq.$$

$$v_R(t) = Ri(t) \qquad \Rightarrow output \;\; eq.$$

$$or, \;\; v_L(t) = -Ri(t) + v(t) \qquad \Rightarrow output \;\; eq.$$

From (1)



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = v(t) \qquad (1)$$

$$\frac{dq}{dt} = i(t) \qquad (2)$$

Ex.2)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t) \qquad (3)$$

R

+

v(t)

-

Since the system is a second order

state equations

$$\begin{cases} \dfrac{dq}{dt} = i \\ \dfrac{di}{dt} = -\dfrac{1}{LC}q - \dfrac{R}{L}i + \dfrac{1}{L}v(t) \end{cases}$$

in matrix form

$$\begin{bmatrix} \dfrac{dq}{dt} \\ \dfrac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{LC} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} q \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} v(t)$$

$$\dot{x} = A \quad x + B \quad u$$

$$from \quad (1)$$

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t)$$

in matrix form for output equation

$$[v_L(t)] = \begin{bmatrix} -\dfrac{1}{C} & -R \end{bmatrix} \begin{bmatrix} q \\ i \end{bmatrix} + [1]\,v(t)$$

$$y = C \quad x + D \quad u$$

State variable form

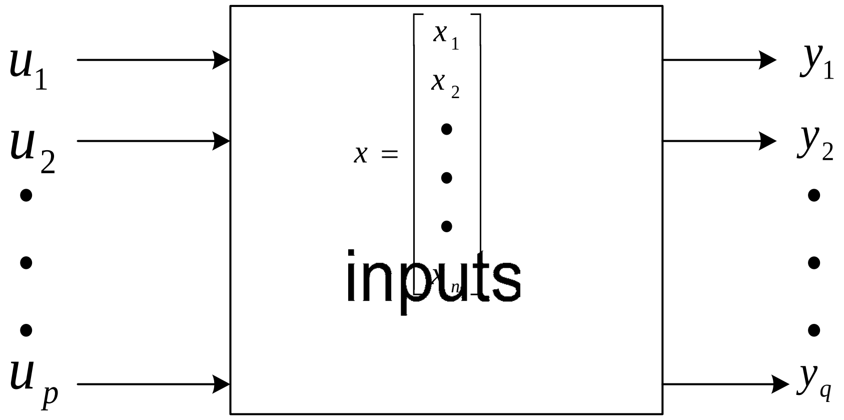
State variables (상태변수)
: (Linearly independent) variables that can completely define the behavior of the system.

Output variable (출력변수): A variable(s) that can be measured.

Note: In the RLC circuit example, q and i should be linearly independent.
Q: What happens if v and i are chosen?
Answer: Will not work because v and i are linearly dependent,
that is, $v = R\, i$.

Q: A state representation is unique?

General state-space representation



$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$(n \times 1)$

$$\dot{x} = derivative \ \ of \ \ x = \frac{dx(t)}{dt} \ (n \times 1)$$

y= output vector $(q \times 1)$

u= input or control vector $(p \times 1)$

A= system matrix $(n \times n)$

B= input coupling matrix $(n \times p)$

C= output matrix $(q \times n)$

D= feedforward matrix $(q \times p)$

linearly independent state variables

Converting a D.E. to a state equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \bullet\bullet\bullet + a_1\frac{dy}{dt} + a_0 y = u(t)$$

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \dot{x}_1$$

$$x_3 = \frac{d^2 y}{dt^2} = \dot{x}_2 = \ddot{x}_1$$

$$\vdots$$

$$x_n = \frac{d^{n-1} y}{dt^{n-1}} = \dot{x}_{n-1}$$

$$\frac{d^n y}{dt^n} = \dot{x}_n = -a_0 x_1 - a_1 x_2 \bullet\bullet\bullet - a_{n-1} x_n + u(t)$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \bullet \\ \bullet \\ \bullet \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 1 & 0 & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \bullet \\ \bullet & \bullet & \bullet & \bullet & & \bullet & & \bullet \\ \bullet & \bullet & \bullet & \bullet & & & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \bullet & \bullet & \bullet & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \bullet \\ \bullet \\ \bullet \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bullet \\ \bullet \\ \bullet \\ 0 \\ 1 \end{bmatrix} u(t)
$$

output equations

$$
y = \begin{bmatrix} 1 & 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \bullet \\ \bullet \\ \bullet \\ x_{n-1} \\ x_n \end{bmatrix}
$$

Converting a transfer function to a state equation

$$\frac{C(s)}{R(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

Assuming zero initial conditions, convert the transfer function to a differential equation as follows:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r(t)$$

$$x_1 = c$$

$$x_2 = \dot{c} = \dot{x}_1$$

$$x_3 = \ddot{c} = \dot{x}_2$$

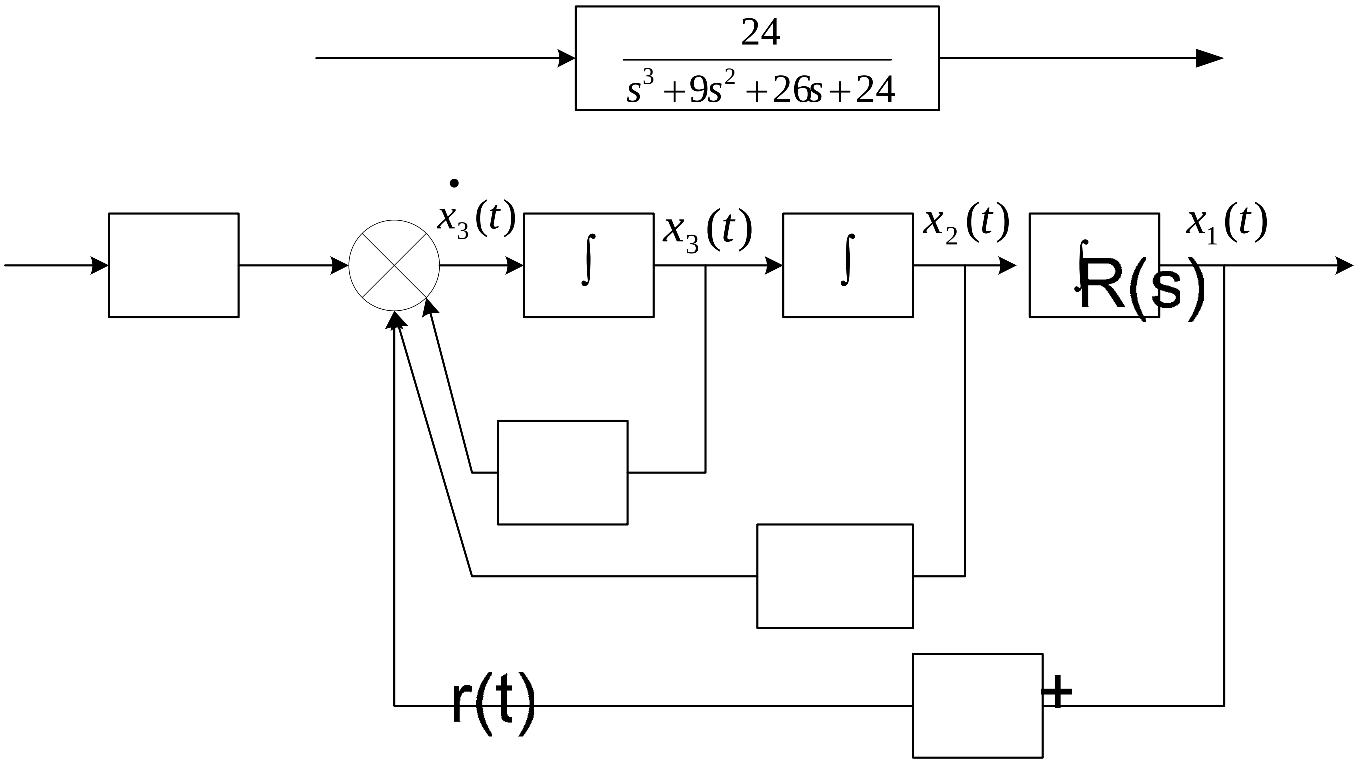
$$\dot{x}_3 = \dddot{c} = -9\ddot{c} - 26\dot{c} - 24c + 24r$$

$$= -24x_1 - 26x_2 - 9x_3 + 24r$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r(t)
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

Block diagram



24

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- - -

Converting a transfer function with a polynomial numerator

$n \geq m$

$$\frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Assume

$$\frac{1}{s^n + \frac{a_{n-1}}{a_n} s^{n-1} \cdots + \frac{a_1}{a_n} s + \frac{a_0}{a_n}} \xrightarrow{x_1(s)} \frac{b_m}{a_n} s^m + \frac{b_{m-1}}{a_n} s^{m-1} + \cdots \frac{b_1}{a_n} s + \frac{b_0}{a_n}$$

R(s) *Internal* variables

$x_2(s), x_3(s) \cdots$

$$Y(s) = C(s) = \left(\frac{b_m}{a_n} s^m + \frac{b_{m-1}}{a_n} s^{m-1} + \cdots \frac{b_1}{a_n} s + \frac{b_0}{a_n}\right) x_1(s)$$

Taking ILT with zero initial conditions

$$y(t) = \frac{b_m}{a_n} \frac{d^m x_1}{dt^m} + \frac{b_{m-1}}{a_n} \frac{d^{m-1} x_1}{dt^{m-1}} + \cdots + \frac{b_1}{a_n} \frac{dx_1}{dt} + b_0 x_1$$

R(s)

$$y(t) = \frac{b_0}{a_n} x_1 + \frac{b_1}{a_n} x_2 + \cdots + \frac{b_{m-1}}{a_n} x_m + \frac{b_m}{a_n} x_{m+1}$$

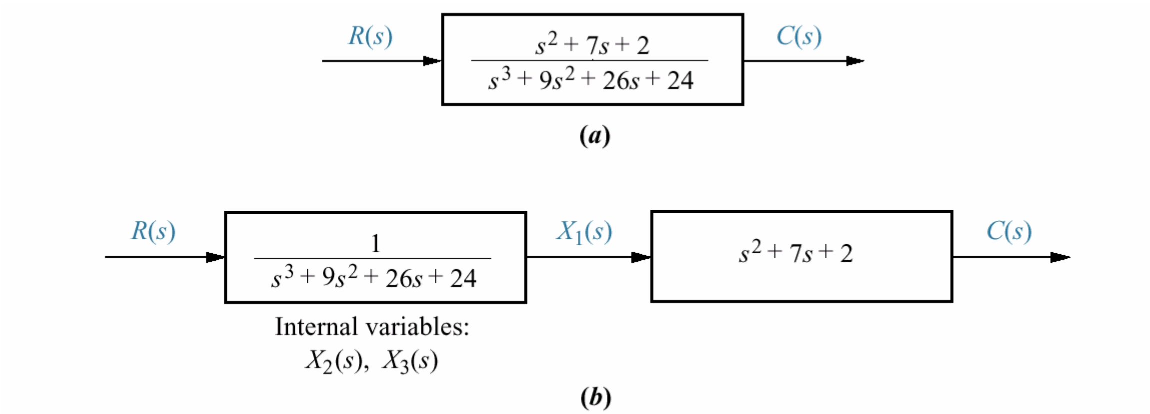
The output equation is

$$y = \begin{bmatrix} \frac{b_0}{a_n} & \frac{b_1}{a_n} & \cdot & \cdot & \cdot & \frac{b_m}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{m+1} \end{bmatrix}$$

The state equation is the same as before

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdot & \cdot & \cdot & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} r$$

Example: Converting a transfer function with a polynomial numerator



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad (3.63)$$

$$C(s) = \left(b_2 s^2 + b_1 s + b_0\right) X_1(s) = (s^2 + 7s + 2) X_1(s) \qquad (3.64)$$

$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1 \qquad (3.65)$$

$$x = x_1 \;, \quad \dot{x}_1 = x_2 \;, \quad \ddot{x}_1 = x_3$$

$$y = c(t) = b_2 x_3 + b_1 x_2 + b_0 x_1 = x_3 + 7x_2 + 2x_1 \qquad (3.66)$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (3.67)$$

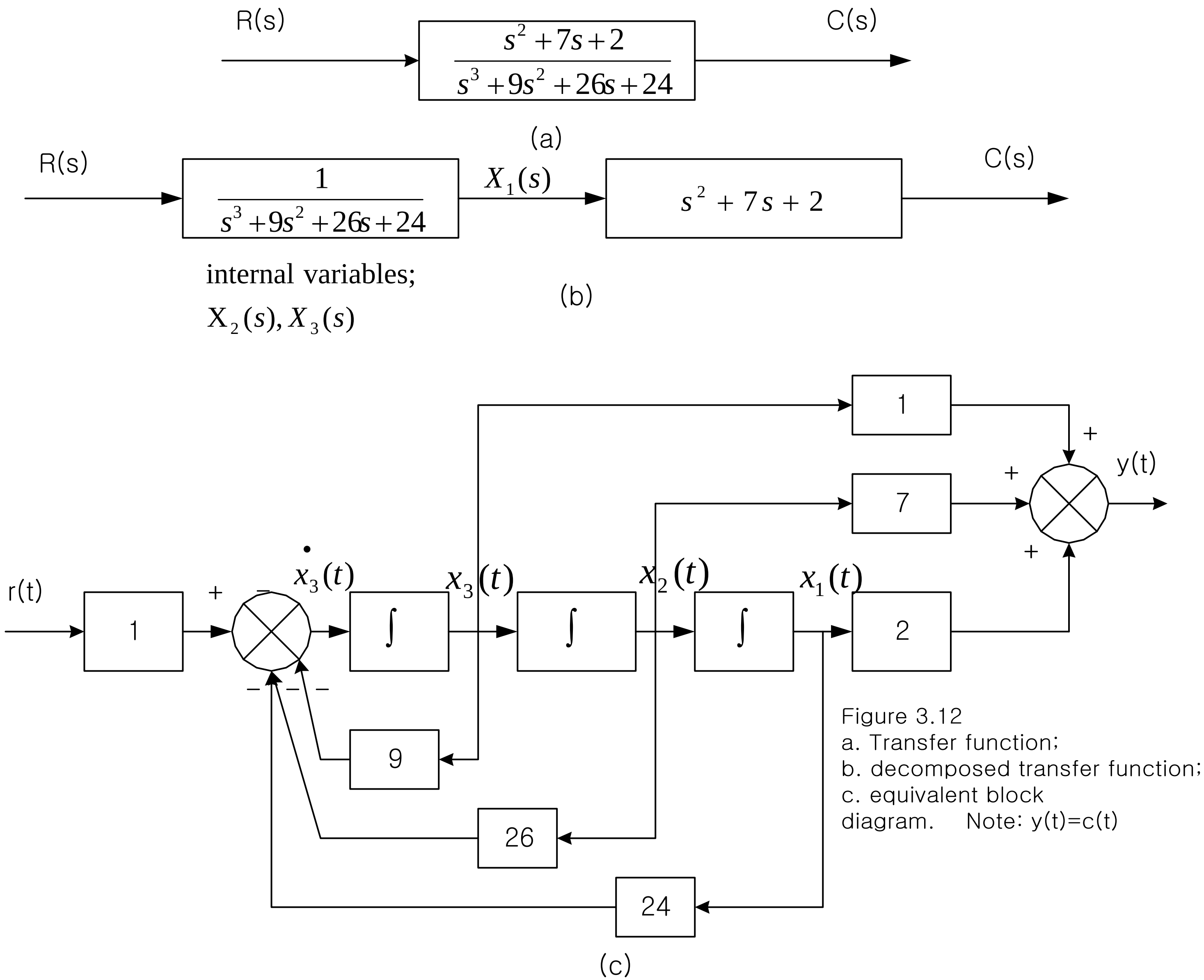


Figure 3.12
a. Transfer function;
b. decomposed transfer function;
c. equivalent block diagram. Note: y(t)=c(t)

Converting a state equation to a transfer function

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Taking the Laplace transform with zero initial conditions yields

$$sX(s) = AX(s) + BU(s), \quad Y(s) = CX(s) + DU(s)$$
$$(sI - A)X(s) = BU(s)$$
$$or \;\; X(s) = (sI - A)^{-1}BU(s)$$

where I is the identity matrix. Hence

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$
$$= [C(sI - A)^{-1}B + D]U(s)$$

$$Y(s) = T(s)U(s)$$

where

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \quad : \text{transfer function}$$

Example: Converting a state equation to a transfer function

Find $T(s) = \dfrac{Y(s)}{R(s)}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{adj\,(sI - A)}{\det(sI - A)} = \frac{\begin{bmatrix} s^2 + 3s + 2 & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$\det(sI - A) = s\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 1 & s+3 \end{vmatrix} + 0\begin{vmatrix} 0 & s \\ 1 & 2 \end{vmatrix}$$

$$= s(s^2 + 3s + 2) + 1$$

$$= s^3 + 3s^2 + 2s + 1$$

$$cofactor \;\; c_{ij} = (-1)^{i+j} M_{ij} \quad \text{(signed minor)}$$

$$Adj \;\; A = \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2n} \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ c_{n1} & c_{n2} & \cdot & \cdot & \cdot & c_{nn} \end{bmatrix}^T \quad (Adj\, A = c_{ij}{}^T)$$

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

Solving the state equation using the Laplace transform

Taking the Laplace form

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$X(s) = [sI - A]^{-1}[x(0) + BU(s)] \qquad (1)$$

$$Y(s) = CX(s) + DU(s) \qquad (2)$$

$(1) \rightarrow (2)$

$$Y(s) = C[sI - A]^{-1}[x(0) + BU(s)] + DU(s)$$

When $x(0) = 0$,

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

The transfer function is defined as

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

or $$T(s) = C\left[\frac{adj(sI - A)}{\det(sI - A)}\right]B + D$$

or $$T(s) = \frac{C\, adj(sI - A)B + D\det(sI - A)}{\det(sI - A)} \qquad (3)$$

Example

$$Ex)\ A = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$Find \quad T(s)\ and \quad t(t)$$

$$T(s) = \frac{1}{(s+2)(s+3)}$$

$$t(t) = L^{-1}\{T(s)\} = L^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+3}\right\}$$

$$t(t) = e^{-2t} - e^{-3t}$$

Poles, zeros, and eigen values

Transfer function $T(s) = \frac{N(s)}{D(s)} = C(sI - A)^{-1}B + D$

Poles: the roots of $D(s) = 0$

Zeros: the roots of $N(s) = 0$

Eigen values: the roots of $\det(sI\text{-}A) = 0$ $\Rightarrow$ poles

Definition: For a given operator A, if there exists a non-zero vector x_i that satisfies

$$Ax_i = \lambda_i x_i$$

then, λ_i is called an eigen value of A and x_i is called the associated eigen vector.

Note: λ_i is called a natural frequency (or an eigenvalue) of the system $\dot{x} = Ax$.

Why : Assume that $x(t) = e^{\lambda_i t} x_0$, where x_0 is an initial condition. Then

$$\dot{x}(t) = \lambda_i e^{\lambda_i t} x_0 = Ax = Ae^{\lambda_i t} x_0$$

$$\lambda_i x_0 = Ax_0$$

$$(\lambda_i I - A)x_0 = 0 \qquad (1)$$

Hence, λ_i is an eigenvalue and x_0 is the corresponing eigenvector of the matrix A.

For a nonzero x_0, (1) is satisfied if and only if $\det(\lambda_i I - A) = 0$.

A zero is a value of frequency s such that the input is $u(t) = u_0 e^{st}$ then the output is zero. That is, the system has a nonzero input signal, but nothing comes out.

$$\text{Let} \quad x(t) = e^{st} x_0$$

$$\dot{x} = Ax + Bu$$

$$\dot{x} = s x_0 e^{st} = A x_0 e^{st} + B u_0 e^{st}$$

$$[(sI - A) x_0 - B u_0] e^{st} = 0$$

For a nontrivial solution,

$$e^{st} \neq 0$$

$$\text{then}, (sI - A) x_0 - B u_0 = 0 \qquad (2)$$

$$y = Cx + Du$$

$$= C e^{st} x_0 + D u_0 e^{st} = 0$$

$$(C x_0 + D u_0) e^{st} = 0$$

$$\text{Since} \quad e^{st} \neq 0$$

$$C x_0 + D u_0 = 0 \qquad (3)$$

combining (2) *and* (3) *yields*

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution

$$\begin{vmatrix} sI - A & -B \\ C & D \end{vmatrix} = 0$$

$$\det\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0$$

The roots are zeros .

$$\therefore T(s) = \frac{\det\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}}{\det[sI - A]}$$

Ex .) Given that

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$A = \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

The zeros are such that

$$\det\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0$$

$$\begin{vmatrix} s+3 & -1 & -1 \\ 0 & s & -2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$= (s+3)\begin{vmatrix} s & -2 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & s \\ 1 & 0 \end{vmatrix}$$

$$= 0 + 2 + s = s + 2 = 0$$

$$zero \quad is \quad s = -2$$

$$\det[sI - A] = \begin{vmatrix} s+3 & -1 \\ 0 & s \end{vmatrix} = s(s+3)$$

$$T(s) = \frac{s+2}{s(s+3)}$$

Previously

$$\mathrm{T(s)} = \mathrm{C}\left[\frac{\mathrm{adj(sI - A)}}{\det(sI - A)}\right]B + D$$

$$= \frac{1}{s(s+3)}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & 1 \\ 0 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} s & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{s(s+3)} = \frac{s+2}{s(s+3)} \Rightarrow The \quad same \quad result \quad .$$

Solving the state equation directly in the time domain

Consider $\dot{x}(t) = Ax(t)$ (*)

If $x(t) = e^{At}x(0)$ (1)

then, $\dot{x}(t) = Ae^{At}x(0) = Ax(t)$ (2)

Hence, (1) is the solution of (*)

*Mathematical interpretation for $x(t) = e^{At}x(0)$

$$Let\ x(t) = b_0 + b_1 t + \cdots + b_k t^k + b_{k+1} t^{k+1} + \cdots \quad (3)$$

$$\dot{x}(t) = b_1 + 2b_2 t + \cdots + k b_k t^{k-1} + (k+1) b_{k+1} t^k + \cdots \quad (4)$$

$$= A(b_0 + b_1 t + b_2 t^2 \cdots + b_k t^k + b_{k+1} t^{k+1} + \cdots) \quad (5)$$

$$= Ax(t)$$

Equating (4) and (5) yields

$$b_1 = Ab_0$$

$$b_2 = \frac{1}{2} Ab_1 = \frac{1}{2} A^2 b_0$$

$$b_3 = \frac{1}{3} Ab_2 = \frac{1}{3} A\left(\frac{1}{2} A^2 b_0\right) = \frac{1}{1 \times 2 \times 3} A^3 b_0 = \frac{1}{3!} A^3 b_0 \quad (6)$$

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$$b_k = \frac{1}{k!} A^k b_0$$

$$b_{k+1} = \frac{1}{(k+1)!} A^{k+1} b_0$$

$(6) \rightarrow (3)$

$$x(t) = b_0 + Ab_o t + \frac{1}{2} A^2 b_o t^2 + \cdots + \frac{1}{k!} A^k b_o t^k + \frac{1}{(k+1)!} A^{k+1} b_o t^{k+1} + \cdots$$

$$= (I + At + \frac{1}{2} A^2 t^2 + \cdots + \frac{1}{k!} A^k t^k + \cdots) b_0$$

$$x(t) = e^{At} x(0)$$

<Review>

If f(z) is analytic at z_0

then it has a derivative at every point of G

- $Power \;\; series$

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

- $Taylor \;\; series$

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)(z - z_0)^j}{j!}$$

- $Maclaurin \;\; series$

$when, \; z_0 = 0 \; in \; Taylor \;\; series.$

State transition matrix (or fundamental matrix) $\Phi(t)$

$\Phi(t) = e^{At}$: Definition

Then the solution is given by $x(t) = e^{At}x(0) = \Phi(t)x(0)$

– Properties of $\Phi(t)$

(1) $\Phi(0) = I$ since $x(0) = \Phi(0)x(0)$

$$or \quad \Phi(0) = e^{A(0)} = I$$

(2) $\dot{\Phi}(0) = A$ since $\dot{x}(t) = \dot{\Phi}(t)x(0) = Ae^{At}x(0)$

(3) $\Phi(-t) = \Phi^{-1}(t)$ or $\Phi^{-1}(t) = e^{-At} = \Phi(-t)$

since $\Phi(t)\Phi^{-1}(t) = e^{At}e^{-At} = e^{0t} = I$

* Homogenous (or unforced) system solution

For a homogeneous system $\dot{x}(t) = Ax(t)$

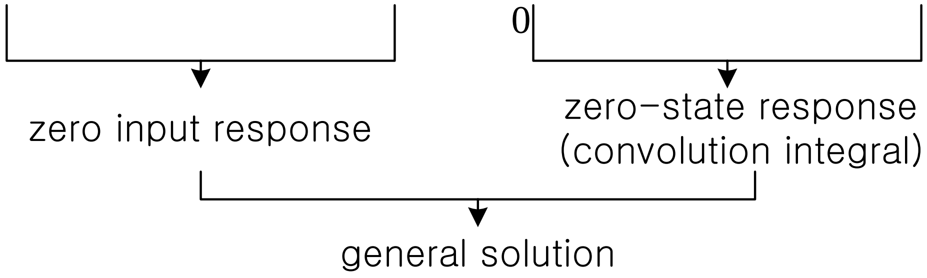
The solution is $x(t) = \Phi(t)x(0)$

where $\Phi(0) = I$ and $\dot{\Phi}(0) = A$

Solution of a nonhomogeneous (or forced) system

$$system : \dot{x}(t) = Ax(t) + Bu(t) \qquad (1$$

$$solution : x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$



< Proof 1 > − Time Domain

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\dot{x}(t) - Ax(t) = Bu(t)$$

$$e^{-At}\left[\dot{x}(t) - Ax(t)\right] = e^{-At} bu(t) \qquad (2)$$

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = -Ae^{-At}x(t) + e^{-At}\dot{x}(t)$$

$$= e^{-At}\left[\dot{x}(t) - Ax(t)\right] \qquad (3)$$

From (2) and (3)

$$\int_o^t \frac{d}{d\tau}\left[e^{-A\tau}x(\tau)\right]d\tau = e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$e^{-At}x(t) = x(0) + \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

< Proof 2 > – Transform Domain

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Taking LT

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = x(0) + BU(s)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

Taking ILT

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$L\{\Phi(t)\} = \int_0^\infty \Phi(t) e^{-st} dt$$

$$= \int_0^\infty e^{At} e^{-st} dt$$

$$= \int_0^\infty e^{-(sI-A)t} dt = \left. \frac{e^{-(sI-A)t}}{-(sI-A)} \right|_0^\infty$$

$$= 0 + \frac{1}{(sI-A)} = (sI-A)^{-1}$$

$$< note > L\{\Phi(t)\} = \Phi(s) = (sI-A)^{-1}$$

$$L^{-1}\{\Phi(s)\} = \Phi(t) = L^{-1}\left\{ \frac{adj(sI-A)}{\det(sI-A)} \right\}$$

Each term of $\Phi(t)$ would be the sum of exponentials generated by the system's poles.

Example: state transition matrix $\Phi(t)$

Find the state transition matrix and the solution $x(t)$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}$$

$$\det[sI - A] = s(s+6) + 8 = s^2 + 6s + 8 = (s+2)(s+4)$$

$$\text{poles}: s_1 = -2\ ,\ s_2 = -4$$

$$\Phi(t) = \begin{bmatrix} \left(k_1 e^{-2t} + k_2 e^{-4t}\right) & \left(k_3 e^{-2t} + k_4 e^{-4t}\right) \\ \left(k_5 e^{-2t} + k_6 e^{-4t}\right) & \left(k_7 e^{-2t} + k_8 e^{-4t}\right) \end{bmatrix}$$

$$\Phi(0) = I$$

$$k_1 + k_2 = 1\ ,\ k_3 + k_4 = 0, k_5 + k_6 = 0, k_7 + k_8 = 1$$

$$\dot{\Phi}(0) = A$$

$$-2k_1 - 4k_2 = 0\ ,\ -2k_3 - 4k_4 = 1,\ -2k_5 - 4k_6 = -8,\ -2k_7 - 4k_8 = -6$$

$$k_2 = 1 - k_1\ ,\ -2k_1 - 4(1 - k_1) = 0$$

$$k_1 = 2$$

Similarly, $k_2 = -1, k_3 = \frac{1}{2}, k_4 = -\frac{1}{2}, k_5 = -4, k_6 = 4, k_7 = -1, k_8 = 2$

$$\Phi(t) = \begin{bmatrix} (2e^{-2t} - e^{-4t}) & \left(\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}\right) \\ (-4e^{-2t} + 4e^{-4t}) & (-e^{-2t} + 4e^{-4t}) \end{bmatrix}$$

$$\Phi(t-\tau)B = \begin{bmatrix} \frac{1}{2}e^{-2(t-\tau)} & -\frac{1}{2}e^{-4(t-\tau)} \\ -e^{-2(t-\tau)} & +4e^{-4(t-\tau)} \end{bmatrix}$$

$$\Phi(t)x(0) = \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ -4e^{-2t} + 4e^{-4t} \end{bmatrix}$$

$$\int_0^t \Phi(t-\tau)Bu(\tau)d\tau = \begin{bmatrix} \frac{1}{2}e^{-2t}\int_0^t e^{2\tau}d\tau - \frac{1}{2}e^{-4t}\int_0^t e^{4\tau}d\tau \\ -e^{-2t}\int_0^t e^{2\tau}d\tau + 2e^{-4t}\int_0^t e^{4\tau}d\tau \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{8} - \frac{1}{4}e^{-2t} + \frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}$$

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$= \begin{bmatrix} \frac{1}{8} + \frac{7}{4}e^{-2t} - \frac{7}{8}e^{-4t} \\ -\frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t} \end{bmatrix} \quad \text{(The solution)}$$

Obtaining the state transition matrix using $(sI-A)^{-1}$

$$(sI-A)=\begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix} \tag{4.148}$$

$$(sI-A)^{-1}=\frac{\begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix}}{s^2+6s+8}=\begin{bmatrix} \dfrac{s+6}{s^2+6s+8} & \dfrac{1}{s^2+6s+8} \\ \dfrac{-8}{s^2+6s+8} & \dfrac{s}{s^2+6s+8} \end{bmatrix} \tag{4.149}$$

$$(sI-A)^{-1}=\begin{bmatrix} \left(\dfrac{2}{s+2}-\dfrac{1}{s+4}\right) & \left(\dfrac{1/2}{s+2}-\dfrac{1/2}{s+4}\right) \\ \left(\dfrac{-4}{s+2}+\dfrac{4}{s+4}\right) & \left(\dfrac{-1}{s+2}+\dfrac{2}{s+4}\right) \end{bmatrix} \tag{4.150}$$

$$\Phi(t)=\begin{bmatrix} \left(2e^{-2t}-e^{-4t}\right) & \left(\dfrac{1}{2}e^{-2t}-\dfrac{1}{2}e^{-4t}\right) \\ \left(-4e^{-2t}+4e^{-4t}\right) & \left(-e^{-2t}+2e^{-4t}\right) \end{bmatrix} \tag{4.151}$$

Solving DE:
Comparison of three methods

$$\dddot{y} + 4\ddot{y} + 29\dot{y} = 29u$$

$$\begin{pmatrix} y(0) = 0, \dot{y}(0) = 17, \ddot{y}(0) = -122 \\ u(t) = 1(t) \end{pmatrix}$$

Method 1: Conventional method

general solution ⟨ homogeneous solution / particular solution : due to forcing term

$$y = y_h + y_p$$

i) homogeneous solution : y_h

$$\dddot{y}_h + 4\ddot{y}_h + 29\dot{y}_h = 0, \quad y(0) = 0, \quad \dot{y}(0) = 17, \quad \ddot{y}(0) = -122$$

put $y_h(t) = e^{\lambda t}$

$$(\lambda^3 + 4\lambda^2 + 29\lambda)e^{\lambda t} = 0$$

$$\therefore \lambda^3 + 4\lambda^2 + 29\lambda = 0$$

$$(\lambda^2 + 4\lambda^1 + 29)\lambda = 0$$

$$\lambda = 0, \text{ or } \lambda = \frac{-4 \pm \sqrt{16 - 116}}{2} = -2 \pm j5$$

$$\therefore y_h(t) = Ae^{(-2 \pm j5)t} + C_3$$

$$= e^{-2t}(C_1 \cos 5t + C_2 \sin 5t) + C_3$$

ii) particular solution : y_p

If the input is a unit step function, we expect tha t the output may be the form of step function because the system is linear. But once we plug $y_p = D$ into the R.H.S. of D.E. then it becomes zero.
Therefore we increase the order of y_p, like $y_p = D_1 t + D_2$.

then by substituting

$$0 + 0 + 29D_1 = 29 \qquad \therefore D_1 = 1$$

Therefore the general solution is

$$y(t) = y_h(t) + y_p(t)$$

$$= e^{-2t}(C_1 \cos 5t + C_2 \sin 5t) + t + C_4$$

where

$$C_4 = C_3 + D_2$$

By applying I.C.

$$y(0) = C_1 + C_4 = 0$$

$$\dot{y}(t) = -2e^{-2t}(C_1\cos 5t + C_2\sin 5t) + e^{-2t}(-5C_1\sin 5t + 5C_2\cos 5t) + 1$$

$$\ddot{y}(t) = 4e^{-2t}(C_1\cos 5t + C_2\sin 5t) - 2e^{-2t}(-5C_1\sin 5t + 5C_2\cos 5t)$$

$$-2e^{-2t}(-5C_1\sin 5t + 5C_2\cos 5t) + e^{-2t}(-25C_1\cos 5t - 25C_2\sin 5t)$$

$$= e^{-2t}(-21C_1\cos 5t - 21C_2\sin 5t) + e^{-2t}(20C_1\sin 5t - 20C_2\cos 5t)$$

$$\dot{y}(0) = -2C_1 + 5C_2 + 1 = 17$$

$$\ddot{y}(0) = -21C_1 - 20C_2 = -122$$

$$\therefore C_1 = 2, C_2 = 4, C_4 = -2$$

$$\therefore y_g(t) = 2e^{-2t}\cos 5t + 4e^{-2t}\sin 5t + t - 2$$

Method 2: Taking the Laplace transform to both sides, i.e.

$$L(\dddot{y} + 4\ddot{y} + 29\dot{y}) = L(29u)$$

$$s^3Y(s) - s^2y(0) - s\dot{y}(o) - \ddot{y}(0)$$
$$+ 4\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + 29\left[sY(s) - y(0)\right] = 29U(s)$$

$$\Rightarrow (s^3 + 4s^2 + 29s)Y(s) = s^2y(0) + s\left[\dot{y}(0) + 4y(0)\right] + \left[\ddot{y}(0) + 4\dot{y}(0) + 29y(0)\right] + 29U(s)$$
$$= (17s - 54) + \frac{29}{\mathrm{s}}$$

$$Y(s) = \frac{17s - 54 + 29/s}{s^3 + 4s^2 + 29s} = \frac{17s^2 - 54s + 29}{\left[(s+2)^2 + 5^2\right]s^2}$$

The denominator polynomial has two real roots at $s = 0$,

a pair of complex roots at $s = \text{-}2 \pm j5$

Expand in partial fractions

$$\frac{c_1(s+2)+s_1(5)}{(s+2)^2+5^2}+\frac{R_{21}}{s}+\frac{R_{22}}{s^2}$$

where

$$S_1 + jC_1 = \frac{1}{5}\left[\frac{17s^2-54s+29}{s^2}\right]_{s=-2\pm j5} = 4 + j2$$

$$R_{22} = \left[\frac{17s^2-54s+29}{(s+2)^2+5^2}\right]_{s=0} = 1$$

$$R_{21} = \frac{1}{1!}\left[\frac{122s^2+928s-1682}{((s+2)^2+5^2)^2}\right]_{s=0} = -2$$

By the inverse Laplace transform

$$y(t) = 2e^{-2t}\cos 5t + 4e^{-2t}\sin 5t - 2 + t$$

$$= \sqrt{20}e^{-2t}\sin(5t+0.464) - 2 + t$$

Method 3. State space method

$$\dddot{y} + 4\ddot{y} + 29\dot{y} = 29u$$

$$\left(\begin{array}{l} y(0) = 0,\ \dot{y}(0) = 17,\ \ddot{y}(0) = -122 \\ u(t) = 1(t) \end{array} \right)$$

Define a state variable

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}, \quad then \quad \dot{X} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} X2 \\ X3 \\ -4\ddot{y} - 29\dot{y} + 29u \end{bmatrix}$$

$$\Rightarrow \quad \dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -29 & -4 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 29 \end{bmatrix} u, \qquad X(0) = \begin{bmatrix} 0 \\ 17 \\ -122 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

$$\left(\begin{array}{l} \text{Note that the above is of the form} \\ \quad \dot{X} = AX + Bu,\ X(0) = X_0 \\ \quad y = CX \end{array} \right)$$

The Laplace transform of a vector is the Laplace transform of individual components as

$$L(X(t)) = L\left(\begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{bmatrix}\right) = \left(\begin{bmatrix} L(X_1(t)) \\ L(X_2(t)) \\ \vdots \\ L(X_n(t)) \end{bmatrix}\right) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$

Taking the Laplace transform

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = X(0) + BU(s)$$

$$\therefore \quad X(s) = (sI - A)^{-1}X(0) + (sI - A)^{-1}BU(s)$$

where

$(sI - A)^{-1}$ is called a resolvent matrix

and $y(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}X(0)$

$$(sI - A)^{-1} = \left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -29 & -4 \end{bmatrix}\right)^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 29 & s+4 \end{bmatrix}^{-1}$$

Recallthat

$$A^{-1} = \frac{\text{adj}A}{\det A} \qquad \text{(Seetextp.35)}$$

$$\det A = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 29 & s+4 \end{vmatrix} = s\begin{vmatrix} s & -1 \\ 29 & s+4 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 0 & s+4 \end{vmatrix} + 0$$

$$= s(s^2+4s+29)$$

$$\text{adj}A = \begin{bmatrix} s^2+4s+29 & 0 & 0 \\ s+4 & s^2+4s & -29s \\ 1 & s & s^2 \end{bmatrix}^T$$

$$\therefore \ (sI-A)^{-1} = \frac{1}{s(s^2+4s+29)}\begin{bmatrix} s^2+4s+29 & s+4 & 1 \\ 0 & s^2+4s & s \\ 0 & -29s & s^2 \end{bmatrix}$$

Continue....